Constraints on the Existence of Chiral Fermions
in Interacting Lattice Theories

by

Yigal Shamir
Department of Physics
Weizmann Institute of Science, Rehovot 76100, ISRAEL
email: ftshamir@weizmann.bitnet

ABSTRACT

It is shown that an interacting theory, defined on a regular lattice, must have a vector-like spectrum if the following conditions are satisfied: (a) locality, (b) relativistic continuum limit without massless bosons, and (c) pole-free effective vertex functions for conserved currents.

The proof exploits the zero frequency inverse retarded propagator of an appropriate set of interpolating fields as an effective quadratic hamiltonian, to which the Nielsen-Ninomiya theorem is applied.

PACS: 11.15Ha, 11.30.Rd, 11.20.Fm.
The only rigorous way, presently known to us, to define non-abelian gauge theories, relies on the lattice as a regulator. The observed fermion spectrum fits into a chiral representation of SU(3) × SU(2) × U(1), and so the construction of a consistent chiral gauge theory on the lattice has been a major goal in theoretical physics.

In spite of extensive efforts, this program has been unsuccessful to date. The basic stumbling block is the doubling problem [1]. A naive discretization of the continuum hamiltonian of a Weyl fermion gives rise to eight Weyl fermions in the classical continuum limit of the lattice hamiltonian. If one starts with a Dirac fermion, the doublers can be eliminated by introducing the Wilson term. But the price is that the axial symmetry of the classical continuum hamiltonian is lost.

Following the work of Karsten and Smit [2], the precise conditions for the presence of doublers in a free fermionic theory defined on a regular lattice were stated by Nielsen and Ninomiya as a no-go theorem [3]. They assume the existence of a set of exactly conserved, locally defined charges which admit discrete eigenvalues. The Nielsen-Ninomiya theorem then asserts that there must be an equal number of positive helicity and negative helicity fermions in every complex representation of the symmetry group, provided the Fourier transform of the free hamiltonian has a continuous first derivative. (Recall that chirality equal helicity for massless fermions). The Nielsen-Ninomiya theorem applies in particular when the hamiltonian has a short range, and the charges are constructed canonically and generate a compact Lie group.

The absence of chiral fermions is essentially a counting theorem about the zeros of the free hamiltonian in the Brillouin zone. In three space dimensions, a massless fermion corresponds to level crossing which is described by the effective two-by-two hamiltonian

\[ H_{\text{eff}}(p) = \pm \sigma \cdot (p - p_\text{c}) + O((p - p_\text{c})^2) . \]  

The point \( p_\text{c} \) is called a degeneracy point. The \( \pm \) signs correspond to the helicity of the fermion. The Nielsen-Ninomiya theorem is then a consequence of theorems in algebraic topology which exploit the fact that the Brillouin zone is topologically a three-torus.

The continuum limit of asymptotically free gauge theories is achieved at vanishing bare coupling. Together with the success of perturbative QCD in deep inelastic scattering, this leads to the generally accepted view that the fermionic spectrum can be correctly determined by setting the gauge couplings to zero. In the absence of other interactions, the Nielsen-Ninomiya theorem implies that the fermionic spectrum must be vector-like provided the hamiltonian has a short range.

Attempts to avoid fermion doubling by using long range lattice derivatives lead to various inconsistencies at the level of weak coupling perturbation theory [4]. Impor-
tant examples include the SLAC derivative which avoids the extra zeros by creating a discontinuity in the dispersion relation, and a method due to Rebbi which is characterized by the presence of a pole in the dispersion relation. The former suffers from Lorentz violations and non-locality [5] while the latter suffers from the presence of ghosts [6].

As it stands, the Nielsen-Ninomiya theorem does not apply if the lattice model contains some strong interactions. This observation have led to several proposals [7] for constructing chiral gauge theories on the lattice which exploit a common strategy. One starts with a model containing only fermions and (possibly) scalar fields. In addition to standard quadratic terms, one introduces judiciously chosen strong interactions among these fields which are operative at the lattice scale, and vanish in the classical continuum limit. Local symmetries of the desired continuum theory – the target theory – should appear at this stage as exact global symmetries of the model. One then tries to find a point in the phase diagram in which all the doublers decouple. It is crucial that, at the same time, the to-be-gauged global symmetries are not broken spontaneously.

If this program were successful, a consistent chiral gauge theory could be obtained by turning on the gauge interactions in such a model. However, explicit model calculations have lead to negative conclusions in all cases studied so far [8].

Our purpose in the present letter is to provide a general treatment of the problem, which applies for example also to similar models based on the recently proposed domain wall fermions [9]. We will prove that under mild assumptions, which are directly related to the physical properties of a consistent continuum limit, the spectrum is necessarily vector-like. In this letter we report our main results, whereas the full details will appear elsewhere [10].

We will consider a hamiltonian defined on a regular space lattice (time is continuous) that has a compact global symmetry group which is not spontaneously broken. The conserved generators of the global symmetries are assumed to be the sum over all lattice points of a local density. The idea is to use the zero frequency inverse retarded propagator of an appropriate set of interpolating fields, denoted $\mathcal{R}^{-1}(p)$, as an effective quadratic hamiltonian which satisfies all the assumptions of the Nielsen-Ninomiya theorem.

The crucial element of our theorem is the demonstration that a sufficiently local anti-commutator, of the kind expected in theories with a short range hamiltonian, gives rise to an analytic $\mathcal{R}(p)$. The only singularities occur at generalized degeneracy points, which are those points in the Brillouin zone where the hamiltonian admits eigenstates of vanishing energy. The proof of analyticity invokes the “edge of the
wedge” theorem [11], and it is an adaptation to the lattice context of classic results from the theory of dispersion relations.

In more detail, if a particle can be created in a causal process, there should exist a local interpolating field which has a finite probability to create the particle by acting on the vacuum. The particle should then generate a singularity in the two point function of the interpolating field.

The two point function we will consider is the retarded anti-commutator. Suppressing possible colour and flavour indices it is defined by

\[ R_{\alpha\beta}(x,t) = \theta(t) \langle 0 | \{ \psi_{\alpha}(x,t), \psi_{\beta}^{\dagger}(0,0) \} | 0 \rangle . \]  

We also introduce the space and space-time Fourier transforms

\[ \tilde{R}_{\alpha\beta}(p,t) = \sum_{x} e^{-ip \cdot x} R_{\alpha\beta}(x,t) , \]

\[ \tilde{R}_{\alpha\beta}(p,\omega) = \int_{0}^{\infty} dt e^{i\omega t} \tilde{R}_{\alpha\beta}(p,t) , \]

and define

\[ R_{\alpha\beta}(p) = \lim_{\epsilon \to 0} \tilde{R}_{\alpha\beta}(p,\omega = i\epsilon) . \]

We first observe that \( R(x,t) \) is bounded. This is a trivial consequence of translation invariance and the fact that \( \psi_{\alpha}(0,0) \) is a well defined operator on the Hilbert space. Thus, there exists \( 0 < b_{1} < \infty \) such that

\[ |R_{\alpha\beta}(x,t)| \leq \left\| \psi |0\rangle \right\|^2 + \left\| \psi^{\dagger} |0\rangle \right\|^2 \leq b_{1} . \]

For similar reasons, at fixed \( x \), \( R(x,t) \) is an analytic function of \( t \). As a result, \( R(x,t) \) cannot vanish identically outside the light cone, for then it would be zero everywhere.

We should therefore discuss the rate at which \( R(x,t) \) tends to zero at large space-like separations. We will say that \( R(x,t) \) is local if it can be bounded by an exponential, i.e. if there are positive constants \( c, b_{2}, \) and \( \mu \) such that for all \( |x| > ct \)

\[ |R_{\alpha\beta}(x,t)| \leq \min \left\{ b_{1}, b_{2} e^{-\mu(|x| - ct)} \right\} . \]

We will say that \( R(x,t) \) is strongly local if it decreases faster than an exponential, i.e. if it satisfies a bound of the form (7) for every \( \mu \).

We will assume below that \( R(x,t) \) is local or strongly local. This will allow us to prove the analyticity of \( \mathcal{R}(p) \). In view of the intimate relation between causality and analyticity in the continuum, an exponentially bounded anti-commutator on the lattice should be a necessary condition for causality in the continuum limit. We
comment, however, that to exclude the existence of chiral fermions, all that is needed is that \( R^{-1}(p) \) have a continuous first derivative. Consequently, it is sufficient to assume a much weaker form of locality, namely, that the anti-commutator is bounded by an appropriate inverse power of \( x \) [10].

The last thing we need is the notion of a \textit{generalized degeneracy point}. Introducing the \textit{advanced} anti-commutator \( \tilde{A}(p, \omega) \) we define for real values of \( p \) and \( \omega \)

\[
E_0(p) = \sup \{ \omega \mid \tilde{R}(p, \omega') = \tilde{A}(p, \omega') \text{ if } |\omega'| < \omega \} .
\] (8)

The physical meaning of this definition is that \( E_0(p) \) is the lowest possible energy for eigenstates with momentum \( p \). We define a generalized degeneracy point by the condition \( E_0(p_c) = 0 \). Thus, \( p_c \) is a generalized degeneracy point if it is the end point of a gap-less continuous spectrum.

\textit{Lemma.} Assume that \( R(x, t) \) is local in the sense of eq. (7). Then (a) \( \tilde{R}(p, \omega) \) is holomorphic in the domain \( \text{Im} \omega > 0, |\text{Im} p| < \min\{c^{-1} \text{Im} \omega, \mu\} \); (b) \( \mathcal{R}(p) \) is analytic with singularities only at generalized degeneracy points.

\textit{Proof.} By assumption, the r.h.s. of eq. (3) is bounded by the r.h.s. of eq. (7) times \( e^{x|\text{Im} p|} \). Hence, the sum in eq. (3) converges absolutely and \( \hat{\mathcal{R}}_{\alpha\beta}(p, t) \) is holomorphic in the domain \( |\text{Im} p| < \mu \). Moreover, \( \hat{\mathcal{R}}_{\alpha\beta}(p, t) \) can be bounded by a polynomial of third degree in \( t \) times \( e^{ct|\text{Im} p|} \). The presence of the damping factor \( e^{-t|\text{Im} \omega|} \) then implies that the integral on the r.h.s. of eq. (4) converges absolutely for \( |\text{Im} p| < \min\{c^{-1} \text{Im} \omega, \mu\} \). This proves (a). Notice that if \( R(x, t) \) is strongly local than \( \hat{\mathcal{R}}_{\alpha\beta}(p, t) \) is an entire function of \( p \). In this case \( \tilde{R}(p, \omega) \) is holomorphic in the forward cone \( |\text{Im} p| < c^{-1} \text{Im} \omega \).

In order to prove (b) we notice that the \textit{advanced} anti-commutator has similar properties except that the sign of \( \text{Im} \omega \) is now negative. A straightforward application of the edge of the wedge theorem [11] now implies that the common boundary function \( \mathcal{R}(p) \) is analytic, with singularities only at generalized degeneracy points. This proves (b).

The analyticity of \( \mathcal{R}_{\alpha\beta}(p) \) away from generalized degeneracy points, implies that there can be no obstructions to the smooth motion throughout the Brillouin zone from one zero of \( \mathcal{R}_{\alpha\beta}^{-1}(p) \) to another, \textit{provided} we exclude the possibility of \textit{poles} in \( \mathcal{R}_{\alpha\beta}^{-1}(p) \). We forbid this situation by \textit{assuming} that the elements of the matrix \( \mathcal{R}_{\alpha\beta}(p) \) are bounded. We comment that, together with the assumption that symmetries are not broken spontaneously, this is equivalent via the Ward identities to the requirement that effective vertex functions, defined as correlation functions of conserved currents and the interpolating fields, are pole free.
The presence of a pole in $R^{-1}_{\alpha\beta}(p)$ may reflect a kinematical singularity, arising from a bad choice of interpolating fields. It may happen if (a) one uses two interpolating fields for two fermions in different corners of the Brillouin zone, whereas actually they can both be interpolated by a single field, or (b) if not all fermions are interpolated. In both cases it is possible to identify the kinematical nature of the singularity, and to construct a new, *admissible* set of interpolating fields which is free of this problem. Details will be given elsewhere [10].

If a pole in $R^{-1}_{\alpha\beta}(p)$ is not an artifact of an inadmissible set of interpolating fields, it cannot arise unless the hamiltonian is highly non-local. In a very general context, it has been shown that such poles give rise to the appearance of ghosts in one loop diagrams once gauge fields are introduced [6]. The reason is that, via the Ward identity, such poles appear in the vertex function, but they contribute to the vacuum polarization with the wrong sign. Interpreted in a hamiltonian language, this result implies that the action of a local current on the vacuum takes one outside the Hilbert space, which is unacceptable. We thus expect that it should be possible to extend our theorem and to rigorously exclude the presence of such poles in a consistent quantum theory.

By construction, $R^{-1}(p)$ is a hermitian matrix. In order to show that it qualifies as an effective hamiltonian which satisfies the assumptions of the Nielsen-Ninomiya theorem, what is left for us to do is to show that it has a continuous first derivative at generalized degeneracy points and that its zeros can be identified with massless fermions.

In order to characterize the singularities of $R^{-1}(p)$, we now assume that at sufficiently large distances physics is correctly described by an effective lagrangian of massless fermions interaction only via non-renormalizable couplings. The justification of this assumption is that, in all models which attempt to decouple the doublers dynamically, the aim is to achieve a continuum limit with the above properties (before the gauge interactions are turned on). We demand that all singularities of $R^{-1}(p)$ should be compatible with those allowed by the effective lagrangian.

Let us denote by $p_{\text{phys}}$ the momentum variable which transforms homogeneously under the Lorentz group in the low energy limit. The only singularity compatible with the assumed form of the low energy effective lagrangian is

$$\pm \frac{1}{\sigma \cdot p_{\text{phys}} \left(1 + O(p_{\text{phys}}^4 \log p_{\text{phys}}^2)\right)}.$$  \hspace{1cm} (9)

We have $p_{\text{phys}}^4$ in front of the logarithmic term because this term involves at least two powers of coupling constants, and all coupling constants have a negative mass dimension which is at least two.
Clearly, an allowed singularity of $\mathcal{R}(p)$ can be obtained by substituting $p - p_c$ instead of $p_{\text{phys}}$ in eq. (9). We call such a singularity a *primary singularity*. The precise definition is as follows. A generalized degeneracy point $p_c$ is a primary singularity if there exists a unitary transformation $U$ such that

$$\lim_{p \to p_c} \left[ \sigma \cdot (p - p_c) \otimes I_1 \right] U \mathcal{R}(p) U^\dagger = [I \otimes A].$$

(10)

In eq. (10), $I$ is the identity matrix in spin space, $I_1$ is the identity matrix in colour and flavour space, and $A$ is a diagonal matrix $A = \text{diag}(Z_1, \ldots, Z_k, 0, \ldots, 0)$. The $Z$-s are non-zero constants, which are in one-to-one correspondence with massless fermions. The helicity of the massless fermion is determined by the sign of the corresponding $Z$.

In addition, there will in general be *secondary singularities* at points which are integer multiples of the primary singularities. These points correspond to multi-particle spectra having the same quantum numbers as the original particle. The leading contribution of the gap-less spectrum to $\mathcal{R}(p)$ at a secondary singularity point can take the form

$$\pm \sigma \cdot p_{\text{phys}} p_{\text{phys}}^2 \log p_{\text{phys}}^2.$$  

(11)

As before, $p_{\text{phys}} = p - p_c$. This form is dictated by the requirement that, once we sum $\mathcal{R}(p)$ over all points that correspond to the same $p_{\text{phys}}$, we will obtain an expression compatible with the expansion of the denominator of eq. (9) in powers of the coupling constant. Notice that there is no reason that $\mathcal{R}(p)$ should vanish at a secondary singularity point, because it always receives additional, regular contributions from finite energy branches of the spectrum.

We now collect all our intermediate results together in the following theorem.

*Theorem.* Consider a hamiltonian defined on a regular lattice. Assume the existence of a compact global symmetry group which is not spontaneously broken. Assume also that the continuum limit is relativistic and that the only massless particles are fermions. Under these assumptions, there is an equal number of left handed and right handed fermions in every complex representation of the global symmetry group, provided $R(x, t)$ is local and $\mathcal{R}^{-1}(p)$ is bounded for every admissible set of interpolating fields.

*Proof.* Consider all sets of interpolating fields which satisfy the above assumptions and which belong to a given complex representation. Choose a maximal set. By this we mean that the total number of $Z$-s, as determined by the limiting procedure (10) and summed over all primary singularity points, is maximal. This number is then the total number of massless fermions in that representation.
Locality of $R(x,t)$ and boundedness of $R^{-1}(p)$ imply that $R^{-1}(p)$ is analytic except at generalized degeneracy points. Furthermore, the allowed forms of singularities, eqs. (9) and (11), imply that $R^{-1}(p)$ has a continuous first derivative at the generalized degeneracy points, and that all zeros of $R^{-1}(p)$, which occur at primary singularity points, are of the relativistic form (1). In addition, $R^{-1}(p)$ is hermitian. Hence $R^{-1}(p)$ satisfies all the assumptions of the Nielsen-Ninomiya theorem. Applying the theorem, we conclude that $R^{-1}(p)$ has an equal number of left handed and right handed zeros. Since the set of interpolating fields we have chosen is maximal, this implies that the spectrum contains an equal number of left handed and right handed fermions in the given complex representation.

For completeness, we note that if a fermion belongs to a real representation, it can generate both a left handed and a right handed pole in its two point function. This is the only way to violate the one-to-one correspondence between poles of the two point function and massless fermions. Of course, this exceptional situation is of no help if we are trying to construct chiral fermions.

Perhaps the most striking consequence of our theorem is the constraints it puts on any attempt to reproduce the standard model on the lattice without violating gauge invariance. It asserts any such attempt must fail, if the spectrum can be correctly determined by switching off the Electro-Weak interactions, and provided that the effective vertex functions of the Electro-Weak currents are pole-free in the symmetric phase. In order to reach this conclusion there is no need to switch off QCD! The reason is that the spectrum of QCD does not contain massless bosons, and so, in the absence of the photon, an effective lagrangian of the kind described above is valid at distances larger than one Fermi. We comment that it should be possible to accommodated a massless pion without changing the conclusions because a Goldstone boson has only derivative couplings.

The fact that gauge invariant lattice theories are necessarily vector-like raises an intriguing question concerning the relation between fermion doubling and the anomaly. If we are careful to work with an anomaly free theory and to break explicitly at the lattice scale all global symmetries which are anomalous in the target continuum theory (e.g. Baryon number), then there is no “need” for the appearance of doublers [12]. This is the case, for example, in the Eichten-Preskill model [7]. The lattice theory does not have a bigger symmetry compared to the target theory, and so one could expect to obtain the latter in the continuum limit of the lattice theory. Nevertheless, the doublers do appear. The resolution of this paradox, which is clearly of a non-perturbative nature, must await future investigations.
I thank A. Casher for numerous discussions of the subject. This research was supported in part by the Basic Research Foundation administered by the Israel Academy of Sciences and Humanities, and by a grant from the United States – Israel Binational Science Foundation.


