\( \mathcal{N} \)-point and higher-genus \( osp(1|2) \) fusion

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Abstract

We study affine \( osp(1|2) \) fusion, the fusion in \( osp(1|2) \) conformal field theory, for example. Higher-point and higher-genus fusion is discussed. The fusion multiplicities are characterized as discretized volumes of certain convex polytopes, and are written explicitly as multiple sums measuring those volumes. We extend recent methods developed to treat affine \( su(2) \) fusion. They are based on the concept of generalized Berenstein-Zelevinsky triangles and virtual couplings. Higher-point tensor products of finite-dimensional irreducible \( osp(1|2) \) representations are also considered. The associated multiplicities are computed and written as multiple sums.
1 Introduction

The representation theory of finite-dimensional irreducible representations of the Lie superalgebra $osp(1|2)$ is well-known [1]. That includes the decomposition of ordinary three-point tensor products. However, to the best of our knowledge, the literature does not offer discussions of general $N$-point couplings. Here we shall consider those and compute the associated tensor product multiplicities. They are characterized as discretized volumes of certain convex polytopes (i.e., the number of integer points bounded by the polytope), and are written explicitly as multiple sums measuring those volumes. The results are obtained by extending recent methods developed to treat $su(N)$ tensor products, and are based on the concept of generalized Berenstein-Zelevinsky (BZ) triangles and virtual couplings [2, 3].

The seminal work by Berenstein and Zelevinsky [4] shows how one may represent ordinary $su(N)$ three-point couplings by triangular arrangements of non-negative integers. Their results were extended in ref. [3] to higher-point couplings, by gluing such triangles together. Here we shall associate two types of triangles to $osp(1|2)$ three-point couplings. We denote them $osp(1|2)$ BZ triangles, and “super-triangles”, respectively. The gluing method of [3] is then applied to treat higher-point $osp(1|2)$ couplings.

We then turn to our second objective: affine $osp(1|2)$ fusion, the fusion in $osp(1|2)$ conformal field theory, for example. Ordinary three-point fusion has been studied from various points of view [5, 6, 7]. For integer level, $k$, and associated admissible (or integrable) representations [8, 5], we show that the level dependence of a fusion may be incorporated in the $osp(1|2)$ BZ triangles or super-triangles. That allows us to discuss higher-point fusion along the lines of [9] on higher-point $su(2)$ fusion. As for the tensor products above, the $N$-point $osp(1|2)$ fusion multiplicities are characterized by level-dependent convex polytopes, and written explicitly as multiple sums.

Our approach admits also an extension to higher-genus, $h$, fusion. The associated fusion multiplicities are characterized as discretized volumes of certain convex polytopes, and are written explicitly as multiple sums. To illustrate and demonstrate consistency, we consider in detail the genus-one one- and two-point fusions.

This work presents the first general results on $N$-point $osp(1|2)$ tensor products, and on $N$-point and higher-genus $osp(1|2)$ fusion. The results are general as they cover all integer $N$, $k$ and $h$. They are also very explicit and should therefore be easy to use in applications. Implementation in computer programming is also straightforward.

1.1 $osp(1|2)$ representation theory

Here we recall some basic facts on the Lie superalgebra $osp(1|2)$ and its irreducible representations [1]. A “physicist-friendly” review may be found in ref. [10], while general Lie superalgebra theory is considered in refs. [11, 12].

The Lie superalgebra $osp(1|2)$ is a five-dimensional graded extension of the three-dimensional Lie algebra $su(2)$:

$$
\begin{align*}
[j^3, j^\pm] &= \pm j^\pm, & [j^3, j^\mp] &= \frac{1}{2}j^\pm, & [j^+, j^-] &= 2j^3 \\
\{j^+, j^-\} &= 2j^3, & \{j^\pm, j^\mp\} &= \pm 2j^\pm, & [j^\pm, j^\mp] &= -j^\pm
\end{align*}
$$

1
All other (anti-)commutators vanish. The three even generators $J^+, J^-$ and $J^3$ generate an $su(2)$ subalgebra of $osp(1|2)$, while $J^+$ and $J^-$ are two odd generators. They comprise a spin-1/2 representation of the $su(2)$ subalgebra in the adjoint representation.

Every finite-dimensional irreducible representation has an isospin $j$ associated to it, where

$$2j \in \mathbb{Z}_{\geq}$$

Such a representation $R_j$ has dimension $4j + 1$:

$$R_j : |j, j\rangle, |j, j-1/2\rangle, ... , |j, 0\rangle, ..., |j, -j + 1/2\rangle, |j, -j\rangle$$

The states $|j, m\rangle$ and $|j, m'\rangle$ have the same parity if and only if $m - m' \in \mathbb{Z}$. The parity $p(R_j)$ of the representation $R_j$ is defined as the parity of the state $|j, j\rangle$. The mode $m$ is the eigenvalue of $J^3$: $J^3 |j, m\rangle = m |j, m\rangle$. It is observed that the representation (3) splits into two $su(2)$ representations – one of spin $j$ and one of spin $j - 1/2$. The former consists of the states $|j, m\rangle$ with $j - m \in \mathbb{Z}_{\geq}$, while the latter consists of the states with $j - m \in \mathbb{Z}_{\geq} + 1/2$.

Disregarding the notion of parity, the $osp(1|2)$ representation space (3) becomes analogous to a single $su(2)$ representation space of spin $2j$. That observation will turn out to be useful in the following.

We shall use the same notation $j$ for an $osp(1|2)$ isospin as for an $su(2)$ spin, but refer to them as indicated. An $su(2)$ representation of spin $j$ is indicated by $R_j^{su(2)}$.

## 2 Tensor products

Decompositions of ordinary tensor products of finite-dimensional irreducible representations are easily computed:

$$R_{j_1} \otimes R_{j_2} = R_{|j_1-j_2|} \oplus R_{|j_1-j_2|+1/2} \oplus ... \oplus R_{j_1+j_2-1/2} \oplus R_{j_1+j_2}$$

(4)

Note the resemblance to tensor products of integer-spin $su(2)$ representations:

$$R_{2j_1}^{su(2)} \otimes R_{2j_2}^{su(2)} = R_{2|j_1-j_2|}^{su(2)} \oplus R_{2|j_1-j_2|+1}^{su(2)} \oplus ... \oplus R_{2(j_1+j_2)-1}^{su(2)} \oplus R_{2(j_1+j_2)}^{su(2)}$$

(5)

Instead of considering a tensor product of the form

$$R_{j_1} \otimes R_{j_2} \supset R_{j_3}$$

(6)

we may equivalently consider the symmetric three-point coupling to the singlet:

$$R_{j_1} \otimes R_{j_2} \otimes R_{j_3} \supset R_0$$

(7)

Similar couplings of $su(N)$ representations are neatly described by Berenstein-Zelevinsky (BZ) triangles [4]. In the case of $su(2)$ the BZ triangle is trivial but has led to characterizations of higher-point and higher-genus couplings and fusions as discretized volumes of certain polytopes [9]. Here we shall discuss the generalization to $osp(1|2)$.  

2
2.1 Berenstein-Zelevinsky super-triangle

An $su(2)$ BZ triangle is a triangular arrangement of three non-negative integer entries $a$, $b$ and $c$

$$R^{su(2)}_{j_1} \otimes R^{su(2)}_{j_2} \otimes R^{su(2)}_{j_3} \supset R^{su(2)}_0 \leftrightarrow \begin{array}{c} b \\ c \\ a \end{array}$$

subject to the spin constraints

$$a = -j_1 + j_2 + j_3 \in Z \geq, \quad b = j_1 - j_2 + j_3 \in Z \geq, \quad c = j_1 + j_2 - j_3 \in Z \geq$$

and hence

$$2j_1 = b + c, \quad 2j_2 = c + a, \quad 2j_3 = a + b$$

When all three spins are integer, either $a$, $b$ and $c$ must all be even or all be odd. Exploring the similarity between (4) and (5) we see that we may describe three-point couplings of $osp(1|2)$ representations by standard BZ triangles

$$R_{j_1} \otimes R_{j_2} \otimes R_{j_3} \supset R_0 \leftrightarrow \begin{array}{c} B \\ C \\ A \end{array}$$

with isospins

$$4j_1 = B + C, \quad 4j_2 = C + A, \quad 4j_3 = A + B, \quad A, B, C \in Z \geq$$

or equivalently by BZ super-triangles

$$R_{j_1} \otimes R_{j_2} \otimes R_{j_3} \supset R_0 \leftrightarrow \begin{array}{c} b \\ c + a + \epsilon \end{array}$$

with isospins

$$2j_1 = b + c + \epsilon, \quad 2j_2 = c + a + \epsilon, \quad 2j_3 = a + b + \epsilon, \quad a, b, c \in Z \geq, \quad \epsilon \in \{0,1\}$$

The super-entry $\epsilon$ measures the “parity violation” of the coupling:

$$\epsilon = p(R_{j_1}) + p(R_{j_2}) + p(R_{j_3}) \mod 2 = 2(j_1 + j_2 + j_3) \mod 2$$

Relaxing the isospin-independent constraints on the entries (thereby allowing $a$, $b$, $c$, $\epsilon \in Z$), there are infinitely many generalized super-triangles associated to a three-point coupling. They are all related through additions of integer multiples of the (basis) virtual super-triangle

$$V = \begin{array}{c} 1 \\ 2 \\ 1 \end{array}$$

$$V = \begin{array}{c} 1 \\ 1 \\ 3 \end{array}$$
where \( \bar{n} \equiv -n \). Given an initial generalized super-triangle \( T_0 \) (see (31) for a choice when extended to higher-point couplings), all other generalized super-triangles are of the form

\[
T = T_0 + \sum_{v \in \mathbb{Z}} v \mathcal{V}
\]

(17)

However, due to the constraint on \( \epsilon \), only one super-triangle in this infinite chain of generalized super-triangles will satisfy all the requirements. We shall call it a true super-triangle. By construction, if a coupling of three isospins \((j_1, j_2, j_3)\) to the singlet is not possible, there will be no true super-triangle associated to that isospin triplet.

A motivation for introducing super-triangles is that they seem to indicate how one may generalize the representation of \( su(N) \) couplings by BZ triangles to a representation of Lie superalgebra couplings by (extended) super-triangles. Even though the \( osp(1|2) \) super-triangles are slightly more complicated to work with than the \( osp(1|2) \) BZ triangles, we shall consider them throughout this paper alongside the BZ triangles. They provide us with alternative characterizations of tensor product couplings and fusions – representations that are more “supersymmetric”. Furthermore, in the Conclusion we will indicate how super-triangles appear natural from the point of view of three-point functions in \( osp(1|2) \) conformal field theory.

2.2 Higher-point couplings

In a decomposition of a higher-point tensor product, the singlet may occur more than once, i.e., the associated tensor product multiplicity, \( T_{j_1, \ldots, j_N} \), may be greater than one:

\[
\mathcal{R}_{j_1} \otimes \ldots \otimes \mathcal{R}_{j_N} \supset T_{j_1, \ldots, j_N} \mathcal{R}_0
\]

(18)

The similar situation for \( su(2) \) couplings is described in [3, 9] (ref. [3] covers all \( su(N) \) but does not discuss fusion). There it is discussed how BZ triangles may be glued together to form \( N \)-sided diagrams representing the \( T_{j_1, \ldots, j_N}^{su(2)} \) different \( su(2) \) couplings. Likewise, we can associate an \( N \)-sided diagram to each of the \( T_{j_1, \ldots, j_N}^{osp(1|2)} \) different \( osp(1|2) \) \( N \)-point couplings. Due to the existence of two types of triangles (11) and (13), we may represent an \( osp(1|2) \) \( N \)-point coupling by two different types of diagrams. We shall call the ones based on super-triangles super-diagrams.

The general method for computing higher-point \( su(N) \) tensor product multiplicities outlined in [3], is based on gluing BZ triangles together using gluing diagrams (we refer to [3] for details). This idea extends readily to \( osp(1|2) \) tensor products (18). To be explicit, let us consider the following \( N \)-point diagram (in this example \( N \) is assumed odd):

\[
\begin{align*}
4j_{N-2} & \quad 4j_N & \ldots & \quad 4j_1 \\
4j_{N-1} & \quad 4j_{N-3} & \quad 4j_2 & \quad 4j_{N-1} \\
4j_{N-2} & \quad 4j_N & \ldots & \quad 4j_1
\end{align*}
\]

(19)
The role of the gluing is to take care of the summation over internal isospins in a tractable way. The dual picture of ordinary (Feynman tree-) graphs is shown in thinner lines. Along a gluing, the opposite isospins must be identified.

Let us begin by considering the diagrams obtained by extending (11). The starting point in [3] and here is to relax the constraint that the entries should be non-negative integers. As for the super-triangles above, a diagram of that kind is called a generalized diagram. Any such generalized diagram, respecting the gluing constraints and the outer isospin constraints (19), will suffice as an initial diagram. All other diagrams that are associated to the same outer isospins may then be obtained by adding integer linear combinations of so-called virtual diagrams: adding a basis virtual diagram changes the value of $4j$ of a given internal isospin by two, leaving all other internal isospins and all outer isospins unchanged. Thus, the basis virtual diagram associated to a particular gluing is of the form:

$$G = \ldots \begin{array}{c} \bar{1} \\ \bar{1} \\ \bar{1} \\ \bar{1} \\ 1 \end{array}$$

Enumerating the gluing diagrams (20) in (19) from right to left, the associated integer coefficients in the linear combinations are $g_1, \ldots, g_{N-3}$. If $D_0$ denotes an initial diagram, all generalized diagrams will then be of the form

$$D = D_0 + \sum_{l=1}^{N-3} \sum_{g_l \in \mathbb{Z}} g_l G_l$$

(21)

It remains to be accounted for how to write down an initial diagram $D_0$. However, that is straightforward:

$$D_0 = A$$

with

$$e_l = 4(j_1 + \ldots + j_l), \quad 1 \leq l \leq N-3$$

$$A = -S + 4j_{N-1} + 4j_N, \quad B = S - 4j_{N-1}, \quad C = S - 4j_N$$

(23)

and

$$S \equiv 2(j_1 + \ldots + j_N)$$

(24)
Re-imposing the condition that all the entries in $\mathcal{D}$ (21) must be non-negative, results in a set of inequalities defining a convex polytope in the Euclidean space $\mathbb{R}^{N-3}$:

$$
\begin{align*}
0 & \leq g_1, \ 4j_1 - g_1, \ 4j_2 - g_1 \\
0 & \leq g_2 - g_1, \ 4j_3 - g_2 + g_1, \ 4(j_1 + j_2) - g_2 - g_1 \\
& \vdots \\
0 & \leq g_{N-3} - g_{N-4}, \ 4j_{N-2} - g_{N-3} + g_{N-4}, \ 4(j_1 + \ldots + j_{N-3}) - g_{N-3} - g_{N-4} \\
0 & \leq S - 4j_{N-1} - g_{N-3}, \ S - 4j_N - g_{N-3}, \ -S + 4(j_{N-1} + j_N) + g_{N-3}
\end{align*}
$$

(25)

By construction, its discretized volume is the tensor product multiplicity $T_{j_1, \ldots, j_N}$. The volume may be measured explicitly, expressing the multiplicity as a multiple sum. In order to avoid discussing intersection of faces we have to choose an “appropriate order” of summation (see refs. [2, 3, 9]). Making such a choice is straightforward, and we find that the $osp(1|2)$ tensor product multiplicity $T_{j_1, \ldots, j_N}$ may be written as

$$
T_{j_1, \ldots, j_N} = \min\{S-4j_{N-1}, S-4j_N\} \min\{g_{N-3}, 4(j_1+\ldots+j_{N-3})-g_{N-3}\} \\
\times \sum_{g_{N-3}=S-4(j_{N-1}+j_N)} \sum_{g_{N-4}=-4j_{N-2}+g_{N-3}} \min\{g_3, 4(j_1+\ldots+j_3)-g_3\} \min\{j_1, j_2, g_2, 4(j_1+j_2)-g_2\} \\
\times \sum_{g_2=-4j_4+g_3} \sum_{g_1=\max(0, -4j_3+g_2)} 1
$$

(26)

This is the first general result for higher-point $osp(1|2)$ tensor product multiplicities.

Following methods discussed in [2, 3, 9], it is not difficult to derive necessary and sufficient conditions determining when an $osp(1|2)$ $\mathcal{N}$-point tensor product multiplicity is non-vanishing. The conditions are

$$
2j_l, \ S - 4j_l \in \mathbb{Z}_+, \quad l = 1, \ldots, \mathcal{N}
$$

(27)

with $S$ defined in (24).

Gluing super-triangles together to represent higher-point couplings, is not a lucrative alternative to the method above. Nevertheless, we give here the associated gluing super-diagram:

$$
\mathcal{G} = \begin{array}{cccc}
0 &  &  & 0 \\
 & 0 &  & \\
\ddots & \ddots & \ddots & \\
0 & 1 &  & 0
\end{array}
$$

(28)

There is a virtual super-triangle associated to each of the glued super-triangles, i.e., there are $\mathcal{N} - 2$ (basis) virtual super-diagrams associated to an $\mathcal{N}$-point super-diagram. In a self-explaining notation we then have that any generalized super-diagram may be written

$$
\mathcal{D} = D_0 + \sum_{l=1}^{\mathcal{N}-2} \sum_{v_l \in \mathbb{Z}} v_l \mathcal{V}_l + \sum_{l=1}^{\mathcal{N}-3} \sum_{g_l \in \mathbb{Z}} g_l \mathcal{G}_l
$$

(29)

6
Now, recall that the super-entry measures the parity violation as indicated in (15). For $N$-point couplings it is the sum of the $N - 2$ super-entries that measures the parity violation. It is therefore natural to introduce the parity parameter $\eta$

$$2\eta = \left( \sum_{i=1}^{N-2} \epsilon_i \right) \mod 2 = \begin{cases} 0 & \text{for } S \in 2\mathbb{Z} \\ 1 & \text{for } S \in 2\mathbb{Z} + 1 \end{cases}$$

which of course must depend only on the outer isospins (through $S$ (24)). We may now write down an initial super-diagram:

$$D_0 = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
2\eta
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
b \quad 2j_{N-2} \quad 0 \quad e_{N-4}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
a
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
c \quad e_{N-3} \quad 2j_{N-3} \quad 0
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\cdots
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
e_1
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
2j_2
\end{array}
\end{array}
\end{array}$$

with

$$\begin{align*}
el_l &= 2(j_1 + \ldots + j_l), \quad 1 \leq l \leq N - 3 \\
a &= -\frac{S}{2} - \eta + 2(j_{N-1} + j_N), \quad b = \frac{S}{2} - \eta - 2j_{N-1}, \quad c = \frac{S}{2} - \eta - 2j_N
\end{align*}$$

Since $\frac{S}{2} - \eta = [S/2]$ ([x] denotes the integer value of x, i.e., the greatest integer less than or equal to x), the entries a, b and c are integers. Imposing the condition that the diagram (29) must be true, leads to a set of inequalities in the parameters $v$ and $g$ defining a convex polytope as (25). This polytope is embedded in the Euclidean space $\mathbb{R}^{2N-5}$. The inequalities are straightforward to write down, but are not given here.

### 2.3 Four-point couplings

To illustrate the results above we shall compute the $osp(1|2)$ four-point tensor product multiplicity $T_{j_1,j_2,j_3,j_4}$. We shall do it in two ways: first by reducing the general result (25) and (26) to $N = 4$, and then by gluing super-triangles together.

It follows from (25) that

$$0 \leq g, \ 4j_1 - g, \ 4j_2 - g, \ S - 4j_3 - g, \ S - 4j_4 - g, \ -S + 4(j_3 + j_4) + g$$

and therefore

$$T_{j_1,j_2,j_3,j_4} = \min \{2(j_1+j_2+j_3+j_4), \ 2(j_1+j_2-j_3+j_4), \ 4j_1, \ 4j_2 \}$$

$$= \begin{array}{c}
\sum_{g=\max\{0, \ 2(j_1+j_2-j_3-j_4)\}} 1 \\
\end{array}$$

$$= 1 + \min \{4j_1, \ ..., \ 4j_4, \ S - 4j_1, \ ..., \ S - 4j_4 \}$$

7
provided the conditions (27) are satisfied.

Now we turn to the super-triangle approach. For \( \mathcal{N} = 4 \), the convex polytope defined by (29) and (31) becomes

\[
0 \leq v_1 + g, \quad 2j_1 + v_1, \quad 2j_2 + v_1 \\
0 \leq -g - 2v_1 \leq 1 \\
0 \leq -\left[ \frac{S+1}{2} \right] + 2(j_3 + j_4) + v_2 + g, \quad \left[ \frac{S}{2} \right] - 2j_3 + v_2, \quad \left[ \frac{S}{2} \right] - 2j_4 + v_2 \\
0 \leq 2\eta - g - 2v_2 \leq 1
\]

Note that the inequalities \( 0 \leq \epsilon_1, \epsilon_2 \leq 1 \) fix \( v_1 \) and \( v_2 \) in terms of \( g \):

\[
v_1 = -\left[ \frac{g+1}{2} \right], \quad v_2 = -\left[ \frac{g+1 - 2\eta}{2} \right]
\]

That means that the set of inequalities in \( g, v_1 \) and \( v_2 \) reduces to a set of inequalities in the gluing coordinate \( g \) alone. It is not hard to verify that the associated (one-dimensional) polytope is identical to (33). Thus, the two ways of counting the tensor product multiplicity \( T_{j_1,j_2,j_3,j_4} \) are essentially equivalent. That generalizes to \( \mathcal{N} \)-point couplings.

3 Fusion

Here we shall extend the above discussion on tensor products to affine fusion, fusion in \( \mathfrak{osp}(1|2) \) conformal field theory, for example. To distinguish this consideration from the similar one concerning tensor products, we denote finite-dimensional irreducible affine modules of isospin \( j \) by \( M_j \). The fusion of three such modules to the singlet is written (cf. the analogous three-point coupling (7))

\[
M_{j_1} \times M_{j_2} \times M_{j_3} \supset \mathcal{N}_{j_1,j_2,j_3}^{(k)} M_0
\]

The fusion multiplicity \( \mathcal{N}_{j_1,j_2,j_3}^{(k)} \) depends on the level \( k \), where \( k \) characterizes the affine extension of \( \mathfrak{osp}(1|2) \) that turns it into a level-\( k \) affine Lie superalgebra. We shall consider only \( k \) a positive integer, and the so-called admissible (or integrable) representations \([8, 5]\). They are (for \( k \) a positive integer) characterized by

\[
2j \in \mathbb{Z}_+, \quad 2j \leq k
\]

The ordinary fusion multiplicities are well-known in that case \([5, 6]\):

\[
\mathcal{N}_{j_1,j_2,j_3}^{(k>\epsilon_1+j_2+j_3-1/2)} = T_{j_1,j_2,j_3}, \quad \mathcal{N}_{j_1,j_2,j_3}^{(k<\epsilon_1+j_2+j_3-1/2)} = 0
\]

We recall that a non-vanishing three-point tensor product multiplicity is one. The non-vanishing conditions follow immediately from (4).

The threshold level, \( t \), of a three-point coupling is the minimum level at which the coupling appears in fusion \([13]\). This means, in particular, that \( t \) is integer and that \( t \leq k \) for the coupling to appear. From (39), it is straightforward to determine the threshold level of an \( \mathfrak{osp}(1|2) \) coupling of three isospins \((j_1, j_2, j_3)\):

\[
t = \left[ \frac{S}{2} \right]
\]
One may also assign a threshold level to an $osp(1|2)$ BZ triangle or super-triangle. It is known how to do that for $su(N \leq 4)$ [14, 15] and has been explored further in [16]. To the BZ $osp(1|2)$ triangle (11) we may assign the threshold level

$$t = \left\lfloor \frac{A + B + C}{2} \right\rfloor$$  \hspace{1cm} (41)$$

and to the super-triangle (13) we may assign the threshold level

$$t = a + b + c + \epsilon$$  \hspace{1cm} (42)$$

Since $t$ is integer, the condition $t \leq k$ on (41) is equivalent to

$$A + B + C - 1 \leq 2k$$  \hspace{1cm} (43)$$

A higher-point coupling can also be assigned a threshold level [9]. It is defined in the same way as for three-point couplings.

Recently, efforts have been made to characterize fusion multiplicities in terms of polytopes. Most results so far pertain to three-point fusion [16, 17], but also higher-genus and higher-point $su(2)$ fusions have been discussed [9]. Below we shall extend the latter results to $osp(1|2)$.

### 3.1 Higher-point fusion

We are now in a position to discuss $N$-point fusion. Using $osp(1|2)$ BZ triangles, we see that fusion is described by supplementing the set of inequalities (25) by $N - 2$ conditions like (43) – a condition associated to each of the $N - 2$ participating triangles. Thus, an $N$-point fusion is characterized by the inequalities

$$0 \leq g_1, 4j_1 - g_1, 4j_2 - g_1, 2k - 4(j_1 + j_2) + g_1 + 1$$

$$0 \leq g_2 - g_1, 4j_3 - g_2 + g_1, 4(j_1 + j_2) - g_2 - g_1, 2k - 4(j_1 + j_2 + j_3) + g_1 + g_2 + 1$$

$$\vdots$$

$$0 \leq g_{N-3} - g_{N-4}, 4j_{N-2} - g_{N-3} + g_{N-4}, 4(j_1 + \ldots + j_{N-3}) - g_{N-3} - g_{N-4}, 2k - 4(j_1 + \ldots + j_{N-2}) + g_{N-3} + g_{N-4} + 1$$

$$0 \leq S - 4j_{N-1} - g_{N-3}, S - 4j_N - g_{N-3}, -S + 4(j_{N-1} + j_N) + g_{N-3}, 2k - S + g_{N-3} + 1$$  \hspace{1cm} (44)$$

defining a convex polytope embedded in $\mathbb{R}^{N-3}$. Its discretized volume is the fusion multiplicity $N^{(k)}_{j_1,\ldots,j_N}$. It may be measured explicitly, expressing the multiplicity as a multiple sum:

$$N^{(k)}_{j_1,\ldots,j_N} = \min\{S - 4j_{N-1}, S - 4j_N\}$$

$$\sum_{g_{N-3} = \max\{S - 4(j_{N-1} + j_N), -2k + S - 1\}} \min\{g_{N-3}, 4(j_1 + \ldots + j_{N-3}) - g_{N-3}\}$$

$$\times \sum_{g_{N-4} = \max\{-4j_{N-2} + g_{N-3}, -2k + 4(j_1 + \ldots + j_{N-2}) - g_{N-3} - 1\}} \ldots$$
This is a new result.

4 Higher-genus fusion

Here we will discuss the extension of our results above on genus-zero fusion to generic genus-$h$ fusion. The results here generalize the similar ones in ref. [9] on higher-genus $su(2)$ fusion. $N_{j_1,\ldots,j_N}^{(k,h)}$ denotes the genus-$h$ $N$-point fusion multiplicity.

A simple extension of (19) is the following genus-$h$ $N$-point diagram (in this example $N$ is assumed even, while $h$ is arbitrary):

![Diagram](image)

The dual trivalent fusion graph is represented by thinner lines and loops. $h$ is the number of such loops or handles. The role of the two zeros in (46) will be discussed below. The number of internal isospins or gluings is $N + 3(h - 1)$, while the number of vertices or triangles is $N + 2(h - 1)$.

First we consider the tadpole diagram

![Diagram](image)

In terms of $osp(1|2)$ BZ triangles the basis diagram associated to it is

\[
\begin{array}{c}
0 \\
0 & 2
\end{array}
\]

We call (48) a loop-gluing diagram. Since we are gluing over even integers, the initial tadpole diagram will depend on $2j$ being even (indicated by $p = 0$) or odd (indicated by $p = 1$). With $l$ being the coefficient to (48), the polytope is defined by

\[
0 \leq 2j, \ 2j, \ p + 2l, \ 2k - 4j - p - 2l + 1
\]
Thus, the genus-one one-point fusion multiplicity becomes

\[ N_j^{(k,1)} = \sum_{l=\left[\frac{p+1}{2}\right]} \frac{2k - 4j + p + 1}{2} 1 = k - 2j + 1 \]  

(50)

irrespective of \(2j\) being even or odd. That independence is novel compared to the similar situation for \(su(2)\) [9].

The basis loop-gluing super-diagram associated to (47) is

\[
\begin{array}{c}
0 \\
0 \ 1 \\
0 \\
\end{array}
\]  

(51)

Let us also describe the basis loop-gluing diagrams associated to the genus-one two-point fusion

\[
\begin{array}{c}
0 \\
0 \ 1 \\
0 \\
\end{array}
\]  

(52)

In terms of \(osp(1|2)\) BZ diagrams there are two basis loop-gluing super-diagrams associated to this fusion. They may be represented by the diagrams

\[
\mathcal{L} = \begin{array}{ccc}
\bar{1} & 1 & 1 \\
1 & \bar{1} & 1 \\
\end{array} \quad \mathcal{L}' = \begin{array}{ccc}
\bar{1} & 1 & 1 \\
1 & \bar{1} & 1 \\
\end{array}
\]  

(53)

They differ significantly from the \(su(2)\) basis loop-gluing diagrams [9], as they do not constitute a basis of \(su(2)\) loop-gluing diagrams. Similarly, the two loop-gluing super-diagrams are

\[
\begin{array}{ccc}
\bar{1} & 0 & 0 \\
0 & \bar{1} & 1 \\
\end{array} \quad \begin{array}{ccc}
1 & 0 & 0 \\
0 & \bar{1} & 1 \\
\end{array} 
\]  

(54)

It is noted that the choice of loop-gluing basis (53) is not a convenient one. Had we only been interested in the polytope characterization of the fusion multiplicity and not an explicit measure of its discretized volume, this symmetric basis would suffice. But in order to be able to choose an appropriate order of summation (i.e., avoid discussing intersection of faces), we can not allow both diagrams to affect all the entries of the two triangles. A good but less symmetric basis is

\[
\begin{array}{ccc}
\bar{1} & 1 & 1 \\
1 & \bar{1} & 1 \\
\end{array} \quad \mathcal{L}^+ = \begin{array}{ccc}
0 & 2 & 0 \\
2 & 0 & 2 \\
\end{array}
\]  

(55)

where \(\mathcal{L}^+ = \mathcal{L} + \mathcal{L}'\).
As a non-trivial check of our procedure, we now consider the genus-one two-point fusion in detail using the two different channels

\[ \begin{align*}
4j_2 & \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 
the higher-genus part to the right of them are zero, while the \( N \)-point part follows the pattern of the initial diagram (22) (assuming \( N \geq 3 \), see the comments above). Enumerating the gluings from right to left (and \( \mathcal{L}^+ \) before \( \mathcal{L} \)), the integer coefficients in the linear combinations are \(-g_1, ..., -g_h, g_{h+1}, ..., g_{N+h-2}\) (the sign convention is merely for convenience), and \( l_1^+, l_1, l_2^+, l_2, ..., l_{h-1}^+, l_{h-1}, l_{h-2}^+, l_{h-2}, \) while \( l \) is associated to the tadpole at the extreme right. Listing the inequalities associated to the triangles from right to left, we have the following convex polytope (assuming \( h \geq 1 \)):

\[
\begin{align*}
0 & \leq g_1, g_1, 2l - g_1, 2k - g_1 - 2l + 1 \\
0 & \leq g_1 - l_1, g_1 + l_1, -g_1 + 2l_1^+ + l_1, 2k - g_1 - 2l_1^+ - l_1 + 1 \\
0 & \leq g_2 - l_1, g_2 + l_1, -g_2 + 2l_1^+ + l_1, 2k - g_2 - 2l_1^+ - l_1 + 1 \\
& \vdots \\
0 & \leq g_{h-1} - l_{h-1}, g_{h-1} + l_{h-1}, -g_{h-1} + 2l_{h-1}^+ + l_{h-1}, 2k - g_{h-1} - 2l_{h-1}^+ - l_{h-1} + 1 \\
0 & \leq g_h - l_{h-1}, g_h + l_{h-1}, -g_h + 2l_{h-1}^+ + l_{h-1}, 2k - g_h - 2l_{h-1}^+ - l_{h-1} + 1 \\
0 & \leq g_{h+1} + g_h, -g_{h+1} + g_h, 4j_1 - g_{h+1} - g_h, 2k - 4j_1 + g_{h+1} - g_h + 1 \\
0 & \leq g_{h+2} - g_{h+1}, 4j_1 - g_{h+2} - g_{h+1}, 4j_2 - g_{h+2} + g_{h+1}, 2k - 4(j_1 + j_2) + g_{h+2} + g_{h+1} + 1 \\
& \vdots \\
0 & \leq g_{N+h-2} - g_{N+h-3}, 4(j_1 + ... + j_{N-3}) - g_{N+h-2} - g_{N+h-3}, 4j_{N-2} - g_{N+h-2} + g_{N+h-3}, 2k - 4(j_1 + ... + j_{N-2}) + g_{N+h-2} + g_{N+h-3} + 1 \\
0 & \leq S - 4j_{N-1} - g_{N+h-2}, S - 4j_{N} - g_{N+h-2}, -S + 4(j_{N-1} + j_{N}) + g_{N+h-2}, 2k - S + g_{N+h-2} + 1
\end{align*}
\]  

By construction, its discretized volume is the fusion multiplicity \( N_{j_1, ..., j_N}^{(k,h)}\), which then provides the first characterization of general \( osp(1|2) \) fusion multiplicities. The volume may be measured explicitly expressing \( N_{j_1, ..., j_N}^{(k,h)} \) as a multiple sum:

\[
N_{j_1, ..., j_N}^{(k,h)} = \sum_{g_{N+h-2}} \cdots \sum_{g_h} \left( \sum_{l_{h-1}} \sum_{l_{h-1}^+} \sum_{g_{h-1}} \right) \cdots \left( \sum_{l_1} \sum_{l_1^+} \sum_{g_1} \right) \sum l.
\]  

(61)

The integer summation variables are bounded according to

\[
\begin{align*}
\left\lfloor \frac{g_1 + 1}{2} \right\rfloor & \leq l \leq \left\lfloor \frac{2k - g_1 + 1}{2} \right\rfloor \\
|l_1| & \leq g_1 \leq \min\{2l_1^+ + l_1, 2k - 2l_1^+ - l_1 + 1\} \\
\left\lfloor \frac{g_2 - l_1 + 1}{2} \right\rfloor & \leq l_1^+ \leq \left\lfloor \frac{2k - g_2 - l_1 + 1}{2} \right\rfloor \\
-g_2 & \leq l_1 \leq g_2 \\
& \vdots \\
|l_{h-1}| & \leq g_{h-1} \leq \min\{2l_{h-1}^+ + l_{h-1}\}
\end{align*}
\]  

13
This constitutes the first explicit result for the general genus-\( h \) \( N \)-point fusion multiplicities. An advantage of using super-triangles instead of the \( \text{osp}(1|2) \) BZ triangles employed above, is that the variables \( v, g \) and \( l \) all appear with unit coefficients in the polytope-defining inequalities similar to (60). However, it is not straightforward to measure the discretized volume of that polytope. The reason is similar to the one excluding the basis (53) as a “good basis”.

5 Conclusion

We have considered higher-point couplings of finite-dimensional irreducible representations of \( \text{osp}(1|2) \). The associated tensor product multiplicities were characterized as discretized volumes of certain convex polytopes, and written explicitly as multiple sums. The results are general.

We have also considered affine \( \text{osp}(1|2) \) fusion. By extending the results on tensor products, we characterized a general genus-\( h \) \( N \)-point fusion multiplicity as a discretized volume of a certain convex polytope, and wrote down an explicit multiple sum measuring that volume. That result is also general.

It has been demonstrated, though not emphasized explicitly, that a fusion polytope may be embedded in the associated tensor product polytope. The reason is that the set of defining inequalities of a fusion polytope is obtained by supplementing the set of defining inequalities of the associated tensor product polytope by level-dependent inequalities. That offers a geometric interpretation of affine fusion being a truncated tensor product.

In the derivation of our results we have described three-point couplings by triangular arrangements of non-negative integers similar to the \( \text{su}(2) \) BZ triangles. We introduced two types. We based most of our results on a direct adaption of the ordinary \( \text{su}(2) \) BZ triangle. However, we also introduced a super-triangle and discussed some of its alternative features. Here we will indicate how it appears natural from the point of view of correlators in \( \text{osp}(1|2) \) conformal field theory. Three-point functions in conformal field theory with affine Lie group symmetry have been considered in ref. [18]. Their level-dependence was subsequently addressed in ref. [19]. The idea is to associate so-called elementary polynomials to the elementary couplings appearing in an expansion of a three-point coupling. The three-point functions are then constructed.
as (linear combinations of) products of those polynomials. The algebraic relations (syzygies) among the elementary couplings complicate the construction. In some cases they may be taken into account at the level of BZ triangles by forbidding certain configurations. In terms of polynomials that amounts to forbidding certain products, as there is a correspondence between BZ triangles and polynomials. As we will show elsewhere [7], the situation for \(osp(1|2)\) is most easily handled using our super-triangles. The constraint on the super-entry \(\epsilon\) (14) is neatly encoded by associating a Grassmann odd polynomial to a super-triangle with \(\epsilon = 1\). This also introduces a natural way of implementing the \(osp(1|2)\) syzygy [20, 21].

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References
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