We study the presence of topological defects in models of scalar fields with a Lagrangian density of the form $L = \frac{1}{2} \partial \phi^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{4} \lambda \phi^4$, where $\phi$ is a real scalar field, $m$ is a constant mass parameter, and $\lambda$ is a coupling constant. The model describes a system of two real scalar fields $(\phi_1, \phi_2)$ with a quartic interaction term.

The action for the system is given by $S = \int d^4x \sqrt{-g} \left( \frac{1}{2} \partial \phi^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{4} \lambda \phi^4 \right)$, where $g$ is the determinant of the metric tensor.

The field equations can be derived from the action by variation with respect to the fields $\phi_1$ and $\phi_2$.

In the present work, we consider a specific class of models described by two real scalar fields and follow the methodology of the investigation of such systems.

The field equations are

$$\partial^2 \phi_1 - m^2 \phi_1 + \frac{1}{2} \left( \partial^2 \phi_2 - m^2 \phi_2 \right) \frac{\lambda \phi_1^2 \phi_2^2}{(\phi_1^2 + \phi_2^2)^2} = \phi_1^2 \phi_2^2,$$

and

$$\partial^2 \phi_2 - m^2 \phi_2 + \frac{1}{2} \left( \partial^2 \phi_1 - m^2 \phi_1 \right) \frac{\lambda \phi_1^2 \phi_2^2}{(\phi_1^2 + \phi_2^2)^2} = \phi_1^2 \phi_2^2.$$
These solutions are BPS states, which solve the first order differential equations and the equations of motion of the model.

To search for defect solutions we first investigate the minima of the potential, which are critical points of \( W \) obtained by requiring that \( W_\phi = 0 \) and \( W_\chi = 0 \). We suppose the model has the discrete set of \( n \) minima, \( (\phi_1, \chi_1), (\phi_2, \chi_2), \ldots, (\phi_n, \chi_n) \). In principle, each pair of minima forms a topological sector, so we use the subscripts \( ij \) to identify the topological sectors. Some of the topological sectors may have solutions that obey the first order equations (3). In this case they are named BPS sectors, and we can identify the BPS sectors with the energy of the topological sector, with \( E_{ij} = [\Delta W_{ij}] \), for \( \Delta W_{ij} = W(\phi, \chi_i) - W(\phi, \chi_j) \).

We eliminate the spatial coordinate in the first order equations (3) to write \( W_\phi \partial_\phi - W_\chi \partial_\chi = 0 \). This equation is exactly solvable for \( W \) harmonic, for \( W_\phi + W_\chi = 0 \). In this case the solution is given by the orbit \( F(\phi, \chi) = \), with \( \partial F/\partial \phi = W_\phi \) and \( \partial F/\partial \chi = -W_\chi \). In Ref. [21] one has obtained that when \( W \) is harmonic all the topological solutions are of the BPS type, that is, are solutions of the first order Bogomol’nyi equations. We use this result together with the above result to introduce a new result, which ensures that the equations of motion for topological solutions are exactly solved when the potential is of the form (1), with \( W \) harmonic.

If \( W \) is not harmonic, we can yet search for some integrating factor, \( I = I(\phi, \chi) \), such that the solution is now given by \( F(\phi, \chi) = 0 \), with \( \partial F/\partial \phi = I(\phi, \chi) W_\phi \) and \( \partial F/\partial \chi = -I(\phi, \chi) W_\chi \). The problem is that it is not always easy to find an integrating factor for the first order equation. This difficulty has led us to revisit the trial orbit method, to see if it can be of some help in the process of finding solutions to the first order equations. As we will show, the trial orbit method is of good use for searching for topological defects in the BPS sectors that appear from the first order Bogomol’nyi equations. Better than that, the trial orbit method is more efficient than it appears in the original work [3], since how it relies on searching for solutions of first order equations, circumventing the weakness that appears in Ref. [3], in which one deals with the equations of motion, which are second order differential equations.

The trial orbit method may be seen as a procedure based on the three steps:

**Step 1.** We select the BPS sector. We do this by supposing the pair of minima \( (\phi_j, \chi_j) \) and \( (\phi_j, \chi_j) \) is such that \( W(\phi_j, \chi_j) \neq W(\phi_j, \chi_j) \), implying that one is dealing with a BPS sector. This means that the first order equations has topological solutions connecting the points \( (\phi_j, \chi_j) \) and \( (\phi_j, \chi_j) \).

**Step 2.** We choose an orbit. We do that by writing the equation of the orbit, say \( F_{ij}(\phi, \chi) = 0 \), and checking compatibility between the chosen orbit and the minima that specify the BPS sector, that is, validating the statements \( F_{ij}(\phi_j, \chi_j) = 0 \) and \( F_{ij}(\phi_j, \chi_j) = 0 \).

**Step 3.** We test compatibility between the orbit and the first order equations. We do that by differentiating the trial orbit. We get \( (\partial F_{ij}/\partial \phi)(d\phi/dx) + (\partial F_{ij}/\partial \chi)(d\chi/dx) = 0 \). We use the first order equations (3) to obtain

\[
\frac{\partial F_{ij}}{\partial \phi} W_\phi + \frac{\partial F_{ij}}{\partial \chi} W_\chi = 0
\]  

This new statement is similar to the trial orbit itself. Thus, we have to check compatible between the chosen orbit and the new statement (4). We say that \( F_{ij}(\phi, \chi) = 0 \) is a good orbit when every pair \( (\phi, \chi) \) that solves the orbit also solves the new statement (4). We then use the good orbit to solve the first order equations.

We exemplify the procedure with some specific investigations. For simplicity we deal with natural units, and with dimensionless fields and coordinates. The first example is given by

\[
W(\phi, \chi) = \phi - \frac{1}{3} \theta^3 - r \phi^2
\]  

where the parameter \( r \) is real. This is the model first investigated in Ref. [18]. We use Eq. (5) to write the first order equations

\[
\frac{d\phi}{dx} = 1 - \phi^2 - r \phi^2
\]  

\[
\frac{d\chi}{dx} = -2r \phi \chi
\]  

We first consider \( r \) real and negative. The potential has minima at \( \epsilon_{1,2} = (\pm 1, 0) \), which define a topological sector of the BPS type. An explicit solution describes a straight line orbit connecting the two minima; it corresponds to \( \phi(x) = \tanh(x) \) and \( \chi = 0 \). For \( r = -1 \) we see that \( W \) is harmonic, and the model can be solved exactly. There is no orbit connecting the minima \( (\pm 1, 0) \), but the single straight line segment with \( \chi = 0 \). This result is valid not only for the first order equations, but also for the equations of motion, that is, the system has no topological solutions connecting the minima \( (\pm 1, 0) \) for \( r = -1 \), unless the kinks with \( \phi(x) = \pm \tanh(x) \) and \( \chi = 0 \) .

We now consider \( r \) real and positive. In this case one can find an integrating factor and exactly solve the model [22]. However, since we want to show how the trial orbit method works we proceed using the methodology explained above. The minima of the potential are now at \( \epsilon_{1,2} = (\pm 1, 0) \) and \( \epsilon_{3,4} = (0, \pm \sqrt{1/7} r) \). The model has one BPS sector with energy \( E_{12} = 4/3 \), and four BPS sectors with energy \( E_{13} = E_{14} = E_{23} = E_{24} = 2/3 \). To find non trivial solutions we use the above procedure.

We follow the first step to select the BPS sector defined by the minima \( (\pm 1, 0) \), with energy \( E_{12} = 4/3 \). The second step leads us to choose the orbit. We try \( 1 + a \phi^2 + b \chi^2 = 0 \). The ending points \( (\pm 1, 0) \) demand that \( 1 + a = 0 \), so we can write \( 1 - \phi^2 + b \chi^2 = 0 \). We follow the third step and we differentiate this orbit to get \( -\phi(d\phi/dx) + b \chi(d\chi/dx) = 0 \). We use the first order
equations to get $1 - \phi^2 + r(2b - 1)\chi^2 = 0$. This new statement is fully compatible with the orbit for $b = r(2b - 1)$, that is, for $b = r/(2r - 1)$. This finally gives the good orbit

$$\frac{\phi^2 + r^2}{1 - 2r^2} = 1$$

which connects the minima $(\pm 1, 0)$ for $0 < r < 1/2$.

We now use the orbit (8) to find explicit solutions. It allows rewriting Eq. (3) in the form $d\phi/dx = 2r(1 - \phi^2)$, which is solved by $\phi(x) = \tanh(2rx)$. The other field is then given by

$$\chi(x) = \pm \sqrt{\frac{(1 - 2r^2)/r}{\cosh(2rx)}}$$

We notice that the limit $r \to 1/2$ leads to $\chi = 0$ and $\phi(x) = \tanh(x)$, which also solve the first order equations (3). This pair corresponds to a straight line equation connecting the minima $(\pm 1, 0)$. It is different from the other pair, which connects the same minima by an elliptic segment. The presence of the elliptic orbit manifests the possibility of domain walls having internal structure. This possibility emerges after examining the masses of the two fields, which are: $m_\phi^2 = 4$, $m_\chi^2 = 4r^2$ at the minima $(\pm 1, 0)$, and $m_\phi^2 = 4r$, $m_\chi^2 = 4r$ at $(0, \pm \sqrt{1/r})$. Also, the energies of the topological defects are given by: for $\chi = 0$ we have $E_{12} = 4/3$ and for $\phi = 0$ we have $E_{34} = 4/3\sqrt{r}$. We see that for $r \in (0, 1)$ energy considerations favor the BPS defect to be the host defect, and for $r > 1$ the non BPS defect becomes the host defect. In both cases the elementary $\phi$ mesons prefer to live inside the host defect, while the $\chi$ mesons prefer to live outside.

In the other BPS sectors we can consider orbits like $r\chi^2 = 1 \pm \phi$, which requires $r = 1/4$. For $r\chi^2 = 1 - \phi$ we have $2r\chi(dx/d\chi) = -d\phi/dx$. In this case we use Eqs. (3) to obtain $d\phi/dx = \phi - \phi^2$, which is solved by $\phi(x) = \big(1/2\big)[1 + \tanh(x/2)]$. The other field is given by $\chi(x) = \pm \sqrt{2[1 - \tanh(x/2)]}$. For $r\chi^2 = 1 + \phi$ the investigation is similar. We get $\phi(x) = \big(-1/2\big)[1 - \tanh(x/2)]$ and $\chi(x) = \pm \sqrt{2[1 + \tanh(x/2)]}$.

In order to further illustrate our procedure we consider other models, and we explore the most interesting BPS sectors they have. The first model is defined by

$$W(\phi, \chi) = -1/3(1 + s)\phi^3 + 1/5 s \phi^5 - r \phi \chi^2$$

It can be seen as an expansion of the former model, which includes the fifth order power on $\phi$. We suppose the parameters $r$ and $s$ are real. The first order equations are

$$\frac{d\phi}{dx} = 1 - (1 + s)\phi^2 + s \phi^4 - r \phi \chi^2$$

$$\frac{d\chi}{dx} = -2r \phi \chi$$

The potential $V(\phi, \chi)$ is given by

$$V(\phi, \chi) = \frac{1}{2}(\phi^2 - 1)^2(s\phi^2 - 1)^2 + \frac{1}{2}r^2 \chi^4 - r(1 - \phi^2)(1 - s\phi^2)\chi^2 + +2r^2 \phi^2 \chi^2$$

It can be projected in the $\chi = 0$ direction to give

$$V(\phi, 0) = \frac{1}{2}(\phi^2 - 1)^2(s\phi^2 - 1)^2$$

It is of the eight order power in $\phi$ and admits symmetry breaking for $s$ positive and negative. It can also be projected in the $\chi = 0$ direction; in this case it gives

$$V(0, \chi) = \frac{1}{2}(r\chi^2 - 1)^2$$

which is of the fourth order power in $\chi$ and admits symmetry breaking for $r$ positive. The model contains the pair of minima $v_{1,2} = (\pm 1, 0)$ and also $v_{3,4} = (\pm \sqrt{1/s}, 0)$ if $s$ is positive, and also $v_{5,6} = (0, \pm \sqrt{1/r})$ if $r$ is positive.

We use the potential to calculate the mass matrix, and to write the masses of the two fields at the diverse minima of the potential. They are: $m_\phi^2 = 4(1 - s^2)$, $m_\chi^2 = 4r^2$ at the minima $(\pm 1, 0)$, $m_\phi^2 = 4(1 - s^2)/s$, $m_\chi^2 = 4r^2/s$ at $(\pm \sqrt{1/s}, 0)$, and $m_\phi^2 = 4r$, $m_\chi^2 = 4r$ at $(0, \pm \sqrt{1/r})$. This model admits the presence of domain walls having internal structure. We examine this possibility in the case $r > 0$ and $s < 0$, where we have the same four minima of the former model. Here, for $\chi = 0$, in the BPS sector defined by the minima $v_{1,2} = (\pm 1, 0)$ the energy is given by $(4/3)(1 + |s|/5)$; for $\phi = 0$, in the non BPS sector defined by $v_{3,4} = (0, \pm \sqrt{1/r})$ the energy is $4/3\sqrt{r}$. Thus, the ratio $(5 + |s|)/5\sqrt{r}$ shows that one favors the presence of the BPS defect with $\chi = 0$ as the host defect for $\sqrt{r} > 1 + |s|/5$, and for $\sqrt{r} < 1 + |s|/5$ the host defect is the non BPS defect with $\phi = 0$. Thus, for $r \in (0, 1)$ and for $s < 0$, the host defect is necessarily the non BPS defect that connects the minima $(0, \pm \sqrt{1/r})$. And in this case the $\phi$ meson prefers to live outside the host defect, while the $\chi$ meson prefers to live inside. This situation is similar to the one that appears in the former model. However, for $r > 1$ and for $s < 0$, if one considers $\sqrt{r} > 1 + |s|/5$ the host defect becomes the BPS defect that connects the minima $(\pm 1, 0)$. In this case, the $\chi$ mesons prefer to live inside the host defect. But for the $\phi$ mesons we have two possibilities: they prefer to live outside the host defect if $\sqrt{r} > 1 + |s|$, and for $1 + |s|/5 < \sqrt{r} < 1 + |s|$ they also prefer to live inside the host defect. The present model is more general than the former one, giving rise to the case where the host defect entraps both the $\phi$ and $\chi$ mesons.

We notice that $W$ in Eq. (10) is not harmonic, so the model is not exactly solvable. Moreover, since we have been unable to find an integrating factor for Eqs. (11) and (12), we could not be sure that the BPS states of the model can be solved exactly. For this reason, let us now use the trial orbit method to explore the presence of BPS states in this model. Because this new model contains an extra term, of the fourth order power in $\phi$, we try the orbit $a + \beta \phi^3 + \gamma \phi^2 + d\phi^2 = 0$. We follow the former steps to get the good orbit $r\chi^2 = s(1 - \phi^2)^2$.
for solutions connecting the minima $(\pm 1, 0)$, with energy $(4/3)(1 - s/5)$. We use this orbit to rewrite Eq. (11) as $\phi(x) = \tanh[(1 - s)x]$. We use this orbit to get for the other field the solutions

$$\chi(x) = \pm \frac{\sqrt{s/r}}{\cosh^2[(1 - s)x]}$$

which requires that $s/r > 0$. This pair of solutions exists for $r$ and $s$ positive or negative, so it appears in the sector connecting the minima $(\pm 1, 0)$ despite the presence of the other minima. It goes to the solution which describes a straight line orbit in the former model in the limit $s \to 0$.

Let us now suppose that $s$ is positive, so that there are minima at $v_5, v_6 = (\pm \sqrt{1/s}, 0)$. In this case we can find BPS solutions connecting those minima, with energy $(4/3)\sqrt{1 - 1/5s}$. We follow the same procedure to find the orbit $r \chi^2 = (1 - s \phi^2)^2 / s$. This orbit allows obtaining the solutions

$$\phi(x) = \sqrt{1/s} \tanh[\sqrt{1/s} (s - 1)x]$$

$$\chi(x) = \pm \frac{\sqrt{1/s}}{\cosh^2[\sqrt{1/s} (s - 1)x]}$$

which requires $r > 0$, so that there are minima also at $(0, \pm \sqrt{1/s}, 0)$. This last solution appears if both $r$ and $s$ are positive, so that the model must contains all the six minima $(\pm 1, 0)$, $(\pm \sqrt{1/s}, 0)$, and $(0, \pm \sqrt{1/s}, 0)$.

We further illustrate the trial orbit method investigating another model, defined by

$$W(\phi, \chi) = -\frac{1}{3} \phi^3 - \frac{r}{2} \frac{\chi^2}{\phi} - \frac{r}{s} \phi + \frac{s}{3} \phi^3$$

where $r$ and $s$ are real and positive, $s \in (0, 1)$. The presence of interactions leading to negative power in the fields is not unusual. They appear in several different contexts, for instance in vortices in planar Abelian systems involving generalized permeabilities [23, 24], in effective Yang-Mills theories coupled to scalar field, with appropriate color dielectric function to mimic quark and antiquark interactions [25, 26, 27], and in models used to map biological systems, involving an activator and its antagonist, the inhibitor [28].

In this model the first order equations are

$$\frac{d\phi}{dx} = 1 - \phi^2 + \frac{r}{2} \frac{\chi^2}{\phi^2} + \frac{s}{2} \frac{\chi^2}{\phi^2} + \frac{s}{2} \frac{\chi^2}{\phi^2}$$

$$\frac{d\chi}{dx} = -r \frac{\chi}{\phi}$$

We investigate the BPS sector defined by the minima $(\pm 1, 0)$. Its energy is given by $(4/3)(1 - s)$. In this model, the presence of interactions leading to negative power in the $\phi$ field makes the search for defects harder than before, so this example offers another good illustration of the trial orbit method.

We follow the former steps and try the orbit $a + b \phi^2 + c \phi^4 + d \phi^2 \chi^2 = 0$. It gives the good orbit $\phi^2 \chi^2 = 2s(1 - \phi^2) = 0$ for $r = 1$. We use this orbit in the first order equation (20) to get $d\phi/dx = 1 - \phi^2$, which is solved by $\phi(x) = \tanh(x)$. This means that the other field $\chi$ is

$$\chi(x) = \pm \frac{\sqrt{s/r}}{\sinh(x)}$$

We see that $\chi(x)$ diverges for $x \to 0$, splitting the orbit into two segments. Because of this singularity we cannot calculate the corresponding energy, and this indicates that this pair of solutions is unstable.

The last example is obtained as a two-field generalization of a model introduced in Ref. [29], which describes a vacuumless potential. The potential has a maximum at $\phi = 0$, and monotonically decreases to zero at $\phi \to \infty$. Potentials with such asymptotic behavior can arise in supersymmetric gauge theories [30], and more recently they have been used in the form of quintessence models [31]. They support defects which present quite different properties and evolution, as compared to the usual defects [29]. To generalize the model introduced in [29] to the case of two real scalar fields we follow Ref. [32]. We introduce the superpotential

$$W(\phi, \chi) = \text{arctan}[\sinh(\phi)] + W(\phi, \chi)$$

where

$$\overline{W}(\phi, \chi) = \frac{\sinh(s\chi) - \sinh(\phi)}{\cosh(\phi) \cosh(s\chi)}$$

with $r$ and $s$ as real parameters. The first order equations are

$$\frac{d\phi}{dx} = \frac{1}{\cosh(\phi)} - r \left[ \frac{1 + \sinh(\phi) \sinh(s\chi)}{\cosh(\phi) \cosh(s\chi)} \right]$$

$$\frac{d\chi}{dx} = r s \left[ \frac{1 + \sinh(\phi) \sinh(s\chi)}{\cosh(\phi) \cosh(s\chi)} \right]$$

These equations are much harder to solve, but we can use the trial orbit method to see that the orbit $\phi = s\chi$ is a good orbit for $r = 1/(1 + s^2)$. In this case we get the solutions

$$\phi(x) = s\chi(x) = \text{arcsinh} \left[ \frac{s^2 x}{1 + s^2 x^2} \right]$$

This pair of solutions is of direct interest to high energy physics, as we can see for instance in [29, 33], and in [34] in the case of quintessence with coupled scalar fields.

The above examples illustrate how efficiently the trial orbit method can be used to obtain explicit BPS solutions in specific models, and this will certainly help exploring other systems, involving the yet unknown possibilities of finding BPS states in models described by three or more real scalar fields, with discrete symmetry.

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