Elimination of threshold singularities in the relation between on-shell and pole widths

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In a previous communication by two of us [B. A. Kniehl and A. Sirlin, Phys. Rev. Lett. **81**, 1373 (1998)], the gauge-dependent deviations of the on-shell mass and total decay width from their gauge-independent pole counterparts were investigated at leading order for the Higgs boson of the standard model. In the case of the widths, the deviation was found to diverge at unphysical thresholds, $m_H = 2\sqrt{\xi_V}m_V$ ($V=W,Z$), in the R_ξ gauge. In this Brief Report, we demonstrate that these unphysical threshold singularities are properly eliminated if a recently proposed definition of wave-function renormalization for unstable particles is invoked.

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The unrenormalized propagator of a scalar boson, with four-momentum q , is of the form

$$\mathcal{D}^{(u)}(s) = \frac{i}{s - M_0^2 - A(s)}, \quad (1)$$

where $s = q^2$, M_0 is the bare mass, and $A(s)$ is the unrenormalized self-energy. In the case of the transverse propagator of a vector boson, there is an additional factor $-(g^{\mu\nu} - q^\mu q^\nu/s)$ on the right-hand side of Eq. (1).

In the conventional on-shell formulation, which most analyses in electroweak perturbation theory are based on, the mass M and total decay width Γ of an unstable boson are defined as

$$M^2 = M_0^2 + \text{Re}A(M^2), \quad (2)$$

$$M\Gamma = -\frac{\text{Im}A(M^2)}{1 - \text{Re}A'(M^2)}, \quad (3)$$

respectively. However, in gauge theories, Eqs. (2) and (3) are known to become gauge dependent at the next-to-next-to-leading order, i.e., in $\mathcal{O}(g^4)$ and $\mathcal{O}(g^6)$, respectively, where g is a generic gauge coupling [1–4]. This problem can be solved by defining the mass and width in terms of the complex-valued position of the propagator's pole,

$$\bar{s} = M_0^2 + A(\bar{s}), \quad (4)$$

which is gauge independent to all orders in perturbation theory [1,5–7]. Fixing the pole mass m_2 and width Γ_2 through the parametrization [1]

$$\bar{s} = m_2^2 - im_2\Gamma_2, \quad (5)$$

we have

$$m_2^2 = M_0^2 + \text{Re}A(\bar{s}), \quad (6)$$

$$m_2\Gamma_2 = -\text{Im}A(\bar{s}). \quad (7)$$

Alternative, gauge-independent definitions of mass and width based on $\bar{s}$, with particular merits, were discussed in the literature [1,5]. Recently, also gauge-independent definitions of partial decay widths that properly add up to Γ_2 were introduced [7].

Equation (6) implies that the mass counterterm in the pole scheme is given by $\delta m_2^2 = m_2^2 - M_0^2 = \text{Re}A(\bar{s})$. In order to complete the renormalization of Eq. (1), we also need to specify an appropriate wave-function renormalization constant, $Z = 1 - \delta Z$, so that the renormalized propagator

$$\begin{aligned} \mathcal{D}^{(r)}(s) &= \frac{\mathcal{D}^{(u)}(s)}{Z} \\ &= \frac{i}{s - m_2^2 - \mathcal{S}^{(r)}(s)} \end{aligned} \quad (8)$$

is ultraviolet (UV) finite for all real values of s . An appropriate definition is [8]

$$Z = \frac{1}{1 + [\text{Im}A(\bar{s}) - \text{Im}A(m_2^2)]/(m_2\Gamma_2)}, \quad (9)$$

which allows us to rewrite Eq. (7) as

$$m_2\Gamma_2 = -Z\text{Im}A(m_2^2). \quad (10)$$

In fact, δm_2^2 and δZ thus defined are real and guarantee that the renormalized self-energy,

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$$\mathcal{S}^{(r)}(s) = Z[A(s) - \delta m_2^2] + \delta Z(s - m_2^2), \quad (11)$$

is UV finite to all orders [8–10]. Equation (9) possesses a number of desirable properties. On the one hand, it avoids threshold singularities that, in the conventional on-shell scheme, appear in the radiatively corrected production and decay rates of the Higgs boson as its mass approaches from below the pair-production threshold of a vector boson [8]. On the other hand, it precludes the occurrence of powerlike infrared divergences in the renormalized propagators of unstable particles that couple to massless quanta, like the W bosons and the quarks of the second and third generations [9,11]. Finally, it allows one to systematically organize the order-by-order removal of UV divergences in $\mathcal{S}^{(r)}(s)$ [10]. In this Brief Report, we elaborate yet another virtue of Eq. (9).

Expanding Eqs. (2), (3), (6), and (7) about $s = m_2^2$ and combining the results, one obtains [3]

$$\frac{M - m_2}{m_2} = -\frac{\Gamma_2}{2m_2} \text{Im}A'(m_2^2) + \mathcal{O}(g^6), \quad (12)$$

$$\begin{aligned} \frac{\Gamma - \Gamma_2}{\Gamma_2} &= \text{Im}A'(m_2^2) \left[\frac{\Gamma_2}{2m_2} + \text{Im}A'(m_2^2) \right] \\ &\quad - \frac{m_2 \Gamma_2}{2} \text{Im}A''(m_2^2) + \mathcal{O}(g^6). \end{aligned} \quad (13)$$

In Ref. [3], the gauge dependence of Eqs. (12) and (13) was analyzed for the Higgs boson in the standard model adopting the R_ξ gauge [12]. In the case of Eq. (13), it was found that, for an arbitrary value of m_2 , unphysical threshold singularities, proportional to $(m_2 - 2\sqrt{\xi_V}m_V)^{-1/2}$, occur as ξ_V approaches from below the point $m_2^2/(4m_V^2)$ ($V=W,Z$). Here and in the following, m_2 and Γ_2 refer to the Higgs boson, while m_V denotes the pole mass of the intermediate boson V . The purpose of this Brief Report is to demonstrate that the unphysical threshold singularities encountered in Ref. [3] are eliminated if Eq. (9) is employed in a judicious manner. For the time being, we disregard physical threshold singularities, which occur independently of the choice of gauge if the Higgs-boson mass happens to have the specific values $m_2 = 2m_V$ [8,13] or $m_2 = 2m_f$, where m_f is a generic fermion mass, and we assume that the value of m_2 is sufficiently far away from the points $2m_V$ and $2m_f$. We shall return to the issue of physical threshold singularities in Eqs. (12) and (13) at the end of this Brief Report.

A one-loop expression for the unrenormalized Higgs-boson self-energy $A(s)$ in the R_ξ gauge may be found in Eq. (8) of Ref. [8]. Detailed inspection reveals that the unphysical threshold singularity in Eq. (13) originates in $\text{Im}A''(m_2^2)$, which contains the term $G_\mu m_2^2/(2\pi^2\sqrt{2})B'_0(m_2^2, \xi_V m_V^2, \xi_V m_V^2)$, where G_μ is Fermi's constant, B_0 is the scalar one-loop two-point integral in $D=4-2\epsilon$ space-time dimensions as given, e.g., in Eq. (9) of Ref. [8], and the prime indicates differentiation with respect to the first argument. In fact,

$$\text{Im}B'_0(m_2^2, \xi_V m_V^2, \xi_V m_V^2) = \frac{\pi a}{2m_2^2\sqrt{1-a}} \theta(1-a) + \mathcal{O}(\epsilon), \quad (14)$$

where $a = 4\xi_V m_V^2/m_2^2$, exhibits the type of singularity mentioned above.

We now illustrate how this singularity is eliminated by consistently working in the pole scheme [8,9]. We start by observing that Eq. (13) is based on the expansion [3]

$$\begin{aligned} m_2 \Gamma_2 &= -\text{Im}A(m_2^2) \left\{ 1 + \text{Re}A'(m_2^2) + [\text{Re}A'(m_2^2)]^2 \right. \\ &\quad \left. - \frac{1}{2} \text{Im}A(m_2^2) \text{Im}A''(m_2^2) + \mathcal{O}(g^6) \right\}. \end{aligned} \quad (15)$$

Here, it is tacitly assumed that $A(s)$ is analytic near $s = m_2^2$, so that the Taylor expansion can be performed. In most cases, this assumption is valid. However, $A(s)$ possesses a branch point if s is at a threshold. As a consequence, at a given two-particle threshold $m_2 = m_A + m_B$, the derivatives $A^{(n)}(m_2^2)$ ($n=1,2,\dots$) develop threshold singularities proportional to $|m_2 - m_A - m_B|^{-1/2}$ or worse. The latter appear in $\text{Re}A^{(n)}(m_2^2)$ [$\text{Im}A^{(n)}(m_2^2)$] as m_2 approaches the threshold from below (above). In the case of the Higgs boson, the problems start at $n=1$ for $m_2 = 2m_V$ [8,13] and at $n=2$ for the residual two-particle thresholds, $m_2 = 2\sqrt{\xi_V}m_V$ [3] and $m_2 = 2m_f$. The solutions to all these problems emerge by undoing the Taylor expansions. In Ref. [8], this was illustrated for $\text{Re}A'(m_2^2)$ at $m_2 = 2m_V$. Here, we consider $\text{Im}A''(m_2^2)$ at $m_2 = 2\sqrt{\xi_V}m_V$, which is relevant for the investigation of the gauge dependence of Eq. (13) [3].

Inserting Eq. (9) in Eq. (10) and expanding in powers of $[\text{Im}A(\bar{s}) - \text{Im}A(m_2^2)]/(m_2 \Gamma_2)$, we obtain

$$\begin{aligned} m_2 \Gamma_2 &= -\text{Im}A(m_2^2) \left\{ 1 - \frac{\text{Im}A(\bar{s}) - \text{Im}A(m_2^2)}{m_2 \Gamma_2} \right. \\ &\quad \left. + \left[\frac{\text{Im}A(\bar{s}) - \text{Im}A(m_2^2)}{m_2 \Gamma_2} \right]^2 + \mathcal{O}(g^6) \right\}. \end{aligned} \quad (16)$$

It is important to note that $[\text{Im}A(\bar{s}) - \text{Im}A(m_2^2)]/(m_2 \Gamma_2)$ involves a finite difference, rather than a derivative. Because of this fact, and as we shall explicitly show later, it is free from threshold singularities. Comparison of Eqs. (15) and (16) shows then that the threshold singularities emerging from $\text{Im}A''(m_2^2)$ are avoided if this amplitude is replaced according to the substitution rule

$$\begin{aligned} \text{Im}A''(m_2^2) &= -\frac{2}{m_2 \Gamma_2} \left[\frac{\text{Im}A(\bar{s}) - \text{Im}A(m_2^2)}{m_2 \Gamma_2} + \text{Re}A'(m_2^2) \right] \\ &\quad + \mathcal{O}(g^4). \end{aligned} \quad (17)$$

Away from thresholds, the expansion of $A(\bar{s})$ about m_2^2 is valid, and both sides of Eq. (17) are well defined. However, at the unphysical threshold $m_2 = 2\sqrt{\xi_V}m_V$, such an expansion breaks down, $\text{Im}A''(m_2^2)$ diverges, and only the right-

hand side of Eq. (17) remains well defined. The substitution in Eq. (17) is equivalent to replacing Eq. (14) by

$$\begin{aligned} \text{Im}B'_0(m_2^2, \xi_V m_V^2, \xi_V m_V^2) \\ = \frac{\text{Re}B_0(\bar{s}, \xi_V m_V^2, \xi_V m_V^2) - \text{Re}B_0(m_2^2, \xi_V m_V^2, \xi_V m_V^2)}{m_2 \Gamma_2} \\ + \mathcal{O}(g^2). \end{aligned} \quad (18)$$

Using the expression for $B_0(s, m_V^2, m_V^2)$ given in Eq. (13) of Ref. [8] and introducing the auxiliary function

$$f(z) = -2\sqrt{1-z} \text{arsinh} \sqrt{-\frac{1}{z}}, \quad (19)$$

we can rewrite Eq. (18) as

$$\begin{aligned} \text{Im}B'_0(m_2^2, \xi_V m_V^2, \xi_V m_V^2) \\ = \frac{\text{Ref}(4\xi_V m_V^2/\bar{s}) - \text{Ref}(a - i\varepsilon)}{m_2 \Gamma_2} \\ + \mathcal{O}(\varepsilon) + \mathcal{O}(g^2), \end{aligned} \quad (20)$$

where a is defined below Eq. (14). We have

$$\begin{aligned} \text{Ref}\left(\frac{4\xi_V m_V^2}{\bar{s}}\right) \\ = -\frac{\sqrt{2}}{b} \left\{ \frac{1}{2} \sqrt{b(c+b)-a} \right. \\ \times \ln \left[\frac{1}{a} (b+c+\sqrt{(b-1)(c+a-1)}) \right. \\ \left. + \sqrt{(b+1)(c-a+1)} \right] + \sqrt{b(c-b)+a} \\ \left. \times \arctan \frac{\sqrt{b+1} + \sqrt{c-a+1}}{\sqrt{b-1} + \sqrt{c+a-1}} \right\}, \end{aligned} \quad (21)$$

$$\begin{aligned} \text{Ref}(a - i\varepsilon) = -2\sqrt{1-a} \text{arcosh} \sqrt{1/a} \theta(1-a) \\ - 2\sqrt{a-1} \arcsin \sqrt{1/a} \theta(a-1), \end{aligned} \quad (22)$$

where $b = \sqrt{1+\gamma^2}$ and $c = \sqrt{(a-1)^2 + \gamma^2}$, with $\gamma = \Gamma_2/m_2$. We note that the discontinuity $f(a+i\varepsilon) - f(a-i\varepsilon) = -2\pi i \sqrt{1-a} \theta(1-a)$ is purely imaginary, so that $\text{Ref}(a+i\varepsilon) = \text{Ref}(a-i\varepsilon)$. At $m_2 = 2\sqrt{\xi_V} m_V$, Eq. (20) becomes

$$\begin{aligned} \text{Im}B'_0(m_2^2, \xi_V m_V^2, \xi_V m_V^2) \\ = -\frac{1}{m_2^2} \left[\frac{\pi}{\sqrt{2}\gamma} \left(1 - \frac{\gamma}{2} - \frac{3}{8}\gamma^2 \right) + \frac{4}{3}\gamma \right. \\ \left. + \mathcal{O}(\gamma^{5/2}) \right] + \mathcal{O}(\varepsilon) + \mathcal{O}(g^2), \end{aligned} \quad (23)$$

FIG. 1. Relative deviation of the on-shell width Γ from the pole width Γ_2 for a Higgs boson with pole mass $m_2 = 200$ GeV as a function of the gauge parameter ξ in the vicinity of the point $m_2^2/(4m_Z^2)$. The unphysical threshold singularity originating in Eq. (14) (dotted line) is eliminated by applying the substitution rule of Eq. (20) (solid line), which is a consequence of invoking Eq. (9).

i.e., the unphysical threshold singularity is automatically regularized in the pole scheme by the width Γ_2 of the primary particle.

In our numerical analysis, we use the pole-mass values $m_W = 80.391$ GeV and $m_Z = 91.154$ GeV, which are extracted from the measured values [14] as described in Ref. [8], and adopt the residual input parameters from Ref. [14]. For definiteness, we choose $m_2 = 200$ GeV and evaluate Γ_2 in the Born approximation. For simplicity, we set $\xi = \xi_W = \xi_Z$. In Fig. 1, we show the ξ dependence of $(\Gamma - \Gamma_2)/\Gamma_2$ given by Eq. (13) in the vicinity of the point $m_2^2/(4m_Z^2)$. The dotted and solid lines are evaluated using Eqs. (14) and (20), respectively. The dotted line corresponds to the solid line in Fig. 1(a) of Ref. [3] and exhibits the familiar abyss at $\xi = m_2^2/(4m_Z^2)$. Obviously, this unphysical threshold singularity is absent in the solid line, which smoothly interpolates across the threshold region and merges with the dotted line sufficiently far away from the threshold. A similar discussion applies to the second abyss, at $\xi = m_2^2/(4m_W^2)$, which is not visible in Fig. 1.

Finally, we return to the physical threshold singularities. From the discussion below Eq. (15) it follows that threshold singularities analogous to the one displayed in Eq. (14) also affect Eq. (12) for $m_2 = 2m_V$ and Eq. (13) for $m_2 = 2m_V$ and $m_2 = 2m_f$. They may be eliminated in a very similar way, by applying the substitution rule of Eq. (20), with $\xi_V m_V^2$ replaced by m_V^2 or m_f^2 .

In Refs. [8,15], the unrenormalized Higgs-boson self-energy was calculated at one loop in the pinch-technique framework. The result does not involve unphysical thresholds and is, therefore, devoid of unphysical threshold singularities, as was noticed in Ref. [3]. However, it is plagued by similar physical threshold singularities as the treatment based on the conventional result in the R_ξ gauge. As for the relative

deviations of the on-shell mass and total decay width of the Higgs boson from their pole counterparts, this may be understood by comparing Eqs. (8) and (9) of Ref. [3] and considering their contributions to Eqs. (12) and (13) of the present paper at $m_2 = 2m_V$. In fact, a similar observation was already made in Ref. [8] in connection with the radiatively corrected production and decay rates of the Higgs boson.

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