Fermionic Field Theory and
Gauge Interactions on Random Lattices

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Random-lattice fermions have been shown to be free of the doubling problem if there are no interactions or interactions of a non-gauge nature. However, gauge interactions impose stringent constraints as expressed by the Ward-Takahashi identities which could revive the free-field suppressed doubler modes in loop diagrams. After introducing a formulation for fermions on a new kind of random lattice, we compare random, naive and Wilson fermions in two-dimensional Abelian background gauge theory. We show that the doublers are revived for random lattices in the continuum limit, while demonstrating that gauge invariance plays the critical role in this revival. Some implications of the persistent doubling phenomenon on random lattices are also discussed.
1 Introduction

Lattice regularisation is particularly important not only because of its fundamental and traditional role in defining and regularising quantum field theory but also because it opens the way to rigorous, non-perturbative treatments. However, on the lattice chiral fermions suffer the doubling phenomenon of generation of extraneous species [1] in such a way that the net chirality is zero. This long-standing doubling problem is the most important problem of quantum field theory. More than just technical, it is inherently fundamental as emphasised by several no-go theorems, and as the difficulty in defining Chiral Gauge Theory is shared by all other regulators in one way or another.

The doubling problem of lattice fermions is inevitable according to the Nielsen Ninomiya no-go theorem [1] if the bilinear free-field action satisfies the conditions of reflection positivity, locality, global axial symmetry, and translational invariance at a fixed scale. An obvious resolution of the doubling problem is thus to relax one of those conditions to obtain, in the order listed above, non-hermitian [2], non-local [3], Wilson [4], or random-lattice [5, 6, 7, 10, 14] fermion formulations. These formulations are all free of doublers when there are no interactions or when the interactions are of a non-gauge nature [11, 12]: the extra poles in the propagators are removed as the lattice spacing $a$ decreases, leaving a single fermion mode in the continuum limit.

Gauge interactions behave very differently on account of a unique and special property. Local gauge invariance imposes severe constraints on the theory, expressed mathematically in the Ward-Takahashi identities. In particular, for electrodynamics, the fermion-gauge vertex $\Lambda_\mu$ is related to the free inverse propagator $G_0^{-1}$ at zero momentum transfer,

$$\Lambda_\mu(p, p) = -g \frac{\partial}{\partial p^\mu} G_0^{-1}(p),$$

(1)

giving the interaction vertices mode dependency. Modifications made to the action $\bar{\Psi} G \Psi$ must be compensated by an appropriate well prescribed change of conserved current, and thus vertex, in order to respect gauge invariance and the minimal coupling prescription. Such modifications have been shown to revive the doubled modes in studies of some non-local [13] and non-hermitian formulations [15, 16], even though those modes are suppressed at the free-field level. For this reason, we investigate the issue of fermion doubling on random lattices with gauge interactions [17].

In random lattice approaches, suitable quantities are measured on a random lattice then averaged, either quenchedly or annealedly, over an ensemble of lattices. Apart from the extra work involved in generating an ensemble of random lattices, this approach better approximates the scale-free rotational and translational symmetry of the continuum than regular lattices. Thus, the continuum limit may be more easily reached on random lattices than on regular lattices of the same size. More relevant to this discussion, since there is no fixed Brillouin
zone, there need be no extra poles of the propagator. Even if extra poles do exist, the one-to-one correspondence between propagator poles in momentum space and zero modes is not necessarily valid since plane waves are no longer eigenstates of the Dirac operator. Alternatively, one could appeal to the fact that there is no transfer matrix on a random lattice (at least for a finite lattice) since there are no identical timeslices, to argue that there may not be a clear relation between poles of the inverse propagator and the particle spectrum [10].

This expectation of no doubling on random lattices has been realised in various studies of free-field theory in both two and four dimensions [6, 7]. It has similarly been shown that random lattice theories with four-point interactions are also doubler free [11]. The full gauge invariant formulation has not yet been properly considered on account of problems associated with identifying the appropriate conserved gauge current that appears in the action. Even though we will not consider this current explicitly, we consider a quantity which automatically includes the correct current. We focus on the connected gauge field two-point function, which can be computed directly from the action,

\[
- \ln \det (G_A G_0^{-1}) = \Gamma_2 + O(g^4),
\]

\[
\Gamma_2 = \text{Tr} [(G_A^{-1} G_0 - 1) - \frac{1}{2} (G_A^{-1} G_0 - 1)^2] + \ldots,
\]

where \(G_A^{-1}\) is the fermion propagator in a gauge field \(A_\mu\) and \(g\) is the gauge coupling. On the lattice, \(\Gamma_2\) is the required two-point function up to a numeric constant and \(O(A^4 g^4 a^2)\) correction terms. Choosing a small coupling \(g A_\mu \ll 1\) keeps this correction under control even on a finite lattice. We choose to study two dimensional background QED, in which there are no internal photon lines in the loops. Hence, we expect to see a clean signal which simply counts the number of fermion modes in the continuum \((a = 0)\) limit. A comparison with identical calculations for naive and Wilson fermions on two dimensional square lattices, which are known to be four-fold doubling and doubler-free respectively, over a range of \(a\), clarifies the continuum limit behaviour of our random lattices. After introducing the random lattice and fermion action used in this work and comparing with other random lattices, we verify that the free-field case does indeed have doubler suppression. However, in the fully interacting gauge invariant theory, the two-point function does not suppress doubling. If gauge invariance is broken on the lattice, the doubled modes are eliminated.

2 Random lattice construction

Random lattices are constructed using a method borrowed from studies of simplicial gravity: an initial regular triangulated array of \(N\) fixed square lattice vertices is rearranged by a sequence of Alexander ‘flip’ moves [18, 19]. Figure 1
pictorially demonstrates a single flip: A quadrilateral $ABCD$, with a unique internal link $AC$ is randomly chosen, the internal link is deleted, and a new link $BD$ is introduced. In simplicial gravity studies there is no restriction on the local curvature, and bond lengths are generally kept fixed. This flip prescription is sufficient to build an arrangement of links and vertices which is statistically unaffected by further flipping $[19]$. In the case at hand, we require the local curvature to be zero. The most straightforward way to accommodate this is to fix the vertex positions and allow the links to have varying lengths. A further condition must be included: a flip is performed only if the planar local orientability of the links is preserved. This prevents undesirable two-dimensional behaviour such as crossed links (which correspond to parts of the lattice overlapping each other), and zero area simplices from being generated in the flipping procedure. Both the number of vertices and the number of links can be fixed on the lattice. This fixed-vertex construction has several features: the entire process is $O(N)$, each random lattice has a fixed size independent of the exact details of the randomising procedure, so measured quantities do not need to be scaled by the average link length $s$, and the scale below which the lattice ceases to be a meaningful representation of space-time is cleanly identified by $s = a$. We consider a quantity of flips called a ‘scan’, which is defined as $6N$ successful flips. This measure of flipping is independent of lattice size, and allows the generic behaviour of
the flipping procedure to be uncovered in more detail. At first glance, it may seem reasonable that the flipping process leads to equilibrated configurations, however it is not clear that such an equilibrated structure actually exists, or is useful in these studies.

Beginning from the regular configuration, the links are in their shortest possible state. Random flipping will thus tend to increase the link length. In fact, this behaviour persists for a relatively small number of scans even after the initial structure has been well destroyed. Figure 3a. demonstrates the behaviour of $\Delta s = s(n) - s(0)$, the average link extension as a function of scan number $n$, for various lattice sizes, clearly demonstrating the universal nature of the randomising process. The link extension is numerically approximated by

$$ (\Delta s)^2 = \frac{1}{2} \log n, $$

with fluctuations about this value diminishing as the lattice size is increased.

One may expect there would be some retarding of the growth; as the length of a link increases, the paths of other growing links are blocked, which may lead to a grid-lock situation, where the structure is equilibrated to a certain extent. That this equilibration is a global phenomenon is easily understood by contemplating the behaviour on larger lattices, where longer links are required in order to introduce the same effect. Thus the equilibrium, if it exists, is probably induced by a finite size effect. Such notions have already been observed in the quantum gravity case where the fractal dimension of the equilibrated structure in two dimensions diverges with $\log N$ [19], indicating that the establishment of equilibrium is intimately connected with the global lattice size.

Since the vertices of this lattice are anchored and we have chosen toroidal
boundary conditions, another scenario is possible: a fault may form which wraps
about the entire torus, link growth perpendicular to the fault is prevented, and
links within the fault cannot be flipped out, thus the fault is more difficult to
remove when it gets longer, and likely to grow further. This leads to a runaway
situation, which is once again related to the finite lattice size.

Figure 3b. demonstrates both the equilibrated, and runaway situation; initial
generations are identical, only the random seed has been changed. In order
to avoid these undesirable global effects, we choose to flip as little as possible;
randomising with 1...8 scans, see Fig. 2 for a typical lattice. By construction,
the lattice violates translational invariance and is thus immediately suitable for
our purpose.

It is interesting to compare with other random lattice formulations, specifically
the CFL lattice of ref. [5]. For these purposes, we consider the coordination
number distribution, \( \rho_N(C) \) and the link length distribution \( \rho_L(x) \).

Figure 4 details \( \rho_N(C) \). Initially \( \rho_N(6) = 1 \) is the only non-zero value,
but after only one scan, the distribution is approximately normal (a Gaussian
approximation is shown by the shaded line in figure 4), and is also very similar
to the standard lattice studied as detailed in ref. [8]. The difference is notably
that our lattice exhibits a broader distribution of coordination numbers, with
significantly more vertices of high coordination number. Further scans produce
more broadening, with even more vertices of high coordination number formed,
until an equilibrium situation is established where asymptotically,

\[
\rho_N(C) \propto e^{-\alpha C}.
\]

Too many highly coordinated vertices is undesirable particularly on a fixed ver-
tex lattice, since this encourages long links which reduce the degree of locality
of the lattice. With this increased graininess one would expect a greater uncer-


tainty in calculations. However, this uncertainty is expected to vanish in the
large lattice limit since the coordination number distribution has the same form
after an equivalent number of scans. i.e: it is dictated by local lattice properties.
The link length distribution of the standard lattice can easily be computed from
the formalism presented in ref. [5],

\[
\rho_L(x) = 2\pi^2a^{-4}x^3e^{-a^2x^2}
\]

our lattice has very different characteristics, as shown in figure 5. The link
distribution is approximated by

\[
\rho_L(x) = s^{-1}e^{-x/a}.
\]

with many more long links than the CFL lattice, as expected. If there are prob-
lems with link variable specification, this lattice will emphasise these problems.

To further the comparison, we put Ising spins on the vertices of the lattice,
with each bond having equal strength couplings. Since the coordination number
Figure 4: Coordination number density for a 65536 point random lattice.
Figure 5: Link length distribution for a 65536 point lattice.
distributions are similar, we do not expect too dissimilar critical behavior. The magnetic susceptibility and magnetisation are calculated from 5000 spin configurations of a 65536 vertex lattice using the Swendsen–Wang clustering method [20], over a range of \( \beta \) either side of criticality. Passing over the critical point from either direction indicates no detectable hysteresis, see figure 6. The critical point for one scan is at \( \beta = 0.2594 \pm 0.002 \), determined by averaging the values determined by the linear fits in this figure. This compares with \( \beta = 0.2631 \) for the CFL lattice [9] and \( \beta = \frac{1}{4} \log 3 = 0.2746 \) for an unflipped lattice. As previously alluded to, our lattice does not seem to produce as clean a signal as CFL which has only 1000 points compared to 65536 of ours, the scaling region in particular is smaller, and the flipping process worsens the behaviour considerably; \( \beta_c \) is reduced, and the critical exponents are more susceptible to finite size effects, which results in a smaller scaling region. This should not be such a problem for the critical phenomena of the fermion theory which naturally suppresses long links (thus, highly connected vertices) with massive propagator. Another obviously different behaviour is the differing gradients on each side of the critical point. The results show that the lattices are quite probably in the same universality class as the square lattice, and certainly not in the same class as the fixed vertex construction of ref. [21].

Having briefly investigated the lattice construction procedure, we now consider the fermion action on this lattice.

3 The Fermion Action

The (Euclidean) fermion action is derived from the continuum in the most direct way

\[
S = \sum_x \frac{1}{2} \left( \sum_l \overline{\Psi}_x \gamma_\mu \Delta_\mu^{x+l} \Omega_{x,x+l} \Psi_{x+l} \right) - \sum_x \frac{1}{2} \left( \sum_l \overline{\Psi}_{x+l} \gamma_\mu \Delta_\mu^{x-l} \Omega_{x+l,x} \Psi_x \right) + m \sum_x \Omega_{x,x} \overline{\Psi}_x \Psi_x, \tag{7}
\]

where \( \Delta_\mu \) is the lattice derivative, and \( \Omega_{x,y} \) weights the volume contribution of the vertices at each end of a link. Both are chosen in a manner which reduces to the naive result on regular lattices [4, 22]. At a vertex \( x \) with coordination number \( C_x \), the derivative is constructed by averaging the contributions of pairs of orientation-consecutive links \( \{(k,l)\} \).

For a pair of such non-parallel links,

\[
\begin{pmatrix}
\frac{\partial_0}{\partial_1} \Psi(x)
\end{pmatrix} \approx \begin{pmatrix}
k_0 & k_1 \\
l_0 & l_1
\end{pmatrix}^{-1} \begin{pmatrix}
\Psi(x + k) - \Psi(x) \\
\Psi(x + l) - \Psi(x)
\end{pmatrix} \tag{8}
\]

The lattice construction ensures the existence of the inverse for all pairs of links which belong to the same simplex. Since each simplex should be treated on
6a. Magnetisation

6b. Susceptibility

Figure 6: Ising spins on our random lattice
an equal footing, we form the average contribution over all such pairs which surround the vertex,

\[ \sum_{i} \Psi_{x} \gamma_{\mu} \Delta_{\mu}^{x+i} \Psi_{x+i} \Omega_{x,x+i} = \Psi_{x} \sqrt{\omega_{x}} C^{-1} \sum_{\{i(k,i)\}} \frac{\gamma}{k \times l} \times \left[ l \Psi_{x+k} \sqrt{\omega_{x+k}} - k \Psi_{x+i} \sqrt{\omega_{x+i}} + (k - l)\Psi_{x} \sqrt{\omega_{x}} \right] \]

\( \omega_{x} = \Omega_{x,x} \) is determined by taking 1/3 of the area of all triangles which have \( x \) as a vertex, the simplest way of discretising the volume integral. The diagonal \( \Psi_{x} \Psi_{x} \) term in equation (9) is canceled by an identical contribution which emerges when the action is Hermitianised.

Gauge interactions are introduced in the usual gauge-invariant manner using the link variables \( U_{x,x+i} = \exp(i g \int_{x} A(x).dx) \). An alternative formulation \( U_{x,x+i} = \exp(i g l. A(x + l/2)) \), which is not gauge covariant under the usual continuous gauge transformations, is also considered. Note that the long link property of our lattice is fortuitous in emphasizing the difference between these two formulations. The resulting action is hermitian in the Euclidean sense, local, and apart from the mass term, axially-symmetric.

4 Suppression of doublers in the free-field case.

Following ref. [7], we first compute a quantity derived from the free propagator

\( f(\xi) = \text{Tr}_{\nu} \frac{1}{a^{2} N} \int_{x,x'} (1 + \gamma_{0}) G_{0}^{-1}(x,x') \delta_{l}(x_{0} - x'_{0} - \xi), \quad (10) \)

evaluating the average zero momentum real particle propagator projected along the \( x_{0} \) direction. In the continuum theory with toroidal boundary conditions, this can be simply computed from the free continuum fermion propagator on a regular torus of area \( V \),

\[ G_{0}^{-1}(x) = \frac{1}{V} \sum_{k = \{i \nu \}} \frac{\gamma_{\mu} k_{\mu} + m}{k^{2} + m^{2} e^{i k \cdot x}}, \quad (11) \]

\[ f(\xi) = V \frac{e^{-m \xi}}{1 - e^{-m \nu \xi}} \quad (12) \]

Fig. 7 indicates the results with naive fermions on a square lattice, and two random lattices, as well as the continuum, clearly identifying the doubler suppression of free fermions on the random lattice in agreement with [7]. Indeed, apart from some small distance \( O(s) \) deviations in the real particle propagation, the doublers are well suppressed at large distances where the random lattice result matches the continuum completely in both normalisation and mass. The
Figure 7: Fermion propagation, $f(\xi)$. 
small distance fluctuation of the real mode is particularly apparent on the eight scan lattice, which also shows a higher degree of doubler suppression. Thus, our lattice agrees with the conventional picture of the suppression of fermion doubling in the free-field case on a random lattice.

5 Revival of doubling

We consider the specific background Abelian gauge field,

\[ gA_\mu = \delta_{\mu,1} \frac{E \sqrt{N}}{2\pi a} \cos \left( \frac{2\pi x_0}{a \sqrt{N}} \right), \]  

with fixed physical quantities: mass \( m = 0.1 \), area \( N = 64 \), and electric field \( E = 0.05 \), for lattice spacing \( a = \{1.0, 0.5, 0.3333, 0.25\} \), in both covariant and non-covariant link variables.

The different choice of link variables amounts to \( g \rightarrow g \sin (l_0 \pi N^{-1/2}) N^{1/2} / (l_0 \pi) \), a perturbative effect of \( \sim 2\% \) on a 64 vertex \( (a = 1) \) lattice and \( \sim 0.1\% \) on 1024 vertices \( (a = 0.25) \). It should be reiterated that although apparently small, this correction is the difference between exact gauge invariance and no gauge invariance at finite lattice spacing, which has significant consequences.

The calculation of \( \Gamma_2 \) on a random lattice is complicated by its sensitivity to the structure of the lattice. This sensitivity can be considered as having two sources: variations in \( s \), and the variation due to inequivalent arrangements of links which have similar \( s \). The later variation gives some estimate of the uncertainties associated with non-zero \( a \); if it were too large, then it might be difficult to conclude anything about the number of species present. To account for both of these we consider an ensemble of lattices randomised by 1...8 scans. Figure 8 shows \( \Gamma_2 \) for both gauge covariant and non-covariant formulations. The gauge invariant formulation gives a result which is always at least four times greater than the gauge non-invariant case. Variation with \( s \), and fluctuations around a fixed \( s \) are both apparent, even with such a small sample. There also appears to be a clustering of the results along two bands in the gauge invariant case. This is clear in the 1024 vertex data presented here in figure 8, although not so evident on smaller lattices due to large fluctuations. It could indicate the discrete change in number of species which characterises lattice doubling behaviour. The trends \( \Gamma_{2,\text{invariant}} \) increasing and \( \Gamma_{2,\text{non invariant}} \) decreasing with increasing \( s \) are also seen on the smaller lattices. For the purpose of comparison, we ignore the band structure, and linearly extrapolate to \( s = a \), which provides a lower bound on the ratio between the two formulations. Identical calculations with naive and Wilson fermions on square lattices of the same sizes are also included for comparison. See Fig. 9. The naive case approaches the continuum limit quadratically with \( a \) and the Wilson approaches 1/4 of the same result linearly, as expected. The random lattice results include the number of lattice configurations used in the extrapolation next to each point. The gauge invariant
Figure 8: The two-point function for fixed physical quantities and lattice size, varying $s$. 
Figure 9: The two-point function for fixed physical quantities, varying $a$. 

[Cropped graph showing data points for naive, random (not gauge invariant), random (gauge invariant), and wilson methods, with a varying parameter $a$.]
result is clearly more like the naive fermion than the Wilson. With gauge invariance broken, the converse is clearly seen; the result is certainly more like Wilson than naive. Large fluctuations in the gauge invariant calculations are expected, since if there are spurious modes, the number of modes will be sensitive to the detailed lattice structure, which we have no control over. It is also clear that the gauge non-invariant formulation approaches the continuum result more rapidly than either Wilson or naive formulations, as expected using a random lattice approach.

6 Discussion

It is clear from our results that there are doublers on random lattices when gauge invariance is maintained at finite lattice spacing, since the extrapolated determinant is comparable to that of naive fermions.

It can also be seen that the doubling can be avoided if one gives up gauge invariance on the lattice. The hope is that it will be recovered again in the continuum limit. This is certainly true naively, but must be considered more carefully with chiral gauge interactions.

In all cases, the lattice fermion actions are invariant under the global axial transformations. When there are doublers on random lattices, the axial anomalies are canceled in the usual manner among opposite-chirality species. When there is no doubling in the gauge non-invariant formulation, the conserved lattice current being the Noether current of axial symmetry is not gauge invariant. Thus it cannot be identified with the continuum axial current, and should instead be identified with a combination of the continuum current and a gauge-noninvariant term, whose divergence gives us the axial anomalies,

\[ J^\mu_{\text{lattice}}(x) = J^\mu_{\text{continuum}}(x) - \frac{g}{2\pi} \epsilon^{\mu\nu} A_\nu(x). \] (14)

We believe that the results obtained here are also applicable to other kinds of random lattices in so far as translational invariance is broken. The calculations of [14] seem to support this claim, even though the interpretation and thus conclusions reached there are different to ours. Previous approaches remove the doublers by point splitting methods [14] or introducing naive vertex operators in the calculation [7], which are inconsistent with the Ward-Takahashi identities. Such aids explicitly break gauge invariance and thus, in the light of our calculation, it is not surprising that doubling may be removed. Indeed giving up gauge invariance at some scale on par with the lattice dimensions is also addressed indirectly in remarks about the general problem of overlocalisation in some physical theories in ref. [23]. Our calculations are not inconsistent with these ideas.

Our doubling conclusion for random lattices is not plainly disappointing but also points to some serious implications.
The lattice no-go theorem has thus been extended, and the importance of
gauge invariance emphasised in the phenomenon of lattice fermion doubling.
The failure of random lattices to accommodate chiral fermions may undermine
the point of view that at the Planck scale or higher the structure of spacetime
is that of randomness; or may indicate that there is a deep connection between
the structure and scale of space time and gauge invariance; or, taken with
other complete failures in dealing with chiral fermions, could be a hint that
our understanding of chiral gauge theories is incomplete. Correspondingly, the
quantisation of those theories is in need of further studies. One of us has been
pursuing this latter path [24].

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References


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B275[FS17], 39 (1986).

[17] J. F. Wheater, in private communication, has also some doubt about wheather random lattices can avoid the doubling if there are gauge interactions.