Abstract

Matrix elements of the form $<0|Tr g^2 GG|G>$ are calculated using the lattice QCD Monte Carlo method. Here, $|G>$ is a glueball state with quantum numbers $0^{++}$, $2^{++}$, $0^{-+}$ and $G$ is the gluon field strength operator. The matrix elements are obtained from the hybrid correlation functions of the fuzzy and plaquette operators performed on the $12^4$ and $14^4$ lattices at $\beta = 5.9$ and 5.96 respectively. These matrix elements are compared with those from the QCD sum rules and the tensor meson dominance model. They are the non-perturbative matrix elements needed in the calculation of the partial widths of $J/\Psi$ radiative decays into glueballs.

The glueballs predicted from Quantum Chromodynamics (QCD) are the most exotic particles from the point of view of the naive quark model, but they are crucial to the QCD picture of the hadronic interactions. The identification of certain experimentally observed particles as glueballs will be a strong confirmation of QCD. Radiative $J/\Psi$ decays have been regarded as the best hunting ground for glueballs [1]. Several candidates from experiments have existed for some time [2]. In order to compare the experimental findings with the theory, we need to find at least some characteristics of the glueball spectrum, or better yet to calculate some properties of the glueballs from QCD. Previous theoretical studies have concentrated on the estimation of the masses of the glueballs. The prediction of these glueball masses has long been attempted in lattice gauge Monte Carlo calculations [3]. Recent progress using fuzzy loops [4, 5, 6],

*Present address: Physics Dept., Univ. of Rochester, Rochester, NY 14627
smeared loops [4], and inverse Dirac operators [8, 9] to improve the signal to noise ratio have yielded quite consistent results on large volumes. These calculations show that the lowest scalar, tensor, and pseudoscalar glueballs lie in the mass range of 1 - 2.5 GeV. In this letter, we shall report a lattice gauge Monte Carlo calculation of vacuum to glueball transition matrix elements which are needed for estimating the partial widths of $J/\Psi$ radiative decays into glueballs.

The glueball matrix elements we compute are of the form $\langle 0| g^2 G_{\mu\nu} G_{\rho\sigma}(x) |G \rangle$, where $G$ can be the $2^{++}$, $0^{++}$ or $0^{-+}$ glueball and $G_{\mu\nu}$ is the gluon field strength operator. On a lattice, the local operator $g^2 G_{\mu\nu} G_{\rho\sigma}(x)$ can be constructed by small loops consisting of lattice links. One way to do the calculation might be, say for the $2^{++}$ glueball, to calculate the correlation function $\langle O(t) O(0) \rangle$, where $O = 2\Box_{12} - \Box_{31} - \Box_{23}$. Here, $\Box_{\mu\nu}$ is a plaquette with edges in the $\mu$ and $\nu$ directions. Then, for large Euclidean time,

$$\langle O(t) O(0) \rangle \sim (\langle 0|O|1 \rangle)^2 e^{-m_1 t},$$

where $|1\rangle$ is the lowest glueball state with the quantum numbers associated with $O$. The matrix element can thus be extracted. Technically, however, we know from glueball mass calculations [4 - 10] that small loops usually are not very good for creating glueball states. The states they create have small projections to the glueball states resulting in large noises in Monte Carlo calculations. In fact, it has become increasingly standard [4, 5, 6, 10] in mass calculations to use the so-called “fuzzy operators”, which have extended structures, presumably reflecting the size of the glueball. On the other hand, it is difficult, if at all possible, to relate the fuzzy operators to the field strength operators numerically. Therefore, we adopt a method which is a hybrid of the two approaches. We use fuzzy operators to do a variational calculation first, finding the linear combination of different fuzzy operators that maximizes the overlap with a glueball state in the calculation of the correlation function of the fuzzy operators $\langle O(t) O(0) \rangle$. Then, we calculate the correlation functions of small loops with the fuzzy operator $O$

$$\langle O(t) FF(0) \rangle = \sum_n \langle 0|O|n\rangle <n|FF|0\rangle e^{-m_n t},$$

Here, $FF$ is the lattice approximation of the $G^a G^a$ operator. If we assume that the variational calculation gives a good overlap with the lowest glueball state as we let $t >> a$, the lattice spacing, then we could fit the two correlations to the following expressions

$$\langle O(t) O(0) \rangle \sim \langle G|O|0\rangle^2 e^{-m_1 t},$$

$$\langle O(t) FF(0) \rangle \sim \langle G|O|0\rangle <0|FF|G\rangle e^{-m_1 t},$$

for different time $t$ with three parameters, i.e. $\langle G|O|0\rangle$, $<0|FF|G\rangle$, and $m$. This way we could obtain the desired matrix element $\langle 0|FF|G\rangle$ and the glueball
mass $m$. Since these two correlation functions are measured on the same set of gauge configura-tions, we shall use the data-covariance matrix to fit them simulataneously \([11, 12]\). More details to follow.

Gluon configurations are generated by a pseudo-heat-bath algorithm \([13]\). A measurement is made for every ten configurations generated. For each configuration to be measured, fuzzy links are constructed \(a la\) Michael & Teper\([5]\), as in Fig.1. The procedure produces links with length twice of the original links. It can be iterated several times to yield links of length $2^{\ell}$. In our calculations, $\ell = 2$. After this procedure is done, the resultant links are no longer $SU(3)$ matrices. They do have the correct gauge transformation properties, however. In principle, these links can be used to form loop operators to create glueball states. In practice, however, we have found that they lead to huge statistical fluctuations. We use the following steps to convert the links back to $SU(3)$ matrices. Two conditions should be satisfied. First, if the fuzzy link happens to be an $SU(3)$ matrix, the procedure should keep the matrix un-altered. Second, the final matrix should have the correct gauge transformation properties. Let us consider a matrix $M'$ and $\det M' \neq 0$. We first divide it by its determinant $M = M' / \det M'$. This obviously satisfy the two conditions. Now, any complex non-degenerate matrix can be factorized uniquely into a positive definite Hermitian matrix $M' = UH$ (one can factorize the matrix as $M = H'U'$ too resulting in a different set operators to create the glueball states). If under a gauge transformation $M \mapsto g_1 M g_2^\dagger$, by the uniqueness for the factorization, $U \mapsto g_1 U g_2^\dagger$ and $H \mapsto g_2 H g_2^\dagger$. Uniqueness also means that $H = 1$ and $M = U$ if $M$ is unitary. Therefore, when we convert $M'$ to $U$, both conditions are satisfied. In actual computation, we first find the positive definite Hermitian matrix $H^2 = M^\dagger M$. We use the Jacobi method to diagonalize $H^2$ leading to $H^2 T = T \Lambda$, where $T$ is the orthonormalized unitary matrix of eigenvectors of $H^2$ and $\Lambda$ is the diagonal matrix of eigenvalues. The inverse of $H$ can be obtained from $H^{-1} = T \Lambda^{-1/2} T^\dagger$. Finally, $M = M H^{-1}$. This procedure is somewhat different from that used in \([4, 5]\).

To form the $O$ operators for various spins, we use the $U$ fuzzy links to form loops with shapes depicted in Fig.2. The same shapes were used in \([3]\). Using the method outlined in \([15]\), these loops and their rotations about different axes thereof are combined into operators belonging to the $A_1^{++}(0^{++}), E^{++}(2^{++}), T_2^{++}(2^{++}), A_1^{+-}(0^{-+})$ representations of the cubic group. The notations in the parentheses are the presumed quantum numbers for $J^{PC}$ when the rotation invariance is restored. Only the real part of the trace is used because all the operators we consider here have positive charge conjugation. (a) and (b) in Fig. 2 contribute to all four kinds of representations. (c) does not contribute to the pseudoscalar. (d) and (e) only contribute to $A_1^{++}(0^{++})$ and $E^{++}(2^{++})$. These operators are averaged over spatial coordinates for each time slice to obtain the zero-momentum projections.

The $G^a G^a$ operators are constructed from the elementary(not fuzzy) links in Fig.3. Because of all the different shapes of the fuzzy operators and the yet-unknown varia-
tional coefficients, we do not know the polarization of the tensor states created by the fuzzy operators. We therefore need to calculate all the components of the $FF$ tensor. For the pseudoscalar, we also need different components of the tensor in order for the $F\tilde{F}$ pseudoscalar to have all the required symmetry properties. To get these, we first construct a lattice approximation of the field strength tensor.

$$L_{\mu\nu} = L_1 + L_2 + L_3 + L_4.$$  \hspace{1cm} (5)

where $L_i (i = 1, 4)$ are products of link variables $UUU^+U^+$ around the plaquettes in the $\mu\nu$ plane as indicated in Fig. 3.

$$F_{\mu\nu} = (L_{\mu\nu} - L_{\mu\nu}^\dagger)/8 = a^2 A_{\mu\nu} + \frac{1}{6} a^4 (D^2_{\mu} + D^2_{\nu}) A_{\mu\nu} + O(a^6).$$  \hspace{1cm} (6)

Here, repeated indices are not summed over and $A_{\mu\nu} = igG_{\mu\nu}$. $a$ is the lattice spacing. Therefore

$$Tr(F_{\mu\nu}F_{\rho\lambda}) \propto a^4 g^2 C^a_{\mu\nu} C^a_{\rho\lambda} + O(a^6).$$  \hspace{1cm} (7)

This way, we can find all the components of the field strength tensor and all the Lorentz components of (gauge invariant) trace of products of the field strength tensor. The fuzzy operators are chosen according to the size and scale of the glueball. The operators for the field strength tensor, on the other hand, have sizes on the order of one lattice spacing. Once we choose the lattice scale such that the glueball sizes are much larger than the lattice spacing, then we expect that terms in higher powers of the lattice spacing $a$ are relatively unimportant. This is the basis of our strategy to use large, fuzzy links to create glueballs and measure the field strength using small, elementary links. We shall use two $\beta$ values to check the scaling of the results.

A variational calculation which maximizes

$$c_i c_j < O_i(a)O_j(0) > = < \Phi | e^{-aH} | \Phi > / < \Phi | \Phi >$$  \hspace{1cm} (8)

is carried out to find the coefficients $\{c_i\}$. Here, $|\Phi > = \sum_i c_i O_i | 0 >$ and $O_i$ are operators corresponding to different loops in Fig. 2. From the right hand side of the equation, we see that this is really not different from the usual variational calculation in quantum mechanics. Instead of taking the expectation value of the Hamiltonian itself and minimizing it, we take the expectation value of $e^{-aH}$ and maximize it, which corresponds to finding the lowest eigenvalue of $H$. For the scalar case, vacuum expectation values are first subtracted from the correlations, which corresponds to making the variational state $|\Phi >$ orthogonal to the vacuum state. In actual calculation, this is equivalent to finding the eigenvector for the largest eigenvalue for the generalized eigenvalue problem,

$$< O_i(a)O_j(0) > c_j = \lambda < O_i(0)O_j(0) > c_j.$$  \hspace{1cm} (9)
Once this is done, we proceed to evaluate the correlations $\langle O(t) F F(0) \rangle$ and $\langle O(t) O(0) \rangle$ using $O = c_i O_i$. Both the variational calculation and the fitting of Eqs. (3,4) are used to extract the desired matrix elements and glueball masses.

For different spins, only the operators with same quantum numbers have non-zero matrix elements between the vacuum and the glueball states. Specifically, we would like to extract the following matrix elements for different quantum numbers

**Scalar**

$$\frac{s}{a^4} = \langle 0 | \text{Tr}(g^2 G_{\rho\sigma} G_{\rho\sigma}) | G \rangle = 2 \langle 0 | \text{Tr}(g^2 (E \cdot E + B \cdot B)) | G \rangle.$$  \hfill (10)

**Pseudoscalar**

$$\frac{p}{a^4} = \langle 0 | \epsilon_{\mu\nu\rho\sigma} \text{Tr}(g^2 G_{\mu\nu} G_{\rho\sigma}) | G \rangle = 4 \langle 0 | \text{Tr}(g^2 E \cdot B) | G \rangle.$$  \hfill (11)

**Tensor**

$$\frac{t\epsilon_{\mu\nu}}{a^4} = \langle 0 | \Theta_{\mu\nu} | G \rangle$$

where $\epsilon_{\mu\nu} = \epsilon_{\nu\mu}$, $\epsilon_{\mu\mu} = 0$, $\epsilon_{\mu\nu} \epsilon_{\mu\nu} = 1$, and $\Theta_{\mu\nu} = 2 \text{Tr} g^2 (G_{\mu\sigma} G_{\nu\rho} - \frac{1}{4} \delta_{\mu\nu} G_{\rho\sigma} G_{\rho\sigma}) = g^2 T_{\mu\nu}$ where $T_{\mu\nu}$ is the energy-momentum tensor. Here,

$$\frac{t^2}{4} g^4 a^8 = \langle E_i E_j > - \langle B_i B_j > \rangle^2 + 2 \langle E_i B_j > \rangle.$$

and $\langle \ast \rangle = \langle 0 | \ast | G \rangle$.

The $s$, $p$, and $t$ in the above equations are the lattice matrix elements obtained from eqs. (3) and (4).

Up to now, we have assumed the normalization $\langle G | G \rangle = 1$. To go to the continuum normalization $\langle G, \mathbf{p} | G, \mathbf{p}' \rangle = 2E_p (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p}')$, or $\langle G, 0 | G, 0 \rangle = 2M(2\pi)^3 \delta^3(0) = 2MV$, we need to multiply the above matrix elements by $\sqrt{2MV}$.

Calculations are performed for a $12^4$ and a $14^4$ lattices with $\beta$ value equals to 5.9 and 5.96 respectively. The lattice sizes are chosen so that the Lüscher scale parameter $z = m_{A^+_1} L$ \cite{14} is about 10 where it is found \cite{10} that $E^{++}$ and $T_{2^{++}}$ start to become degenerate indicating the restoration of the rotational invariance. Furthermore, their physical sizes are about the same. From a cold start, we begin to make measurements after 4000 thermalization sweeps. The measurements are made for every 10 updates. For the $12^4$ lattice at $\beta = 5.9$, 5000 measurements have been performed. For the $14^4$ lattice at $\beta = 5.96$, 5800 measurements have been performed. Errors that enter the data-covariance matrix for the $\chi^2$ fit of eqs. (3) and (4) are estimated by binning the data into 20 sets and doing jackknife on them. The parameters chosen are in the
Table 1: Glueball Masses in unit of $\sqrt{K}$. $t_1/t_2$ denotes the range of time slices of eq. (3) / eq. (4) that are fitted. Also included are the $\chi^2$ per degree of freedom for $t_1/t_2$.

<table>
<thead>
<tr>
<th>Glueball</th>
<th>$t_1$</th>
<th>$t_2$</th>
<th>$\beta = 5.9$</th>
<th>$\chi^2/N_{DF}$</th>
<th>$t_1$</th>
<th>$t_2$</th>
<th>$\beta = 5.96$</th>
<th>$\chi^2/N_{DF}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_{1}^{++}$ ($0^{++}$)</td>
<td>4.5</td>
<td>1.2</td>
<td>3.08(1)</td>
<td>0.1</td>
<td>1.2,3</td>
<td>2</td>
<td>3.42(1)</td>
<td>7.9</td>
</tr>
<tr>
<td></td>
<td>1.2,3,4,5</td>
<td></td>
<td>3.29(9)</td>
<td>0.14</td>
<td>1.2,3</td>
<td>2</td>
<td>3.40(3)</td>
<td>0.04</td>
</tr>
<tr>
<td>$E^{++}$ ($2^{++}$)</td>
<td>2.3</td>
<td>0.1</td>
<td>4.44(1)</td>
<td>19.2</td>
<td>3.4</td>
<td>0.1</td>
<td>4.14(1)</td>
<td>0.3</td>
</tr>
<tr>
<td></td>
<td>1.2,3</td>
<td>4.68(3)</td>
<td>52.0</td>
<td>2.3,4</td>
<td>4.33(4)</td>
<td>2.8</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$T_{2}^{++}$ ($2^{++}$)</td>
<td>1.2</td>
<td>1.2</td>
<td>4.83(1)</td>
<td>190</td>
<td>1.2,3</td>
<td>1</td>
<td>5.63(1)</td>
<td>138</td>
</tr>
<tr>
<td></td>
<td>1.2,3</td>
<td>5.12(2)</td>
<td>242</td>
<td>1.2,3</td>
<td>5.72(18)</td>
<td>72</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A_{1}^{-+}$ ($0^{-+}$)</td>
<td>1.2,3</td>
<td>1</td>
<td>6.10(3)</td>
<td>49.7</td>
<td>1.2,3</td>
<td>1</td>
<td>6.18(2)</td>
<td>1.0</td>
</tr>
<tr>
<td></td>
<td>1.2,3</td>
<td>6.05(3)</td>
<td>40</td>
<td>1.2,3</td>
<td>6.17(2)</td>
<td>0.8</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2: Scalar matrix elements obtained from Eqs. 3 and 4. They are in the unit of $K^{3/2}$.

<table>
<thead>
<tr>
<th>$\beta = 5.9$</th>
<th>$\beta = 5.96$</th>
<th>$\chi^2/N_{DF}$</th>
<th>$\chi^2/N_{DF}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_1$</td>
<td>$t_2$</td>
<td>$s/a^4$</td>
<td>$t_1$</td>
</tr>
<tr>
<td>4.5</td>
<td>2.3</td>
<td>45.2(3)</td>
<td>4.7</td>
</tr>
<tr>
<td>4.5</td>
<td>1.2</td>
<td>47.2(3)</td>
<td>0.1</td>
</tr>
</tbody>
</table>

We present the results in units of the string tension $K$, which is determined from fitting the correlations of Polyakov loops to the exponential $Ae^{-KLt}$ for the time interval $t = 2$ to 4 for $\beta = 5.9$ and $t = 2$ to 5 for $\beta = 5.96$. The scale determined from the string tension indicates that the lattice spacing decreases by $\sim 19\%$ from $\beta = 5.9$ to 5.96. This makes the physical sizes of the $12^4$ and $14^4$ lattices about the same. The glueball masses determined from both the joint fit of the correlations of eqs. (3) and (4) and the fit of the correlation of fuzzy loops alone in eq. (3) are given in Table 1. They agree with each other and with the higher statistics calculations in this $\beta$ range [5] (i.e. $\beta = 5.9, 6.0$) for each mass determined from the corresponding time slices. However, for the $T_{2}^{++}$ case, we could not find a fit with small $\chi^2$. This presumably reflects the fact that our lattice is still not large enough for the tensor glueball, since it is known that the tensor glueball is about 4 times larger than the scalar glueball [16, 17].

The glueball matrix elements with several fits to eqs. (3) and (4) are listed in Tables 2 to 5. They are in the unit of $K^{3/2}$. These numbers can be converted into physical units using $K \approx (0.42 GeV)^2$ (i.e., multiplying the numbers by $K^{3/2}$). We see from these results that the scalar glueball matrix elements are better calculated, much like the calculation of the glueball masses. The variation from $\beta = 5.9$ to 5.96 is about 10%. This is quite reasonable as the matrix element scales like $K^{3/2}$. The rest of the matrix elements differ more. Again,
Table 3: Tensor matrix elements for the $E^{++}$ representation in units of $K^{3/2}$.

<table>
<thead>
<tr>
<th></th>
<th>$\beta = 5.9$</th>
<th></th>
<th>$\beta = 5.96$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$t_1$</td>
<td>$t_2$</td>
<td>$t/a^4$</td>
</tr>
<tr>
<td></td>
<td>2.3</td>
<td>0.1</td>
<td>7.42(6)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>2.3,4</td>
</tr>
</tbody>
</table>

Table 4: Tensor matrix elements for the $T_2^{++}$ representation in units of $K^{3/2}$.

<table>
<thead>
<tr>
<th></th>
<th>$\beta = 5.9$</th>
<th></th>
<th>$\beta = 5.96$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$t_1$</td>
<td>$t_2$</td>
<td>$t/a^4$</td>
</tr>
<tr>
<td></td>
<td>1.2</td>
<td>1.2</td>
<td>6.38(8)</td>
</tr>
</tbody>
</table>

the case for $T_2^{++}$ is the worst with no fit of small $\chi^2$, enough though the values of the matrix element are not that dissimilar to that of $E^{++}$. Both $E^{++}$ and $A_1^{++}(0^{-+})$ have fits with reasonably small $\chi^2$.

To come to grips with the continuum matrix elements, we need to consider the finite lattice renormalization of the lattice operators. The finite renormalization of the lattice operators considered here have been calculated perturbatively [18, 19, 20, 21]. We can define the renormalization of the operators as follows,

$$ O = Z_O L a^{-d} $$

where the $O_L$ is the lattice-regularized version of the continuum operator $O$ and $d$ is the mass dimension of $O$. For the case of the scalar operator $g^2 G_{\mu\nu} G_{\mu\nu}$ which coincides with the renormalization group invariant trace anomaly $\frac{\beta(g^2)}{g} G_{\mu\nu} G_{\mu\nu}$ to lowest order in $g$, the renormalization constant $Z$ is found [18] to be 1.06 at one-loop order. For the pseudoscalar lattice operator which is the topological charge in the limit of zero lattice spacing, the finite renormalization $Z$ is found to be 2.5 with the two-loop corrections estimated by dominant tadpole graphs [19]. For the energy-momentum tensor operator, the lattice perturbation to one loop gives the $Z$ to be 1.84 for $\beta = 5.9$ [21].

When these renormalization constants are taken into account, the scalar matrix element in eq.(10) is about 50$K^{3/2}$ or about 3.7$GeV^3$ for $K = (0.42GeV)^2$. Based on the scaling properties of QCD and trace anomaly, both the QCD sum rule [22] and the soft meson theorem [23] lead to the estimate that relates the scalar glueball

Table 5: Pseudoscalar matrix elements in units of $K^{3/2}$.

<table>
<thead>
<tr>
<th></th>
<th>$\beta = 5.9$</th>
<th></th>
<th>$\beta = 5.96$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$t_1$</td>
<td>$t_2$</td>
<td>$p/a^4$</td>
</tr>
<tr>
<td></td>
<td>1.2,3</td>
<td>1</td>
<td>19.2(2)</td>
</tr>
<tr>
<td></td>
<td>2.3</td>
<td>1.2</td>
<td>17.4(2)</td>
</tr>
</tbody>
</table>
matrix element to the gluon condensate,

\[ <0|\text{Tr}(g^2 G_{\mu\nu} G_{\mu\nu})|G> = 16\pi^2 \sqrt{\frac{G_0}{2b}} m_G, \quad (15) \]

where \( G_0 = <0|\text{Tr}(g^2 G_{\mu\nu} G_{\mu\nu})|0> \) is the gluon condensate, \( b = 11/3N_c - 2/3N_f \) and \( m_G \) the scalar glueball mass. Taking \( N_f = 0 \) for the pure gauge case, \( G_0 = 0.012 GeV^4 \) from the ITEP value \([24]\), and \( m_G \) in the range of 1 to 1.7 GeV, this matrix element is then estimated to be 3.7 \( \sim \) 6.3 GeV. This agrees with our lattice calculation to within a factor of two for the extreme cases. For the pseudoscalar case, the lattice renormalization \( Z_p = 2.5 \) makes the glueball matrix element in eq. (11) to be about \( 44 \sim 58K^{3/2} \) or \( 3.3 \sim 4.3 GeV^3 \). It has been proposed that there is an approximate chiral symmetry between the scalar and pseudoscalar glueballs \([25]\). A sum rule is derived from an effective action which relates the topological susceptibility \( \chi \) in the pure gauge case to the gluon condensate \( G_0 \)

\[ \chi = \xi^{-2} G_0 \frac{2}{2b} \quad (16) \]

where \( \xi \) denotes the degree of chiral symmetry. From the Witten-Veneziano \([26]\) mass formula for the \( \eta' \) mass which is related to \( \chi \) in the large \( N_c \) analysis, this sum rule predicts \( \xi \) to be 0.7. From our lattice results, \( \xi \) can be obtained from the ratio of the pseudoscalar to scalar matrix elements \([25]\), i.e.

\[ \xi_L = \frac{Z_p b/a^4}{2Z_s s/a^4}. \quad (17) \]

When we take the ratio between the pseudoscalar and scalar matrix elements from Tables (2) and (5), \( \xi_L \) turns out to range from 0.5 to 0.6. This is quite close to the prediction from the sum rule in eq. (16). In the Witten-Veneziano mass formula for the \( \eta' \) mass \([24]\), the glueball contribution to the topological susceptibility is neglected. The glueball contribution to the topological susceptibility is

\[ \chi_{\text{glueball}} = \left( \frac{<0|\text{Tr}(g^2 G\tilde{G})|G>}{16\pi^2m_G} \right)^2 \quad (18) \]

Using our lattice result and the mass of the pseudoscalar glueball at 1.4 GeV, the pseudoscalar glueball turns out to give an appreciable 9% contribution to the topological susceptibility. With the perturbative lattice renormalization \( Z_T \), our lattice result for the tensor glueball matrix element is \( \sim 14K^{3/2} \) or \( \sim 1.0 GeV^3 \). On the other hand, the prediction from the tensor dominance model \([27]\) and QCD sum rule \([28]\) is about 0.16 \( \sim 0.35 GeV^3 \) for the tensor glueball mass ranging from 1.7 to 2.2 GeV. Hence, the lattice result is about 3 to 6 times larger. Since the tensor dominance model gives an reasonable prediction of \( \theta(f_2(1720)) \) production rate in \( J/\Psi \) radiative decay \([27, 29]\), we suspect that the lattice result we obtain for the tensor
case is overestimated. Recent glueball wavefunction study [16, 17] reveals that the size of the tensor glueball is about 4 times larger than those of the scalar glueball and the pion [30]. Hence, we expect that the tensor glueball is more susceptible to the finite size effect. Considering that our choice of the lattice size should be large enough for the scalar and pseudoscalar glueballs but may not be enough for the tensor, the squeezed tensor glueball will lead to an overestimated tensor matrix element. In the constituent glue picture, the matrix element we calculate can be viewed as the wavefunction at the origin which scales as the inverse 3/2 power of the glueball size. We think this is the likely source for the large tensor matrix element.

Finally, we would like to discuss the prediction of the glueball production rates in $J/\Psi$ radiative decays. While the details of the calculation will be presented elsewhere [31], we would like to mention some of the main features of the findings. It is shown that the partial widths of the glueball production in $J/\Psi$ radiative decays can be related to the vacuum to glueball transition matrix elements calculated in this paper [29, 27, 31]. This is based on the approximation that the charm quark is heavy so that $c$ and $\bar{c}$ annihilate at a point. The recent analysis of the $\pi\pi$ and $K\bar{K}$ decays shows that a large component of spin zero is observed in the $\theta(1720)$ region [32]. If this is confirmed, it would be a very good candidate for the scalar glueball [30].

The calculated scalar glueball matrix element in eq. (10) predicts a branching ratio of $5 \times 10^{-3}$ in $J/\Psi$ radiative decay which is a only a few times larger than the the experimental value from the observed decays in $\pi\pi$ and $K\bar{K}$ channels [32]. The unobserved decay modes will make the agreement better. The calculated pseudoscalar matrix element gives a prediction of $4 \times 10^{-3}$ for the branching ratio of the $0^-$ glueball candidate $\eta(1440)$ in $J/\Psi$ radiative decay. This agrees quite well with the observed B.R. of $\sim 3.5 \times 10^{-3}$ form the decay modes of $K\bar{K}\pi$, $K\bar{K}$, $a_0(980)\pi$, and $\rho^0\rho^0$ [33].

On the other hand, the predicted branching ratio for the tensor case, assuming that the $2^+$ component in $\theta(1720)$ is the tensor glueball, is one to two orders of magnitude larger than the experimental one. This could be due to the fact that the tensor glueball matrix element is overestimated as alluded to earlier. Or perhaps the tensor glueball lies higher in mass (e.g. > 2 GeV) leading to a smaller predicted B.R.

To conclude, we have calculated the glueball to vacuum transition matrix elements in quenched lattice QCD. Notwithstanding the fact that our present study does not have high statistics and the Euclidean time separation is limited, the variational method for the hybrid correlation functions of the fuzzy and plaquette operators does seem to yield consistent results for the glueball masses and matrix elements. The matrix elements obtained for the scalar and the pseudoscalar glueballs seem to agree reasonably with those obtained phenomenologically and the prediction for glueball production rates in $J/\Psi$ radiative decays are in line with the experimental results based on the observed decay modes. The tensor case could be an exception. It yields too large a branching ratio for the tensor glueball in the $\theta$ region in $J/\Psi$ radiative decay and is considerably larger than those from the tensor dominance model and the QCD sum rule. We speculate that it is due to the large finite size effect which is
particularly acute for the tensor due its large size \[16, 17\]. To verify this, one needs to conduct a high statistics calculation on a larger lattice which should restore the rotational invariance for the $E^{++}$ and $T_2^{++}$ and allow a larger separation in $t$ in order to have finite size effect under control.

1 ACKNOWLEDGEMENT

We thank W. Wilcox and C.M. Wu for helpful discussions. This work is partially supported by the DOE Grand Challenge Award and DOE Grant No. DE-FG05-84ER40154. Y.L. acknowledges the support of the Center for Computational Sciences at the University of Kentucky. K.I. is partially supported by a Grant-in-Aid for general Scientific Research (03640256) and a Grant-in-Aid for Scientific Research on Priority Area (04231101), the Ministry of Education, Science, and Culture of Japan.

References


**Figure Captions**

Fig. 1 The making of fuzzy links *a la* Michael and Teper.

Fig. 2 Shapes of different loops used in the variational calculation.

Fig. 3 Operators for the local field strength.