Inequality beyond CCR's bound

If the V state violates the CCR's bound, the inequality beyond CCR's bound exists. This is shown when the correlation between two qubits exceeds a certain threshold. Furthermore, this bound is not only a constraint on the local observables but also on the correlations between them. The inequality holds for all correlations, not just those that can be described by local hidden variables.
tion on two of them, namely, those two in which, if we had measured \( \sigma_z \), we would have obtained the result \(-1\). These two qubits will be called \( i \) and \( j \) hereafter, while the corresponding third qubit (the one in which, if we had measured \( \sigma_z \), we would have found the result \( 1 \)) will be called \( k \). In quantum mechanics, the result of measuring \( \sigma_z \) is not predefined and therefore this prescription for choosing pairs is meaningless. However, the prescription makes sense in a local-realistic theory.

For reasons that will be explained in Sec. III, we are interested in the correlations when we choose \( A = Z_i, a = X_i, B = Z_j, b = X_j \), where \( Z_0 \) and \( X_0 \) are the spin of qubit \( q \) along the \( z \) and \( x \) directions, respectively. In addition, the particular CHSH inequality (1) we are interested in is the one in which \( m = n = x_k \), where \( x_k \) is one of the possible results, \(-1 \) or \( 1 \) (although we do not know which one), of measuring \( X_k \). With this choice we obtain the following CHSH inequality:

\[
|C(Z_i, Z_j) - x_k C(Z_i, X_j) - x_k C(X_i, Z_j) - C(X_i, X_j)| \leq 2, \tag{3}
\]

which holds for any local-realistic theory, regardless of the particular value, either \(-1 \) or \( 1 \), of \( x_k \).

The next step is to use quantum mechanics to calculate the four correlations appearing in inequality (3) for the subset of two qubits \( i \) and \( j \) taken from three qubits prepared in the \( W \) state (2).

For the subset of two qubits \( i \) and \( j \) defined above,

\[
C(Z_i, Z_j) = 1, \tag{4}
\]

because, for the \( W \) state (2),

\[
P_{Z_i Z_2 Z_3}(1, -1, -1) = \frac{1}{3}, \tag{5}
\]

\[
P_{Z_i Z_2 Z_3}(-1, 1, -1) = \frac{1}{3}, \tag{6}
\]

\[
P_{Z_i Z_2 Z_3}(-1, -1, 1) = \frac{1}{3}, \tag{7}
\]

where \( P_{Z_i Z_2 Z_3}(1, -1, -1) \) means the probability of qubit \( 1 \) giving the result \( 1 \), and qubits 2 and 3 giving the result \(-1 \) when measuring \( \sigma_z \) on all three qubits.

By the definition of qubits \( i \) and \( j \),

\[
C(Z_i, X_j) = -x_k, \tag{8}
\]

because, for the \( W \) state (2),

\[
P_{Z_i X_2 X_3}(-1, 1, -1) + P_{Z_i X_2 X_3}(-1, -1, 1) = 0, \tag{9}
\]

\[
P_{X_1 Z_2 Z_3}(1, -1, -1) + P_{X_1 Z_2 Z_3}(-1, -1, 1) = 0, \tag{10}
\]

\[
P_{X_1 Z_2 Z_3}(1, -1, 1) + P_{X_1 Z_2 Z_3}(-1, 1, -1) = 0. \tag{11}
\]

Analogously, using Eqs. (9)-(11),

\[
C(X_i, Z_j) = -x_k. \tag{12}
\]

Finally, qubit \( k \) is the one in which, if we had measured \( \sigma_z \), we would have found the result \( 1 \). The other two are qubits \( i \) and \( j \). For the \( W \) state (2),

\[
P(X_2 = X_3 Z_1 = 1) = P(X_2 = -X_3 Z_1 = 1), \tag{13}
\]

\[
P(X_1 = X_3 Z_2 = 1) = P(X_1 = -X_3 Z_2 = 1), \tag{14}
\]

\[
P(X_1 = X_3 Z_3 = 1) = P(X_1 = -X_3 Z_3 = 1). \tag{15}
\]

where \( P(X_2 = X_3 Z_1 = 1) \) is the conditional probability of \( X_2 \) and \( X_3 \) having the same result, given that the result of \( Z_1 \) is 1. Therefore, irrespective of whether \( i \) and \( j \) are qubits 2 and 3, or 1 and 3, or 1 and 2, we conclude that

\[
C(X_i, X_j) = 0. \tag{16}
\]

Correlations (4), (8), (12), and (16) violate the CHSH inequality (3). The violation (3 vs 2) goes beyond Cirel’son’s bound \( 2\sqrt{2} \).

### III. WHY \( X \) AND \( Z \)?

A particular type of local-realistic theories are those in which the only local experiments whose results are assumed to be predetermined are those which satisfy the criterion for “elements of reality” proposed by Einstein, Podolsky, and Rosen (EPR): “If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity” [3].

As can be easily checked, the violation reported in Sec. II is not the maximal violation of the CHSH inequality (3) for two qubits in the \( W \) state (2). For instance, considering local spin observables on plane \( x-z \) and assuming \( A = B \) and \( a = b \), we find a maximum violation of \( 3.046 \) [by choosing \( A = \cos(0.628)\sigma_x - \sin(0.628)\sigma_z \) and \( a = \cos(1.154)\sigma_x + \sin(1.154)\sigma_z \)]. Why then have we chosen \( A = Z_i, a = X_i, B = Z_j, \) and \( b = X_j \)? The reason is that these observables are not only local observables but, for the \( W \) state (2), they also satisfy EPR’s criterion of elements of reality.

From Eqs. (5)-(7), it can be immediately seen that \( z_1, z_2, \) and \( z_3 \) are elements of reality, since any of them can be predicted with certainty from space-like separated measurements of \( \sigma_z \) on the other two qubits. In addition, from Eqs. (9)-(11), it can easily be seen that, if \( z_i = 1 \) then, with certainty, \( x_j = x_k \). Therefore, if \( z_i = 1 \), then by measuring \( x_j \) (or \( x_k \)) one can predict \( x_k \) (or \( x_j \)) with certainty. Therefore, if \( z_i = 1 \), then \( x_j \) and \( x_k \) are elements of reality. If \( z_i = 1 \) then, using Eqs. (5)-(7), it can immediately be seen that \( z_j = 1 \). Therefore, following the previous reasoning, \( x_i \) and \( x_2 \) are elements of reality (although \( x_i \) could have ceased to be an element of reality after measuring \( \sigma_z \) on particle \( i \)). In conclusion, for trios of qubits in the \( W \) state (2), \( z_1, z_2, z_3, x_1, x_2, \) and \( x_3 \), are EPR elements of reality and thus, according to EPR, they should have predefined values \(-1 \) or \( 1 \) before any measurement.
The violation of the CHSH inequality (3) presented in Sec. II is thus not only a proof of the impossibility of local hidden variables, but also proves a more powerful result: the apparently mild condition proposed by EPR is inconsistent with quantum mechanics.

IV. WHY W?

As was shown in Ref. [10], two qubits belonging to a three-qubit system in a GHZ state can provide a higher violation (4 vs 2, instead of 3 vs 2) of the CHSH inequality (3), even using observables that satisfy EPR’s criterion of elements of reality. Why then use a W state?

One reason is because a test of the violation of Bell’s inequalities beyond Cirelson’s bound could be achieved in practice in the near future. Sources of W states based on parametric down-converting photons are now available for real experiments [14] and some new proposals to prepare W states via cavity quantum electrodynamics have recently been presented [15].

Another reason is because this violation beyond Cirelson’s bound is, in one sense, surprising. The W state is the genuine three-qubit entangled state whose entanglement has the highest robustness against the loss of one qubit [12]. In particular, from a single copy of the reduced density matrix for any two qubits belonging to a three-qubit W state, one can always obtain by means of a filtering measurement a state that is arbitrarily close to a Bell state. Therefore, one might think that any two qubits belonging to a W state will not lead to a higher violation of the CHSH inequality (3) than that for two qubits in a Bell state, and thus it is of interest to realize that this is not the case.

There is, however, another subtler reason for preferring the W state instead of the GHZ state for a test of violation of Bell’s inequalities beyond Cirelson’s bound. Any test of this kind requires a prescription for selecting a pair of qubits from each triple prepared in a quantum state. Such a prescription assumes local realism. In the violation of the CHSH inequality (3) presented in Sec. II, this prescription is simple: qubits i and j are those two in which if we had measured \( \sigma_z \), we would have obtained the result \(-1\). However, in the violation of the CHSH inequality (3) using a GHZ state described in Ref. [10], the prescription is not so simple: there, qubits i and j are either those two in which, if we had measured \( \sigma_z \), we would have obtained the result \(-1\), or any two, if we had obtained the result 1 for all three qubits if we had measured \( \sigma_z \). This means that, for the W state, any local observer could know whether or not his qubit belonged to the selected pair just by measuring \( \sigma_z \); while for the GHZ state, the fact that a qubit belongs or not to the selected pair cannot be decided with certainty from a measurement on that qubit, but requires knowledge of the results of measurements on the other two qubits. From the perspective of local realism, for the W state, one of the elements of reality carried by each qubit determines whether or not it belongs to the selected pair; while for the GHZ state, this information is not local since it is distributed among distant elements of reality.

V. EXPERIMENTAL CH INEQUALITY

The result in Sec. II opens the possibility of using sources of three-qubit W states [14, 15] to experimentally test the CHSH inequality. The main advantage of an experiment like this (or that proposed in Ref. [10]) is that it will admit a direct comparison with the dozens of previous experiments with two qubits [4, 6, 7, 8, 9] and thus goes beyond any previous experiments to test local realism using sources of three qubits [6, 17] inspired by proofs of Bell’s theorem without inequalities [11] or by Bell’s inequalities for three qubits [18, 19].

However, in any real experiment using three qubits, the experimental data consist on the number of simultaneous detections by three detectors \( N_{ABC} \) \((a, b, c)\) for various observables \( A, B, \) and \( C \). This number is assumed to be proportional to the corresponding joint probability, \( P_{ABC} \) \((a, b, c)\). Therefore, in order to make inequality (3) useful for real experiments, it would be convenient to translate it into the language of joint probabilities.

Taking into account that

\[
P_{Z_i Z_j}(-1, -1) = \frac{1}{4}[1 - C(Z_i) - C(Z_j)] + C(Z_i, Z_j),
\]

(17)

\[
P_{Z_i X_j}(-1, -x_k) = \frac{1}{4}[1 - C(Z_i) - x_k C(X_j)] + x_k C(Z_i, X_j)],
\]

(18)

\[
P_{X_i Z_j}(-x_k, -1) = \frac{1}{4}[1 - x_k C(X_i) - C(Z_j)] + x_k C(X_i, Z_j)],
\]

(19)

\[
P_{X_i X_j}(x_k, x_k) = \frac{1}{4}[1 + x_k C(X_i) + x_k C(X_j)] + x_k^2 C(X_i, X_j)],
\]

(20)

where \( C(Z_i) \) is the mean of the results of measuring \( \sigma_z \) on qubit \( i \), and assuming physical locality i.e. assuming that \( C(Z_i) \) is independent of whether \( \sigma_x \) or \( \sigma_y \) is measured on qubit \( j \), that is, assuming that the value of \( C(Z_i) \) is the same in Eqs. (17) and (18), etc., the CHSH inequality (3) between correlations can be transformed into a Clauser-Horne (CH) inequality [20] between joint probabilities:

\[
-1 \leq P_{Z_i Z_j}(-1, -1) - P_{Z_i X_j}(-1, -x_k) - P_{X_i Z_j}(-x_k, -1) - P_{X_i X_j}(x_k, x_k) \leq 0.
\]

(21)

As can be easily checked, the bounds of the CHSH inequality (3) are transformed into the bounds \((l - 2)/4\) of the corresponding CH inequality (21). Therefore, the local-realistic bound in the CH inequality (21) is 0 and Cirelson’s bound is \((\sqrt{2} - 1)/2 \approx 0.207\).
For qubits $i$ and $j$ of a system in the $W$ state (2),
\[
\begin{align*}
P_{Z_i Z_j} (-1, -1) &= 1, \\
P_{Z_i X_j} (-1, -x_k) &= 0, \\
P_{X_i Z_j} (-x_k, -1) &= 0, \\
P_{X_i X_j} (-x_k, -x_k) &= \frac{3}{4}.
\end{align*}
\] (22), (23), (24), (25)

Therefore, probabilities (22)–(25) violate the CH inequality (21). Such a violation (0.25 vs 0) is beyond the corresponding Cirel’son’s bound (0.207).

On the other hand, since we do not know which ones are qubits $i$ and $j$, we cannot obtain the four joint probabilities (22)–(25) just by performing measurements on two qubits. Therefore, we must show how the joint probabilities of qubits $i$ and $j$ are related to the probabilities of the three.

As can easily be seen from the definition of qubits $i$ and $j$,
\[
P_{Z_i Z_j} (-1, -1) = P_{Z_i Z_j Z_k} (1, -1, -1) + P_{Z_i Z_j Z_k} (-1, 1, -1) + P_{Z_i Z_j Z_k} (-1, 1, 1) + P_{Z_i Z_j Z_k} (-1, -1, 1). \tag{26}
\]

Therefore, in order to experimentally obtain $P_{Z_i Z_j Z_k} (-1, -1)$, we must measure the four probabilities in the right-hand side of Eq. (26). In the $W$ state (2), the first three probabilities in the right-hand side of Eq. (26) are expected to be 1/3 and the fourth is expected to be zero.

On the other hand, $P_{Z_i X_j} (-1, -x_k)$ and $P_{X_i Z_j} (-x_k, -1)$ are both less than or equal to
\[
\begin{align*}
P_{Z_i X_j X_k} (-1, 1, -1) + P_{Z_i X_j X_k} (-1, 1, 1) + P_{X_i Z_j X_k} (-1, 1, -1) + P_{X_i Z_j X_k} (-1, 1, 1)
\end{align*}
\] (27)

Therefore, in order to experimentally obtain $P_{Z_i X_j} (-1, -x_k)$ and $P_{X_i Z_j} (-x_k, -1)$, we must measure (using three different setups) all six probabilities in sum (27). In the $W$ state (2), each of these six probabilities is expected to be zero.

Finally,
\[
P_{X_i X_j} (x_k, x_k) = P_{X_i X_j X_k} (1, 1, 1) + P_{X_i X_j X_k} (-1, -1, -1). \tag{28}
\]

Therefore, in order to experimentally obtain $P_{X_i X_j} (x_k, x_k)$, we must measure the two probabilities in the right-hand side of Eq. (28). In the $W$ state (2), each of them is expected to be 3/8.

VI. CONCLUSIONS

Two qubits selected from a trio prepared in a $W$ state violate the CHSH inequality, or the corresponding CH inequality, more than two qubits prepared in any quantum state. Such violations beyond Cirel’son’s bound are smaller than those achieved by two qubits selected from a trio in a GHZ state [10]. However, for the $W$ state the argument is simpler, since all local observers can know from their own measurements whether or not their qubits belong to the selected pair.

The importance of these arguments relies on the fact that they suggest how to use sources of three-qubit quantum entangled states to experimentally reveal violations of the familiar two-qubit Bell inequalities beyond those obtained using sources of two-qubit quantum states.

Acknowledgments

This work was prompted by a question made by H. Weinfurter during the Conference Quantum Information: Quantum Entanglement (Sant Feliu de Guíxols, Spain, 2002). I thank D. Collins for pointing out a mistake in a previous version, C. Serra for comments, and the Spanish Ministerio de Ciencia y Tecnología, Grant No. BFM2001-3943, and the Junta de Andalucía, Grant No. FQM-239, for support.