\[(g) \quad \mathfrak{g} = \mathfrak{g}(U)\]

\[(h) \quad \mathcal{B} \mathfrak{g} \mathfrak{d} + \mathfrak{g} \mathfrak{p}(\mathfrak{g}) U V + \mathfrak{g} \mathfrak{p}(\mathfrak{g}) U N = -\varepsilon^{-1} \varepsilon \rho \]

\[(i) \quad \varepsilon \mathcal{B} \mathfrak{g} \mathfrak{d} + \varepsilon \mathfrak{p}(\mathfrak{g}) U V + \varepsilon \mathfrak{p}(\mathfrak{g}) U N = -\varepsilon^{-1} \varepsilon \rho \]

\[(j) \quad \varepsilon \mathfrak{g} \mathfrak{d} + \varepsilon \mathfrak{p}(\mathfrak{g}) U V + \varepsilon \mathfrak{p}(\mathfrak{g}) U N = -\varepsilon^{-1} \varepsilon \rho \]

The above equations are derived from the field equations and the above equations to establish the consistency of the solution. The solution to the equations is given by

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This procedure turns out to be particularly convenient for the problem at hand because it converts Einstein equations (3) into three sets of second order ordinary differential equations in the variable $\zeta$ of the form

$$\frac{d^2 f_n}{d \zeta^2} - \sigma^2 f_n = F_k \zeta^n,$$

(9)

where $f_n$ is any of the functions $a_n(\zeta)$, $a_0(\zeta)$ and $a_1(\zeta)$ ($n \geq 1$), and $F_k \zeta^n$ a function of the lower order terms $f_k \zeta^{n-1}$ and their first and second derivatives. The relevant boundary conditions for Eq. (9) are given by the requirement (8) for $f_n(0)$ and the junction conditions [12] which imply $d f_n / d \zeta(0) = -\sigma f_n(0)$. We shall not display the (cumbersome) $F_k \zeta^n$ for simplicity, however they turn out to be such that the Cauchy problem thus defined admits analytic solutions. The entire procedure can then be executed automatically with the aid of an algebraic manipulator to determine the functions $f_n$ recursively, from the lowest order up [18], with the only limitation of the power and memory of the available computer.

Let us now comment on a few more points regarding the above procedure. First of all, we wish to stress that there is a large freedom in the choice of the metric on the brane. In particular, the coefficients $n_k(0)$ and $a_k(0)$ can be chosen at will, except for the algebraic constraints following from Eq. (5). Since such coefficients are related to the shape of the source, this input represents the physical content of the model. A second, related point is the convergence of the series expansion (7). It is in general difficult to pinpoint one parameter (among the many possible coefficients of the multipole-expansion) whose “smallness” guarantees that orders higher than $n$ may be negligible. Because of this, we should consider our results as reliable for those values of $r$ and $z$ such that

$$\left| \frac{f_n + 1}{r^n + 1} \right| \ll \sum_{k=0}^n \frac{f_k(z)}{r^k},$$

(10)

for given values of the parameters $n_k(0)$ and $a_k(0)$. In general, for a given $z$, such a condition will be satisfied for sufficiently large $r$. As examples of brane metrics, we have considered the solutions given in Refs. [4, 5, 6] which can be expressed in terms of the ADM mass $M$ and the post-Newtonian parameter $(PNP) \eta [14]$ measured on the brane. The case with $\eta = 0$ (exact Schwarzschild on the brane) is the well known black string [15] which extends all along the extra dimension. The BS is known to suffer of serious stability problems [15, 16], e.g. the Kretschmann scalar,

$$R^2 \equiv (5)^2 R_{\mu \nu \lambda \delta} (5) R^{\mu \nu \lambda \delta} = \frac{5}{2} \sigma^4 + 48 M^2 \frac{\epsilon^2 \sigma z}{r^6},$$

(11)

diverges on the AdS horizon ($z \rightarrow \infty$). One is therefore led to conclude that black holes on the brane must depart from Schwarzschild and have $\eta \neq 0$. As was suggested in [6], the interesting cases are those with $\eta < 0$, since $\eta > 0$ implies some sort of anti-gravity effects (see later for further comments). Short distance tests of Newtonian gravity yield the bound $\sigma^{-1} < 1 \text{mm} [1]$ and $|\eta| < 10^{-3}$ from solar system tests [14]. Since we want to study astrophysical sources of solar mass size, in the following we shall often refer to the typical values

$$M = 10^7 \sigma^{-1} \sim 1 \text{km}, \quad \eta = -10^{-4}.$$ (12)

In this range ($M \sigma \gg 1$ and $|\eta| \ll 1$) one finds a qualitatively identical behavior for all brane metrics in Refs. [4, 5, 6], so we shall just give the results for case I of Ref. [6] (see also [5]), that is

$$N_B = 1 - \frac{2 M}{r},$$

$$A_B = \frac{1 - \frac{3}{2} M}{\frac{1}{2} M} \left[ \frac{1 - \frac{3}{2} M}{\frac{1}{2} M} + 4 \eta \right],$$

(13)

where $r \equiv r_B \equiv 2M$ is the event horizon and the remaining (non-vanishing) PNP's are $\beta = \gamma = 1 + \frac{1}{2} \eta$. We applied the above procedure to the brane metric (13) and were able to solve the corresponding Eqs. (9) up to $n = 19$. For brevity, we just display a few terms:

$$N_5 = e^{-\sigma z} \left[ 1 - \frac{2 M}{r} - \frac{\eta M^2}{3 \sigma^2} (1 + \sigma^2 z^2) \left( 1 + \frac{M}{r} \right) \right],$$

$$A_3 = e^{-\sigma z} \left[ 1 + \frac{2 M}{3 r} (3 + \eta) + \frac{4 M^4}{r^2} \left( 1 + \frac{M}{r} \right) \right] - \frac{2 \eta M}{3 \sigma^2} \left( 1 - \sigma^2 z^2 \right),$$

(14)

$$\frac{R^2}{r^2} = e^{-\sigma z} \left[ 1 + \frac{\eta M}{3 \sigma^2} (1 - \sigma^2 z^2) \left( 1 + \frac{2 M}{r} + \frac{7 M^2}{2 r^2} \right) \right].$$

It is important to note the appearance of positive exponentials in the metric functions. Such terms (also show up at higher orders) are non-perturbative in $z$, which makes the expansion in $1/r$ preferable (or at least complementary) to the expansion for small $z$.

For $\eta < 0$ one generally finds that, for every $n \geq 3$ and $r > 0$, there exists a corresponding value of $z$ such that

$$R^2(r, z_{\text{axis}}) = 0.$$ (15)

Since $R^2$ is determined up to corrections of order $n+1$, which in general do not vanish for $z \equiv z_{\text{axis}}(r)$, one cannot consider this as a mathematically rigorous proof [the condition (10) obviously fails for $z = z_{\text{axis}}^n(r)$]. However, we found that in the physically interesting range of the
parameter space \((M, \eta)\) the \(1/r\) expansion yields rather stable values of \(z_{\text{axis}}(r)\) in a wide span of \(n\). The stability improves for larger values of \(r\) [as could be inferred from (10)] and becomes very satisfactory for \(r \gtrsim r_h\) (see Fig. 1). From Eq. (14) and \(M \sigma \gg 1, r \gg r_h\), one finds

\[
z_{\text{axis}}^{\text{axis}} \sim \frac{1}{2} \ln \left( \frac{3 \sigma^2 r^3}{-\eta M} \right),
\]

which numerically agrees fairly well with \(z_{\text{axis}}^{\text{BS}}(r)\).

It is interesting to compare our solutions for \(\eta < 0\) with the BS [15]. In particular, one would like to see the shape of the horizon in the bulk, knowing that for the BS it does not close but extends all the way to the AdS horizon. First we note that, if the horizon closes in the bulk, then it must cross the axis of cylindrical symmetry at a point (the “tip”) of finite coordinates \((x_{\text{tip}}, z_{\text{tip}})\) where \(N = R^2 = 0\). Of course, we just have such equations explicitly at order \(n\),

\[
N_n(x_{\text{tip}}^n, z_{\text{tip}}^n) = \frac{R^2 - x_{\text{tip}}^n}{z_{\text{tip}}^n} = 0.
\]

For large values of \(n\), one can solve Eqs. (16) numerically and find the “tip”. More in detail, for \(n = 19\) one has

\[
N_{19}(r_h + \varepsilon, z) > 0 \quad \text{and} \quad N_{19}(r_h - \varepsilon, z) < 0,
\]

for \(\varepsilon \gtrsim (3/100) r_h\) and \(0 < z \lesssim 0.89 z_{\text{axis}}(r)\). Thus, to a very good approximation, \(N_{19}(r_h, z) \approx 0\) for \(0 \leq z < z_{\text{axis}}(r_h)\) when \(M \sigma \gg 1\) and \(|\eta| \ll 1\) and negative. A good parameterization for the horizon is thus given by \(r \approx r_h\) and \(0 \leq z \lesssim z_{\text{axis}}(r) \approx 0.99 z_{\text{axis}}^{\text{BS}}(r)\) (see Fig. 2). For the typical parameters (12) we have \(z_{\text{tip}}^{\text{tip}} \approx 0.97 r_h\) and \(z_{\text{tip}}^{\text{tip}} \approx 20.8 \sigma^{-1}\) which is very close to \(z_{\text{axis}}^{\text{axis}}(r_h) \approx 20.9 \sigma^{-1}\). This all strongly suggests that the horizon does close in the bulk, as previously obtained by numerical analysis [9, 10] (for a comparison with the BS see Fig. 3).

One can now get an estimate of how flattened the horizon is towards the brane by comparing the proper length of a circle on the horizon which lies entirely on the brane, \(c_{\|} = 2 \pi r_h \approx 1.3 \cdot 10^6 \sigma^{-1}\), with the length of an analogous curve perpendicular to the brane, \(c_{\perp} \approx 4.6 \sigma^{-1} \approx 84 \sigma^{-1}\). Since their ratio is huge, one can in fact speak of a “pancake” horizon as was suggested, e.g., in Ref. [7].

It is interesting to note that for \(n = 5\), one can still solve Eqs. (16) analytically and finds

\[
z_{\text{tip}}^n = \frac{1}{\sigma} \ln \left[ 1 + \frac{\sigma r_h}{\sqrt{-\eta}} \right] \sim \frac{1}{\sigma} \ln \left( \frac{M \sigma}{\sqrt{-\eta}} \right),
\]
where \( a \approx 0.5 \) and we used \( M \sigma \gg 1 \) in the final expression. This yields \( \tilde{z}_5^{(p)} \approx 20.7 \sigma^{-1} \) for the parameters (12), in excellent agreement with the numerical value obtained at order 19. In light of this stability, one can therefore approximate the dependence of the exact \( \tilde{z}_5^{(p)} \) on the black hole ADM mass \( M \) from Eq. (18) and obtains that the area of the (bulk) horizon is approximately equal to the four-dimensional (brane) expression [19],

\[
(5) A = 4 \pi \int_0^{\tilde{z}_5^{(p)}} \tilde{R}^2(r, z) \, dz \approx 2 \pi (2 M)^3.
\]

(19)

where we again used \( M \sigma \gg 1 \). Eqs. (18) and (19) again supports the idea of a “pancake” shape for the horizon (see also [7] for the logarithmic dependence of \( \tilde{z}_5^{(p)} \) on \( M \)).

Drawing upon the above picture, in particular the crossing of lines of constant \( r \) with the axis of cylindrical symmetry at finite \( z \), one can infer that the spacetimes we obtain do not suffer of one of the instabilities of the BS, namely the diverging Kretschmann scalar [15]. In fact, \( K^2 \) is still an increasing function of \( z \) along lines of constant \( r \), but one has

\[
K^2 - \frac{5}{2} \sigma^4 = \tilde{a} \frac{4 \pi z}{L} + O \left( \frac{1}{r} \right) \leq \tilde{b} \sigma^4 + O \left( \frac{1}{r} \right),
\]

where we used Eq. (15) to maximize \( K^2 \) uniformly in \( z \). The coefficients \( \tilde{a} \) and \( \tilde{b} \) depend on the parameters of the multipole expansion, correctly vanish in pure RS and \( \tilde{b} \geq 48 M^2 \) for \( \eta = 0 \) [the BS, cfr. Eq. (11)]. The remaining problem of stability under (linear) perturbations [16] is a difficult one to tackle and we do not attempt at it here.

As we mentioned previously, the cases with \( \eta > 0 \) show a very different qualitative behavior. One in fact finds that \( R_{\mu\nu}(r, z) \) is generically a (monotonically) increasing function of \( z \) for all (sufficiently large) values of \( r \), as one would indeed expect on a stability test for \( \eta = 0 \) [the BS, cfr. Eq. (11)]. However, for any \( r_1, r_2 > 0 \) there now exists \( \tilde{z}_5^{(p)}(r_1, r_2) \) such that \( R_{\mu\nu}(r_1, \tilde{z}_5^{(p)}(r_1, r_2), r_2) \), \( \tilde{z}_5^{(p)}(r_1, r_2) \), i.e. \( \tilde{z}_5^{(p)} \)-like geodesics of constant \( r \) display caustics and the Gaussian coordinates \( (r, z) \) do not cover the whole bulk [13].

We have explained in some detail how to extend the bulk into a bulk a given asymptotically flat static spherically symmetric metric on the line which is based on the multipole \((1/r)\) expansion. The application of our method to candidate metrics [4, 5, 6] for astrophysical sources led us to conclude that black hole horizon closes in the bulk and indeed has the shape of a “pancake”. Our solutions depend on three parameters \((\sigma, M, \eta)\), although one could argue that only two of them are independent.

For instance, in order to recover the four-dimensional Schwarzschild metric when the extra dimension shrinks to zero size \((\sigma \to \infty)\) in the brane equations [11], one can guess \( A_0 \sim 1/M \sigma \). Reducing the number of dimensions is however a singular limit of the five-dimensional metric, so it is critical to obtain precise relations among the parameters by this procedure. It is also difficult to extract sensible results for tiny black holes \((M \sigma \ll 1)\), e.g. from Eq. (18) or now gets \( \tilde{z}_5^{(p)} \sim M \) and Eq. (19) yields the relation for five-dimensional Schwarzschild black holes \( (5) A = M^3 \) [17]. Our expansion however suggests that the horizon departs significantly from the line \( r = r_h \) when \( M \sigma \ll 1 \) and the above estimate is very rough. The dependence of the horizon area on the ADM mass is crucial to study the Hawking evaporation and we hope to return to it in the future.

We thank A. Fabbri for contributing to the early part of the work. R. C. thanks C. Germani and R. Maartens for comments and suggestions.

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[18] For \( n = 0 \) one has a system of three coupled second order ordinary differential equations for the \( f_0 \)'s. The corresponding Cauchy problem is solved by the usual warp factor, \( f_0 = \exp(-\sigma z) \), which is unique as follows from the usual theorems of uniqueness. We further note that for \( r \to \infty \) the bulk solutions reproduce the RS spacetime [1] as one expects for an asymptotically flat brane.
[19] The fundamental possibility of TeV scale five-dimensional gravitational coupling \( G_5 \sim G_N/\sigma \), where \( G_N \) is the four-dimensional Newton constant [1]. Thus, from (19) one has \((5) A/G_5 \sim M^2/G_N \sim (6) A/G_N \).