Bell inequalities in phase space and their violation in quantum mechanics\vspace*{4pt}.

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Abstract

We derive “Bell inequalities” in four dimensional phase space and prove the following “three marginal theorem” for phase space densities $\rho(q, p)$, thus settling a long standing conjecture: “there exist quantum states for which more than three of the quantum probability distributions for $(q_1, q_2), (p_1, p_2), (q_1, p_2)$ and $(p_1, q_2)$ cannot be reproduced as marginals of a positive $\rho(q, p)$”. We also construct the most general positive $\rho(q, p)$ which reproduces any three of the above quantum probability densities for arbitrary quantum states. This is crucial for the construction of a maximally realistic quantum theory.
I. Joint probabilities of conjugate observables

A quantum system permits many different choices \( \{A\}, \{B\}, \{C\} \ldots \) for a complete commuting set (CCS) of observables. If \( \{\alpha\} \) denotes a set of eigenvalues of \( \{A\}, \{\beta\} \) of \( \{B\}, \{\gamma\} \) of \( \{C\} \) etc, quantum theory predicts the probabilities of observing \( \{\alpha\} \) in the experimental context to measure \( \{A\} \), and similarly of \( \{\beta\}, \{\gamma\} \ldots \) in different contexts, but not a joint probability of \( \{\alpha\}, \{\beta\}, \{\gamma\} \ldots \), because they refer to noncommuting observables. Thus quantum theory predicts probabilities for observing “eigenvalues” \( \{\vec{q}\} \) of position operators \( \vec{Q} \) in one context, or \( \{\vec{p}\} \) of momentum operators \( \vec{P} \) in another context, but not their joint probability.

We can ask whether the “contextual” quantum probabilities can be extended so as to encompass joint probabilities of noncommuting observables. The difficulty in extending quantum probabilities for different CCS of observables \( \{A\}, \{B\}, \{C\} \ldots \) (which do not mutually commute) to a joint probability of the different CCS is the lesson learnt from decades of work on quantum contextuality theorems [1]-[3]. Of these the most celebrated is Bell’s theorem [2] where the Einstein-locality postulate in the context of the EPR paradox is equivalent to a postulate of existence of a joint probability for the different CCS \( \{\vec{\sigma}_1, \vec{a}, \vec{b} \}, \{\vec{\sigma}_1, \vec{a}', \vec{b}' \}, \{\vec{\sigma}_1, \vec{a}'', \vec{b}'' \} \) and \( \{\vec{\sigma}_1, \vec{a}' \vec{b} \}, \{\vec{\sigma}_2, \vec{b} \} \) for the system of two spin-half particles. Here, \( \vec{\sigma}_1 \) and \( \vec{\sigma}_2 \) are Pauli spin operators for the two particles, \( \vec{a}, \vec{a}', \vec{b}, \vec{b}' \) are arbitrary unit vectors. The postulate leads to the Bell-CHSH [2] inequalities which are in conflict with quantum spin correlations.

Consider now the conjugate observables position and momentum. The first phase space formulation of quantum mechanics is due to Wigner [4] who defined the phase space distribution \( \rho(\vec{q}, \vec{p}) \) to be

\[
\rho_W(\vec{q}, \vec{p}) \equiv \int \frac{d\vec{q}' d\vec{p}'}{(2\pi)^{3N}} \langle \vec{q}' - \frac{\vec{q}'}{2} | \hat{\rho} | \vec{q}' + \frac{\vec{q}'}{2} \rangle \exp(i\vec{p}' \cdot \vec{q}')
\]

where \( \hat{\rho} \) is the density operator of the quantum state. The marginals of this phase space distribution reproduce the quantum position and momentum probabilities

\[
\int d\vec{p} \rho(\vec{q}, \vec{p}) = \langle \vec{q} | \hat{\rho} | \vec{q} \rangle, \quad \int d\vec{q} \rho(\vec{q}, \vec{p}) = \langle \vec{p} | \hat{\rho} | \vec{p} \rangle. \tag{1}
\]

However, \( \rho_W(\vec{q}, \vec{p}) \) cannot be interpreted as a phase space probability density as it is not in general positive. Cohen and Zaporovanny [5] found the most general positive phase space density function obeying the quantum marginal conditions (1) for configuration space dimension \( N = 1 \), and Cohen [6] found them for general \( N \). These density functions can be considered as generalizations of the simple uncorrelated positive function \( \langle \vec{q} | \hat{\rho} | \vec{q} \rangle \langle \vec{p} | \hat{\rho} | \vec{p} \rangle \) which satisfies (1). These results might raise hopes that quantum probabilities of all CCS can be reproduced as marginals of one phase space density. This is false.

Martin and Roy [3] showed that for 2-dimensional configuration space the postulate of existence of a positive phase space density is in conflict with the hypothesis that appropriate “marginals” of this density reproduce the quantum probability densities of the different CCS \( \{Q_1 \cos \alpha + Q_2 \sin \alpha, -P_1 \sin \alpha + P_2 \cos \alpha \} \) for all \( \alpha \).
On the positive side, Roy and Singh [7] discovered that not only the quantum probabilities of the two CCS \( Q \rightarrow P \) (which have no observables in common) but in fact the quantum probabilities for a chain of \((N + 1)\) CCS, for example \((Q_1, Q_2, \ldots, Q_N), (P_1, P_2, Q_3, \ldots, Q_N), \ldots (P_1, P_2, \ldots, P_N)\) can be simultaneously reproduced as marginals of one positive phase space density:

\[
\int dp_1 dp_2 \ldots dp_N \rho(\vec{q}, \vec{p}) = \langle q_1 q_2 \ldots q_N | \hat{\rho} | q_1 q_2 \ldots q_N \rangle, \\
\int dq_1 dp_2 \ldots dp_N \rho(\vec{q}, \vec{p}) = \langle p_1 q_2 \ldots q_N | \hat{\rho} | p_1 q_2 \ldots q_N \rangle, \\
\vdots\\n\int dq_1 dq_2 \ldots dq_N \rho(\vec{q}, \vec{p}) = \langle p_1 p_2 \ldots p_N | \hat{\rho} | p_1 p_2 \ldots p_N \rangle,
\]

They conjectured that it is impossible to find, for every quantum state, a positive phase space density whose marginals reproduce quantum probabilities of more than \(N + 1\) CCS of observables.

Our main purpose here is: i) to prove this long standing conjecture and thus quantify the extent of simultaneous realizability of noncommuting CCS as marginals of a positive phase space density, ii) to construct explicitly the most general positive phase space density which reproduces the quantum probabilities of the maximum number of CCS as marginals. This will enable the construction of the most general “maximally realistic” quantum theory, generalizing the special construction of Roy and Singh [7]-[8] which reproduces \((N + 1)\) CCS. The earliest realistic quantum theory, viz. that of de Broglie and Bohm [9] (dBB) which reproduces only one CCS (position) is of course not maximally realistic. It would be interesting to compare particle trajectories of maximally realistic quantum theories given here with the dBB trajectories.

From the mathematical standpoint, our basic results are theorems concerning multidimensional Fourier transforms. They can be expected to open up new applications in classical signal and image processing as they vastly improve the earlier results of Cohen [6] (which only considered marginals with no variables in common).

In the present paper, we restrict ourselves to the case \(N = 2\). We formulate the four marginal problem and develop the new tool of “phase space Bell inequalities” to solve the problem. The resulting “three marginal theorem” proves the conjecture of Roy and Singh for \(N = 2\). The most general maximally realistic phase space densities reproducing quantum probabilities of three CCS are then explicitly constructed. Full details are given in [10]. The corresponding results for general \(N\), more involved, will be reported separately [11].

II. Four marginal problem

Consider a four dimensional phase space with position variables \((q_1, q_2)\) and momentum variables \((p_1, p_2)\). Let \(\{\sigma_{qq}(q_1, q_2), \sigma_{qp}(q_1, p_2), \sigma_{pq}(p_1, q_2), \sigma_{pp}(p_1, p_2)\}\) be arbitrary given normalized probability distributions. Is it possible to find a normalized
phase space density $\rho(\overrightarrow{q}, \overrightarrow{p})$ of which $\sigma_{qq}$, $\sigma_{qp}$, $\sigma_{pq}$ and $\sigma_{pp}$ are marginals? i.e.

$$\int dp_1 dp_2 \rho(\overrightarrow{q}, \overrightarrow{p}) = \sigma_{qq}(q_1, q_2), \quad \int dp_1 dq_2 \rho(\overrightarrow{q}, \overrightarrow{p}) = \sigma_{qp}(q_1, p_2),$$

$$\int dq_1 dp_2 \rho(\overrightarrow{q}, \overrightarrow{p}) = \sigma_{pq}(p_1, q_2), \quad \int dq_1 dq_2 \rho(\overrightarrow{q}, \overrightarrow{p}) = \sigma_{pp}(p_1, p_2), \quad (3)$$

$$\rho(\overrightarrow{q}, \overrightarrow{p}) \geq 0, \quad \int d\overrightarrow{q} d\overrightarrow{p} \rho(\overrightarrow{q}, \overrightarrow{p}) = 1.$$  

It is obvious from (3) that the given probabilities $\sigma_{qq}$, $\sigma_{qp}$, $\sigma_{pq}$ and $\sigma_{pp}$ must at least obey the consistency conditions

$$\sigma_{qq}, \sigma_{qp}, \sigma_{pq}, \sigma_{pp} \geq 0, \quad (4)$$

and

$$\int dq_2 \sigma_{qq}(q_1, q_2) = \int dp_2 \sigma_{qp}(q_1, p_2), \quad \int dq_1 \sigma_{qq}(q_1, q_2) = \int dp_1 \sigma_{pq}(p_1, q_2),$$

$$\int dq_1 \sigma_{qp}(q_1, p_2) = \int dp_1 \sigma_{pp}(p_1, p_2), \quad \int dq_2 \sigma_{pq}(p_1, q_2) = \int dp_2 \sigma_{pp}(p_1, p_2), \quad (5)$$

We therefore pose the following problem which we shall call the **four marginal problem**: Given four normalized probability distributions $\sigma_{qq}$, $\sigma_{qp}$, $\sigma_{pq}$ and $\sigma_{pp}$ obeying the consistency conditions (4) and (5), does there exist any positive normalized phase space probability density $\rho(\overrightarrow{q}, \overrightarrow{p})$ with these distributions as marginals? Further, the special case where the four given $\sigma$’s are quantum probability distributions for eigenvalues of the corresponding CCS of observables will be of great interest, and we shall call it the **quantum four marginal problem**. This means that the given probability distributions are of the form

$$\sigma_{qq}(q_1, q_2) = |\langle q_1, q_2 | \psi \rangle|^2, \quad \sigma_{qp}(q_1, p_2) = |\langle q_1, p_2 | \psi \rangle|^2,$$

$$\sigma_{pq}(p_1, q_2) = |\langle p_1, q_2 | \psi \rangle|^2, \quad \sigma_{pp}(p_1, p_2) = |\langle p_1, p_2 | \psi \rangle|^2, \quad (6)$$

for a pure quantum state $|\psi\rangle$, or of the analogous form obtained by replacing $|\langle \xi | \psi \rangle|^2$ by $\langle \xi | \hat{\rho} | \xi \rangle$ for a quantum state with density operator $\hat{\rho}$. In this case, the consistency conditions are automatically satisfied. A positive answer to the quantum four marginal problem would imply simultaneous realizability of the four CCS $(Q_1, Q_2)$, $(Q_1, P_2)$, $(P_1, Q_2)$ and $(P_1, P_2)$. We shall see that in fact at most three CCS can be simultaneously realized.

III. Phase space Bell inequalities

Consider the functions $r(q_1, q_2)$, $s(q_1, p_2)$, $t(p_1, q_2)$ and $u(p_1, p_2)$, defined by

$$r(q_1, q_2) = \text{sgn} F_1(q_1) \text{sgn} F_2(q_2), \quad s(q_1, p_2) = \text{sgn} F_1(q_1) \text{sgn} G_2(p_2),$$

$$t(p_1, q_2) = \text{sgn} G_1(p_1) \text{sgn} F_2(q_2), \quad u(p_1, p_2) = -\text{sgn} G_1(p_1) \text{sgn} G_2(p_2), \quad (7)$$

where $F_1$, $F_2$, $G_1$ and $G_2$ are arbitrary nonvanishing functions. Then, it is obvious that

$$r(q_1, q_2) + s(q_1, p_2) + t(p_1, q_2) + u(p_1, p_2) = \pm 2 \quad (\forall q_1, q_2, p_1, p_2). \quad (8)$$
Given four probability distributions obeying the consistency conditions (4) and (5), suppose that a normalized phase space density $\rho(\vec q, \vec p)$ satisfying the four marginal conditions (3) exists. Multiplying eq.(8) by $\rho(\vec q, \vec p)$ and integrating over phase space, we deduce the phase space Bell inequalities

$$|S| \leq 2,$$

where

$$S \equiv \int dq_1 dq_2 r(q_1, q_2) \sigma_{qq}(q_1, q_2) + \int dq_1 dp_2 s(q_1, p_2) \sigma_{qp}(q_1, p_2)$$

$$+ \int dp_1 dq_2 t(p_1, q_2) \sigma_{pq}(p_1, q_2) + \int dp_1 dp_2 u(p_1, p_2) \sigma_{pp}(p_1, p_2).$$

The necessary conditions (9)-(10) provide us with a proof that the four marginal problem does not always admit a solution: choose the probability distributions

$$\sigma_{qq}(q_1, q_2) = \frac{1}{2} [\delta(q_1 - a_1)\delta(q_2 - a_2) + \delta(q_1 - a_1')\delta(q_2 - a_2')] ,$$

$$\sigma_{qp}(q_1, p_2) = \frac{1}{2} [\delta(q_1 - a_1)\delta(p_2 - b_2) + \delta(q_1 - a_1')\delta(p_2 - b_2')] ,$$

$$\sigma_{pq}(p_1, q_2) = \frac{1}{2} [\delta(p_1 - b_1)\delta(q_2 - a_2) + \delta(p_1 - b_1')\delta(q_2 - a_2')] ,$$

$$\sigma_{pp}(p_1, p_2) = \frac{1}{2} [\delta(p_1 - b_1)\delta(p_2 - b_2) + \delta(p_1 - b_1')\delta(p_2 - b_2')] .$$

Inequality (9) is violated for functions $F$'s and $G$'s such that

$$F_1(a_1), F_2(a_2), G_1(b_1), G_2(b_2) > 0, \quad F_1(a_1'), F_2(a_2'), G_1(b_1'), G_2(b_2') < 0,$$

which yields $S = 4$.

To show that the Bell inequalities can be violated also in the quantum case, and to find by how much, is not a trivial matter.

**IV. Violation of phase space Bell inequalities in quantum theory**

Suppose next that the given probability distributions $\sigma_{qq}$, $\sigma_{qp}$, $\sigma_{pq}$ and $\sigma_{pp}$ are of the form (6) or of the corresponding forms in terms of an operator $\hat \rho$. Notice first that $\chi_1(q_1) \equiv \frac{1}{2} [1 + \text{sgn} F_1(q_1)]$ is the characteristic function of some set $S_1 \subset \mathbb{R}$, and similarly for $F_2, G_1$ and $G_2$, so that eqs.(7) read

$$r(q_1, q_2) = (2\chi_1 - 1)(2\chi_2 - 1), \quad s(q_1, p_2) = (2\chi_1 - 1)(2\chi_2' - 1),$$

$$t(p_1, q_2) = (2\chi_1' - 1)(2\chi_2 - 1), \quad u(p_1, p_2) = -(2\chi_1' - 1)(2\chi_2' - 1),$$

where $\chi_i$ stands for $\chi_i(q_i)$ and $\chi_i'$ for $\chi_i'(p_i), (i = 1, 2)$. Eqs.(8) then become

$$P = 0 \text{ or } 1,$$

i.e. $P(1 - P) = 0$, where $P(q_1, q_2, p_1, p_2)$ is given by

$$P = \chi_1 + \chi_2 + \chi_1'\chi_2' - \chi_1\chi_2 - \chi_1\chi_2' - \chi_1'\chi_2.$$
Let us define a corresponding quantum operator \( \hat{P} \) by

\[
\hat{P} = \hat{\chi}_1 + \hat{\chi}_2 + \hat{\chi}'_1 \hat{\chi}'_2 - \hat{\chi}_1 \hat{\chi}_2 - \hat{\chi}_1 \hat{\chi}'_2 - \hat{\chi}'_1 \hat{\chi}_2,
\]

where

\[
\hat{\chi}_1 = \int_{S_1} dq_1 |q_1\rangle \langle q_1| \otimes 1_2, \quad \hat{\chi}_2 = 1_1 \otimes \int_{S_2} dq_2 |q_2\rangle \langle q_2|, \\
\hat{\chi}'_1 = \int_{S'_1} dp_1 |p_1\rangle \langle p_1| \otimes 1_2, \quad \hat{\chi}'_2 = 1_1 \otimes \int_{S'_2} dp_2 |p_2\rangle \langle p_2|.
\]

(15)

The \( \hat{\chi}_i \)'s are orthogonal projectors, \( (\hat{\chi}_i = \hat{\chi}_i, \hat{\chi}_j^2 = \hat{\chi}_j) \) acting on \( \mathcal{H} = L^2(\mathbb{R}, dq_1) \otimes L^2(\mathbb{R}, dq_2) \). The product of two of them involving different indices commutes, so that \( \hat{P} \) is a (bounded) self-adjoint operator.

The Bell inequalities (9) to be tested in the quantum context then become

\[
0 \leq \langle \Psi | \hat{P} | \Psi \rangle \leq 1 \quad \forall |\Psi\rangle \in \mathcal{H} \text{ with } \langle \Psi | \Psi \rangle = 1,
\]

(17)

or \( 0 \leq \text{Tr} \hat{P} \leq 1 \) in case of mixed states. Equivalently,

\[
\hat{P} \geq 0 \quad \text{and} \quad 1 - \hat{P} \geq 0 \quad \text{in the operator sense.}
\]

(18)

Because \( \hat{\chi}_j \) fails to commute with \( \hat{\chi}'_j \) \( (j = 1, 2) \), \( \hat{P} \) is not an orthogonal projector (see below), in contrast to the classical equality \( P^2 = P \). This leads to the following proposition:

**The operators \( \hat{P} \) and \( 1 - \hat{P} \) cannot be both positive.**

*Proof:* Assume that \( \hat{P} \) and \( 1 - \hat{P} \) are both positive. Then

\[
\hat{P}(1 - \hat{P}) \geq 0,
\]

(19)

(remember that the product of two positive commuting operators is positive).

Now, a straightforward calculation of \( \hat{P}^2 \) yields

\[
\hat{P}^2 = \hat{P} - [\hat{\chi}_1, \hat{\chi}'_1] [\hat{\chi}_2, \hat{\chi}'_2],
\]

(20)

Take a factorized \( |\Psi\rangle \), namely \( |\Psi\rangle = |\Phi_1\rangle \otimes |\Phi_2\rangle \), so that

\[
\langle \Psi | \hat{P}(1 - \hat{P}) | \Psi \rangle = -\langle \Phi_1 | i [\hat{\chi}_1, \hat{\chi}'_1] | \Phi_1 \rangle \langle \Phi_2 | i [\hat{\chi}_2, \hat{\chi}'_2] | \Phi_2 \rangle.
\]

(21)

To prove the proposition, it is enough to show that, for a given choice of the characteristic functions \( \chi \) and \( \chi' \), the real number \( R[\Phi] \equiv \langle \Phi | i [\hat{\chi}, \hat{\chi}'] | \Phi \rangle \) can assume both signs when \( |\Phi\rangle \) is varied. Defining \( |\Phi^+\rangle = \hat{\chi} |\Phi\rangle \) and \( |\Phi^-\rangle = (1 - \hat{\chi}) |\Phi\rangle \), and using the identity \( \hat{\chi} \hat{\chi}' = \hat{\chi} \hat{\chi}'(1 - \hat{\chi}) - (1 - \hat{\chi}) \hat{\chi} \hat{\chi}' \), gives \( R[\Phi] \) the form

\[
R[\Phi] = i\langle \Phi^+ | \hat{\chi} | \Phi^- \rangle - i\langle \Phi^- | \hat{\chi}' | \Phi^+ \rangle.
\]

Obviously, for \( |\Phi\rangle = |\Phi^+\rangle - |\Phi^-\rangle \), one has \( R[\Phi] = -R[\Phi] \). As a consequence, there is at least one \( |\Psi\rangle \neq 0 \) such that the inequalities \( \langle \Psi | \hat{P} | \Psi \rangle \geq 0 \) and \( \langle \Psi | (1 - \hat{P}) | \Psi \rangle \geq 0 \) cannot be simultaneously true, and the four marginal conditions (3) are inconsistent.
Remark: When the wave function $|\Psi\rangle$ factorizes, i.e. $\Psi(q_1, q_2) = \Phi_1(q_1)\Phi_2(q_2)$, a corresponding probability distribution $\rho$ always exists, namely

$$\rho(q_1, q_2, p_1, p_2) = |\Phi_1(q_1)|^2 |\Phi_2(q_2)|^2 |\tilde{\Phi}_1(p_1)|^2 |\tilde{\Phi}_2(p_2)|^2,$$

where the $\tilde{\Phi}_i$'s are the Fourier transforms

$$\tilde{\Phi}_i(p_i) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dq_i e^{-ip_i q_i} \Phi_i(q_i), \quad (i = 1, 2).$$

Of course, this implies that eqs.(17) are automatically satisfied for such factorized $|\Psi\rangle$'s (which can also be checked from eq.(15)).

The above proof is non-constructive and gives no quantitative information about the amount of violation. In order to gain such an information, one needs to construct explicitly some “optimal” wave function $\psi(q_1, q_2)$, which is not a trivial matter as it involves a sort of fine tuning. We shall content ourselves to give here the (surprisingly simple!) result:

$$\Psi_{\pm}(q_1, q_2) = \frac{1}{2\sqrt{2}} \left[ 1 \pm e^{i\pi/4} \text{sgn}(q_1) \text{sgn}(q_2) \right] h(|q_1|)h(|q_2|),$$

(22)

where $h(q)$ stands for some regularized form of $\frac{1}{\sqrt{q}}$, with $\int_0^\infty dq h(q)^2 = 1$, e.g.

$$h_L(q) = \frac{\theta(L - q)}{\sqrt{\ln(L + 1)}} \frac{1}{\sqrt{q + 1}} \quad L \to \infty.$$

One can then check that, with the choice

$$\chi_i(q_i) = \theta(q_i), \quad \chi_i'(p_i) = \theta(p_i), \quad (i = 1, 2)$$

in eq.(12), the inequalities (9) are violated indeed for $L \to \infty : S \to \pm 2\sqrt{2}$.

This opens up the exciting possibility of experimental test of quantum violation of non contextuality postulate in the phase space context.

V. General solution of the three marginal problem

We proved the impossibility of reproducing quantum probabilities of four CCS as marginals. We now give a sketchy description of the most general nonnegative phase space density which reproduces any three given probabilities, say $\sigma_{qq}$, $\sigma_{pq}$ and $\sigma_{pp}$, satisfying consistency constraints as in eq.(3). A precise statement and a full mathematical proof including the required technical details will be published elsewhere [10].

Let us introduce the one variable marginals

$$\sigma_q(q_2) = \int dq_1 \sigma_{qq}(q_1, q_2) = \int dp_1 \sigma_{pq}(p_1, q_2),$$

$$\sigma_p(p_1) = \int dq_2 \sigma_{pq}(p_1, q_2) = \int dp_2 \sigma_{pp}(p_1, p_2).$$

(23)
Let \( E = \{ \vec{q}, \vec{p} \mid \sigma_{qq}, \sigma_{pq} \text{ and } \sigma_{pp} \neq 0 \} \), and
\[
\rho_0(\vec{q}, \vec{p}) = \begin{cases} 
\frac{1}{\sigma_{pq}(q_1, q_2)} \frac{1}{\sigma_{q}(q_2)} \frac{1}{\sigma_{p}(p_1, p_2)} \rho_{pq}(p_1, q_2) \rho_{pp}(p_1, p_2) & \text{if } (\vec{q}, \vec{p}) \in E, \\
0 & \text{otherwise}.
\end{cases}
\] (24)

Clearly \( \rho_0 \) is a particular non negative solution of the given three marginal constraints. We now state the theorem :

The general nonnegative \( \rho(\vec{q}, \vec{p}) \) with prescribed marginals \( \sigma_{qq}, \sigma_{pq} \) and \( \sigma_{pp} \) is given by
\[
\rho(\vec{q}, \vec{p}) = \rho_0(\vec{q}, \vec{p}) + \lambda \Delta(\vec{q}, \vec{p}),
\] (25)
where
\[
\lambda \in [-1/m_+, 1/m_-],
\] (26)
and
\[
\Delta(\vec{q}, \vec{p}) = \int dp_1 dp_2 F(q_1, q_2, p_1, p_2) - \rho_0(\vec{q}, \vec{p}) \left[ \frac{1}{\sigma_{pq}(q_1, q_2)} \int dp_1 dp_2 F(q_1, q_2, p_1, p_2) + \frac{1}{\sigma_{pp}(p_1, p_2)} \int dq_1 dq_2 F(q_1, q_2, p_1, p_2) - \frac{1}{\sigma_{q}(q_2)} \int dq_1 dp_2 F(q_1, q_2, p_1, p_2) - \frac{1}{\sigma_{p}(p_1)} \int dq_1 dq_2 F(q_1, q_2, p_1, p_2) \right],
\] (27)

\( F \) being an function with support contained in \( E \). The \((F\text{-dependent})\) constants \( m_\pm \) in (26) are defined as
\[
m_+ = \sup_{(\vec{q}, \vec{p}) \in E} \frac{\Delta(\vec{q}, \vec{p})}{\rho_0(\vec{q}, \vec{p})}, \quad m_- = -\inf_{(\vec{q}, \vec{p}) \in E} \frac{\Delta(\vec{q}, \vec{p})}{\rho_0(\vec{q}, \vec{p})},
\] (28)
and are both positive if \( \Delta \) does not identically vanish \((m_+ = \infty \text{ or/and } m_- = \infty \text{ are not excluded})\).

The proof goes in two steps. First, it is readily shown that any non negative solution \( \rho_1 \) of the three marginal conditions admits the representation (25). Indeed, choosing \( F = \rho_1 \) in eq.(27) gives \( \Delta = \rho_1 - \rho_0 \) and \( m_- \leq 1 \), allowing to choose \( \lambda = 1 \). Eq.(25) then reads \( \rho = \rho_1 \).

Second, one shows that any function \( \rho \) defined by (25) to (28) is a non negative solution of the three marginal conditions. To do it, it is convenient to rearrange the writing of \( \Delta \) as follows :
\[
\Delta = \left[ F - \frac{\rho_0}{\sigma_{pq}} \int dp_1 dp_2 F \right] - \left[ \frac{\rho_0}{\sigma_{pq}} \int dq_1 dp_2 F - \frac{\rho_0}{\sigma_{q}} \int dq_1 dp_1 dp_2 F \right] - \left[ \frac{\rho_0}{\sigma_{pp}} \int dq_1 dq_2 F - \frac{\rho_0}{\sigma_{p}} \int dq_1 dq_2 dp_2 F \right].
\] (29)

Integrating the right-hand side over \( p_1 \) and \( p_2 \), one finds that the two terms coming from each square bracket cancel each other. Similar results obtain on integrating over \((q_1, p_2)\) or \((q_1, q_2)\). Hence
\[
\left\{ \int dp_1 dp_2, \int dq_1 dp_2, \int dq_1 dq_2 \right\} \Delta(\vec{q}, \vec{p}) = 0,
\]
and the three marginal conditions are satisfied by (25).

Since the integral of $\Delta$ over phase space vanishes, $m_\pm$ in eqs.(28) are both strictly positive if $\Delta$ does not vanish identically. The positivity of $\rho$ then follows from eqs.(25), (26) and (28).

Finally, combining the proposition of section IV with the above theorem, we can state the

**Three marginal theorem**: Any three out of a given set of four probability densities obeying the consistency conditions (5) can be reproduced as marginals of a positive density $\rho(\vec{q}, \vec{p})$. There exist sets of four consistent probability densities which cannot be reproduced as marginals of a positive $\rho$.

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**VI. Conclusions**

We established phase space Bell inequalities from the postulate of existence of a positive phase space probability density. We demonstrated that quantum mechanics violates these inequalities by a factor $\sqrt{2}$, as in the violation of the standard ones, opening the road to experimental tests of quantum contextuality in the position-momentum sector. We also established the three marginal theorem which shows that in 2-dimensional configuration space, three (but not four) noncommuting CCS can be simultaneously realized in quantum mechanics. The simultaneous realization of three CCS (rather than the usual 1 CCS) and the construction of the most general such phase space density sets the stage for construction of maximally realistic quantum theory.

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**References**


