The Behavior of Vortex Loops in the 3-d XY Model

Arjan Hulsebos\

DAMTP, Chadwick Tower, University of Liverpool, P. O. Box 147, Liverpool, L69 3BZ, UK.

The behavior of vortex loops is studied in the 3-d XY model. It is found that the phase transition of the 3-d XY model is caused by percolating vortex loops.

The 3-d XY model is known to allow for vortices and antivortices. If we impose periodic boundary conditions, these vortices and antivortices have to form closed loops. Since the 3-d XY model undergoes a phase transition at finite coupling $\beta_c \approx 0.4542$ [1], we may suspect that those loops behave differently on either side of the phase transition.

Here, we will study the behavior of these loops by defining suitable correlation functions and loop distributions. Our conclusion is that the phase transition coincides with a percolation transition for the vortex loops.

The action for XY models is given by

$$S = \beta \sum_{x, \mu} \cos(\theta_{x+\mu} - \theta_x), \quad \mu = 1, 2, \ldots, d.$$  (1)

The quantity

$$k_{\mu\nu}(x) = \frac{1}{2\pi} \left\{ [\theta_{x+\rho} - \theta_x] + [\theta_{x+\rho+\nu} - \theta_{x+\rho}] + [\theta_{x+\rho} - \theta_{x+\nu}] + [\theta_{x+\rho+\nu} - \theta_x] \right\}, \quad \mu < \nu,$$  (2)

where $[\ldots]$, denotes the restriction to the interval $< -\pi, \pi]$, yields the vortex number on the corresponding plaquette. Turning now to 3-d, we perform a dual transformation by defining

$$j_{\mu}(\bar{x} - \bar{\mu}) = \frac{1}{2\pi} k_{\mu\nu}(x),$$  (3)

where $\bar{x}$ on the dual lattice is identified with $x + \frac{1}{2}\bar{\mu} + \frac{1}{2}\bar{\nu} + \frac{1}{2}\bar{\rho}$ on the original lattice.

The links $j_{\mu}$ have to satisfy the following relation, dropping the tilde from now on,

$$\partial_{\mu} j_{\mu}(x) = 0,$$  (4)

$$\partial_{\nu} f(x) \equiv f(x) - f(x - \hat{\mu})$$

This means that each point $x$ on the dual lattice has to be visited by an even number of non-zero links $j$. Hence the loops have to be closed on a finite lattice. To such a conservation law one usually coins the phrase “what comes in, must go out again”, but this need not be true on a lattice. Consider the following spin configuration on an elementary cube:

$$\theta_x = \theta_{x+\bar{\mu}} = 0, \quad \theta_{x+\bar{\nu}} = \theta_{x+\bar{\mu}+\bar{\nu}} = -\frac{1}{4}\pi,$$

$$\theta_{x+1+3} = \frac{3}{4}\pi, \quad \theta_{x+1+2+3} = \frac{3}{4}\pi,$$

$$\theta_{x+2+3} = \theta_{x+1+2} = \frac{1}{4}\pi.$$  (5)

See also figure 1. This configuration will lead to a vortex $(k = 1)$ on the right face, and an antivortex $(k = -1)$ on the top face. The $z$-tensor conspires in such a way as to have the same sign for both non-zero $k’s$, leading to

$$j_1(x) = 1, \quad j_2(x) = 0, \quad j_3(x) = -1,$$

$$j_1(x - 1) = j_2(x - 2) = j_3(x - 3) = 0.$$  (6)

So we see that “what comes in, doesn’t go out, and what goes out, doesn’t come in”! However,
relation (4) remains satisfied for this spin configuration. This also implies that it does not make much sense to distinguish between vortex loops and antivortex loops, since any loop may consist of links \( j = 1 \) as well as of links \( j = -1 \). We shall therefore make no distinction and call all loops ‘vortex loops’. In order to study these loops, we define two correlation functions:

\[
C_s(r) = \sum_{x,y,i,j} \frac{\delta_{x_i,i_j} \delta_{y_i,i_j}}{\sum_{x,y,i,j} \delta_{x_i,i_j} \delta_{y_i,i_j}} \sum_{x,y,i,j} 1, \quad (7)
\]

\[
C_d(r) = \sum_{x,y,i,j} \frac{\delta_{x_i,i_j} \delta_{y_i,i_j}}{\sum_{x,y,i,j} \delta_{x_i,i_j} \delta_{y_i,i_j}} \sum_{x,y,i,j} 1 - \left( \frac{1}{V} \sum_{x,y,i,j} \delta_{x_i,i_j} \right)^2. \quad (8)
\]

(7) is the probability that two points \( x \) and \( y \), a distance \( r \) away, are on the same loop \( l_i \), and was already studied in [2].

The simulations were done using a 10-hit Metropolis algorithm with site dependent step-size, combined with \( \gamma = 2 \) multigrid, as described in [3]. We made runs consisting of 10 batches of 100 measurements each, after discarding 500 measurements for thermalization.

We made runs for \( \beta = 0.00 \) up to \( \beta = 1.00 \) in steps of \( \Delta \beta = 0.05 \), as well as runs with \( \Delta \beta = 0.01 \) for \( \beta \) between 0.40 and 0.50.

The results for \( C_s(r) \) for \( 16^3 \) are given in figure 2. The errors on \( C_s \) and \( C_d \) were obtained by averaging over the 10 batches.

As can be seen, there is a dramatic change in \( C_s \) around the phase transition.

To determine whether the different vortex loops interact with one another, we have plotted \( C_s \) and \( C_d \) in figures 3a and 3b. We see that \( C_d \) falls off exponentially for both \( \beta \)-values, and for \( \beta = 0.60 \) differs from \( C_s \) only by a constant.

![Figure 2](image1.png)

Figure 2. \( C_s(r) \) for \( \beta = 0.40 \) (upper), \( \beta = 0.44, 0.46, 0.48 \) and \( \beta = 0.50 \) (lower) on \( 16^3 \).

![Figure 3](image2.png)

Figure 3. \( C_s \) and \( C_d \) on \( 16^3 \) for \( \beta = 0.35 \) (top) and for \( \beta = 0.60 \) (bottom).

In table 1, we present the results from fitting \( C_s \) to

\[
C_s = d \frac{\exp(-ar)}{r^b} + c
\]

for \( \beta \)-values around the critical coupling \( \beta_c \).

![Figure 4](image3.png)

Figure 4. Histogram of the length of the vortex loops on a \( 16^3 \) lattice for \( \beta = 0.25 \) and \( \beta = 0.55 \). Here, we also see different behavior on either side of the phase transition. For \( \beta = 0.25 \), and indeed for all \( \beta \leq \beta_c \), we see that there are very long loops present in the system. For \( \beta = 0.55 \), and more general for...
\( \beta \gtrsim \beta_c \), we see that these very long loops have vanished, and the longest loops measure only a few times the lattice extension. This is shown in figure 5, in which the average length of the longest loop is plotted against \( \beta \), for all four lattice sizes used.

These results can be explained in the following way. For \( \beta < \beta_c \), the vortex loops may be very long, and they percolate through the lattice. In that region, \( C_s \) will tend to a constant for large separations \( r \). This constant is related to the density of percolating loops. As \( \beta \) approaches \( \beta_c \), this density decreases, and becomes very small at the phase transition. As the percolating loops disorder the system, the only correlation length for these loops is the lattice size, which causes the small exponential factor \( a \) in table 1. The constants \( a, b \) and \( d \) appear to be independent of \( \beta \).

For \( \beta > \beta_c \), the relevant degrees of freedom are the spin waves, while only short vortex loops are allowed. As these loops have a predominantly planar structure, we see that \( a \) increases with \( \beta \), while \( b \) tends to zero, and it vanishes for \( \beta \gg \beta_c \).

There is no sign for interactions between vortex loops as \( C_d \) falls off exponentially, and differs by a constant from \( C_s \) for \( \beta < \beta_c \). This is hardly surprising, as loops may consist of links \( j = 1 \) as well as links \( j = -1 \).

Our results indicate that the phase transition in the 3-d XY model is caused by percolating, non-interacting vortex loops.

**ACKNOWLEDGEMENTS**

We would like to thank Chris Michael, Jan Smit and Peer Uebelhoer for useful discussions. This work was supported by EC contract SCI *CT91-0642.

**REFERENCES**