HIGH ACCURACY ANALYSIS OF ARBITRARY MODES IN TAPERED
DISK-LOADED STRUCTURES

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Abstract

In this paper, a high-precision eigenmode-computation analysis of arbitrary azimuthal mode numbers in periodic
disk-loaded structure based on variational method will be
discussed. It allows for rounding the edge of a disk hole
without any approximation in shape treatment and
calculates the modes exactly synchronous to the beam. It
converges much faster than the mesh-based computer
code. Good agreement was observed between the results
of variational method and those of other methods.

1 INTRODUCTION

In the past, various numerical methods were developed
for the analysis of the higher order modes in a cell[1-4],
such as KN7C[5], TRANSVRS[6], URMEL[7],
MAFIA[8], and Ω2[9], SUPERFISH[10]. The codes
KN7C and TRANSVRS can calculate the longitudinal
and transverse modes with any phase advance per period,
though imposing two approximations: an infinite periodic
structure and a flat beam hole surface parallel to the beam
axis instead of round edge. Because of the latter
assumption, the calculated wake field is not accurate
enough[1]. Though these two codes have been very
successfully applied to the actual design of structures,
they need to be further improved if they are to be used for
cells with a round edge beam hole. This is one of the
motives for developing the present code.

On the other hand, such codes as SUPERFISH,
URMEL, Ω2 and MAFIA can represent the structure by
filling with a mesh so that these can calculate the modes
in cells of any shape. However, our method still has its
advantages in many aspects. The variational method can
describes the structure geometry without any
approximation. Therefore it is a better way in the
analysis of arbitrary azimuthal mode numbers in periodic
structure (any azimuthal mode, any synchrotron phase,
any frequency or passband). The fields are expressed as
simple formula. Therefore it can be conveniently used to
serve as input for other purpose calculation, such as the
base vectors for equivalent circuit model or open-mode

The present code is named LONGTRANSVRS.

2 THEORY

The theory of variational approach is a well-known
method and is basically a mathematical forerunner of the
finite element method. In the classical formulation the
problem is to find the unknown function or functions that
extremize (maximize, minimize) or get stationary under
the specified boundary condition. The fields with steady-
state sinusoidal time dependence of $e^{-j\omega t}$ in vacuum space
are described by the following value minimum

$$ J = jk \int (\mathbf{z} \mathbf{H}) \times \mathbf{E} \cdot n dS. $$

(1)

where $k = \omega \sqrt{\varepsilon_0 \mu_0}$ is the propagation constant, $n$ is the
unit vector outward normal to the surface $S$. The variational form eq.(1) has no further limitation on the
trial function on the metal boundary, but the non-metal
boundary condition should be satisfied by the trial
function.

The accelerator structure with which we are concerned
is a conventional disk-loaded cylindrical waveguide as
shown in Figure 1. The whole acceleration structure is
divided into the inner and outer regions, separated at the
common boundary at $r = r_c$. The inner region is
characterized by the fact that it has a rounded edge of the
disk hole as part of boundary surface.

The Hertz vectors $\Pi_h = \mathbf{\Pi}_{h,z}$ and $\Pi_e = \mathbf{\Pi}_{e,z}$ are
chosen to simplify the solution. In the inner traveling
wave region, the Hertz vectors $\Pi_{h,z}$ and $\Pi_{e,z}$ take the
following forms

$$
\begin{align*}
&\Pi_h = F_h \mathbf{e}_z + F_v \mathbf{e}_r, \\
&\Pi_e = F_h \mathbf{e}_z - F_v \mathbf{e}_r.
\end{align*}
$$

Figure 1 Cross-section of disk-loaded waveguide.
\[\Pi_{n}^{m} = -\sum_{n=-\infty}^{\infty} \sum_{n=0}^{\infty} A_{n} J_{m}(\alpha_{n} r) e^{-i\beta_{n} z} \cos m\theta,\]  
(2)  
\[\Pi_{n}^{m} = \frac{1}{\Omega_{n}} \sum_{n=0}^{\infty} B_{n} J_{m}(\alpha_{n} r) e^{-i\beta_{n} z} \sin m\theta,\]  
(3)  
where \(\alpha_{n}^{2} = k^{2} - \beta_{n}^{2} \), \(\beta_{n} = \beta_{n}^{0} + 2\pi n/D\). \(J_{m}\) is the first kind Bessel function of order \(m\). When \(\alpha_{n}\) becomes imaginary, the regular Bessel functions can be replaced by modified Bessel functions \(I_{m}\) of a real argument. The above Hertz vectors fulfill the Floquet condition that is required by the trial function in eq. (1).

In the outer standing wave region, we have
\[\Pi_{n}^{m*} = -\sum_{n=0}^{\infty} C_{n} R_{m}(\Gamma) \cos \alpha(z + g) \cos m\theta,\]  
(4)  
\[\Pi_{n}^{m} = \frac{1}{\Omega_{n}} \sum_{n=0}^{\infty} D_{n} S_{n}(\Gamma) \sin \alpha(z + g) \sin m\theta,\]  
(5)  
where the transverse propagation constant \(\Gamma_{n}^{2} = k^{2} - \alpha_{n}^{2}\) with \(\alpha_{n} = \pi n/2g\) and  
\[R_{n}(\Gamma) = J_{m}(\Gamma) Y_{n}(\Gamma) - J_{n}(\Gamma) Y_{m}(\Gamma),\]  
(6)  
\[S_{n}(\Gamma) = J_{m}(\Gamma) Y_{n}(\Gamma) - J_{n}(\Gamma) Y_{m}(\Gamma).\]  
(7)  
Here, \(Y_{m}\) is the second kind regular Bessel function of order \(m\). The regular Bessel functions \(J_{m}\) and \(Y_{m}\) should be replaced by modified Bessel functions \(I_{m}\) and \(K_{m}\) of a real argument when \(\Gamma_{n}\) is imaginary.

The field components, which are used as the trial functions, can be found from the Hertz vectors (2-5) as follows
\[\mathbf{E} = \nabla \times \mathbf{H} - j \omega \mathbf{u}_{c} \nabla \times \mathbf{B},\]  
(8)  
\[\mathbf{H} = \nabla \times \mathbf{E} + j \omega \mathbf{u}_{c} \nabla \times \mathbf{D}.\]  
(9)  
Fields given by eqs.(2-3) don't satisfy the metal boundary condition. However, fields given by eqs.(4-5) do. The metal boundary condition on the iris can be satisfied by making function \(J\) in eq.(1) minimum.

We use different trial functions in the outer region and inner regions. The fields should be continuous across the interface between the two subregions. We can obtain the fields matching conditions by equating the two tangential components of the magnetic fields at the interface. The procedure to make function \(J\) in eq.(1) minimum consists of substituting the trial functions into the functional and thereby expressing the functional in terms of coefficients which are the unknowns, such as \(A, B, C, D\) in eqs.(2-5). The functional is then differentiated with respect to each coefficient, and the resulting equation is set to zero.

It is true that the periodicity theorem, as embodied in eqs. (2-3), help us to obtain the waves only for the infinitely long structure. If, however, we know the value of \(\beta_{n}\) for each wave, and the proper linear combination of the harmonics composing each wave, we can specify completely the excitation in a finite structure. We merely choose the amplitudes of the various waves so as to match boundary conditions at both ends of the structure.

**3 NUMERICAL RESULTS**

The field series should be truncated when we make \(J\) minimum. Figure 2 shows the value of \(J\) around the resonant frequency for the case of disk-loaded structure with round edge beam hole. All examples used in this paragraph have phase 120 degree and structure geometry \(r = 5.0013\) mm, \(D = 3.50088\) mm, \(a = 10.0025\) mm, \(\delta = 0.5076\) mm, \(\rho = 2.6007\) mm. The number of terms used in the outer region is 53 (\(s\) from 0 to 52) and in the inner region 53 (\(n\) from –26 to 26). \(J\) is normalized by
\[\int (Z,\mathbf{H})(\mathbf{Z},\mathbf{H})' dV.\]  
It is shown that \(J\) takes minimum value at the resonance frequency 2855.9777MHz, which means a true field. The truncation errors decrease when the number of the field terms increases. The convergence of the frequency and field can be estimated from the calculated accuracy of the resonance frequency \(\Delta f / f\) and the error of the eigenvector \(\epsilon\):
\[\Delta f = f_{n} - f_{w} \times 100\%\]  
(10)  
\[\epsilon = \frac{\int (\mathbf{E}_{n} - \mathbf{E}_{n-1})(\mathbf{E}_{n} - \mathbf{E}_{n-1})' dV}{\int \mathbf{E}_{n} \mathbf{E}_{n} ' dV}.\]  
(11)  
where \(f_{n}\) is the calculated frequency using \(n\) terms, \(f_{w}\) is the convergence frequency when the number of terms is infinite, and \(\mathbf{E}_{n}\) is the eigenfunction obtained from \(n\) terms. Figure 3 shows the resonance frequency, accuracy of frequency and estimated error \(\epsilon\) of the first longitudinal mode for different numbers of terms in the inner region, where \(f_{w} \approx f_{55}\) is used for the calculation of \(\Delta f / f\). Note that the accuracy in figure 3 is not the exact accuracy. Nevertheless it shows that the convergence rate of the eigenvalue is very rapid. Usually, we choose the series term \(s\) ranging from 0 to 52 in the outer region and \(n\) from –26 to 26 in the inner region, which correspond to the total number of terms 53 in Figure 3, to reach an eigenvalue convergence \(\Delta k / k\) better than \(1\times 10^{-6}\). The estimated error of the eigenvector also approximately decays exponentially with the number of terms. The error can reach to \(10^{-6}\) when \(n\) is from –26 to 26. The CPU time is mainly decided by the number of terms in the inner region. The number of terms in the outer region has little effect. We should choose a suitable number of terms in order to get accurate results within a short calculation time. For the above case, The CPU time needed for LONGTRANSVRS to reach within 0.07MHz of its convergence is about 96 sec while the CPU time for SUPERFISH to reach the same precision is about 4196 sec using a 500MHz Pentium-III computer. Figure 4 shows calculated frequency and CPU time used for the SUPERFISH code with different mesh sizes and LONGTRANSVRS code with different numbers of terms as in Figure 3.
4 SUMMARY AND CONCLUSIONS

Based on the variational method, a code, LONGTRANSVRS was developed which could calculate a disk-loaded structure with rounded edge beam hole shape. It can calculate all modes in disk-loaded structures with high accuracy, which depends on the number of terms used. Our method has good convergence and it's much faster than SUPERFISH code when the accuracy is high. This code provides a powerful tool to design the disc-loaded structure, such as searching for structure geometry (a or b) with fixed field modes.

5 REFERENCES

[8] The MAFIA Collaboration, MAFIA.