Gravitational Energy in Quadratic Curvature Gravities

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Abstract

We define and compute the energy of gravitational systems involving terms quadratic in curvature. While our construction parallels that of Einstein gravity, there are significant differences both conceptually and concretely. In particular, for $D = 4$, all purely quadratic models admit (zero energy) vacua of arbitrary constant curvature. The energy of all quadratic models, including conformal Weyl gravity, necessarily vanishes in asymptotically flat spaces. Instead, in $\Lambda \neq 0$ backgrounds, the energy expressions are proportional to the usual (AD) form of cosmological Einstein gravity, and therefore to the mass parameter in the corresponding asymptotic Schwarzschild-(Anti) deSitter geometry. Combined Einstein-quadratic curvature systems reflect the above results: Absent a cosmological constant term, the only vacuum is flat space, with (ADM) Einstein energy. With an explicit $\Lambda$ term, the energy is just the sum of the separate AD contributions. We also discuss higher curvature invariants and $D > 4$ spaces, where obstacles to a useful energy arise.

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General relativity is no different from other effective low energy models, acquiring higher momentum corrections, represented locally by higher derivative terms, at smaller scales. By coordinate invariance, these consist of powers of curvatures (and their derivatives), the lowest of which are quadratic. We will initially work in $D = 4$, which turns out to be somewhat special. Up to the topological Gauss-Bonnet invariant $\int (R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2)$, the generic action is

$$I = \frac{1}{16\pi\kappa} \int d^4 x \sqrt{-g}(\alpha R^2 + \beta R^2_{\mu\nu}),$$

which for $\beta = -3\alpha$ includes conformal Weyl gravity; here $\kappa$ is of dimension $[ML]^{-1}$, leaving $(\alpha, \beta)$ dimensionless. Although the notion of “charge” in generic non-abelian gauge theories is well-understood [1, 2, 3], its application to this class of models seems to be lacking. We intend here to provide a universal definition of energy, and to evaluate it in appropriate asymptotic geometries for theories quadratic (or higher) in curvature, with or without an Einstein term.

Let us recall that there are two necessary aspects to a proper energy definition: firstly, identification of the “Gauss law” whose existence is guaranteed by gauge invariance, secondly choice of the proper vacuum, with sufficient (Killing) symmetries with respect to which global, background gauge invariant, generators can be defined and will always appear as 2-surface integrals. Historically, the first application was of course to Einstein gravity without cosmological term [1] whose natural vacuum is flat background with its (Poincaré) symmetries. The next case, cosmological gravity, is a bit more involved [2], as its vacuum has constant (rather than zero) curvature, necessarily dictated by the cosmological constant $\Lambda$ in the action. The relevant asymptotic symmetries are $SO(4,1)$ or $SO(3,2)$. Consider first the pure quadratic gravity of (1). Although the equivalents of the Gauss law still exist and are still the $(0^\mu)^V V_\lambda = R^V_{\mu\lambda\sigma} V_\sigma$, $R_{\mu\nu} = R_{\mu\lambda\nu\lambda}$, the field equations are

$$2\alpha R (R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R) + (2\alpha + \beta)(g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu) R + \beta \Box (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) + 2\beta (R_{\mu\sigma\nu\rho} - \frac{1}{4} g_{\mu\nu} R_{\sigma\rho}) R^{\sigma\rho} = \kappa \tau_{\mu\nu},$$

(2)
where we have introduced a matter source $\tau_{\mu\nu}$. Now decompose the metric into the sum of a background (that solves the source-free equation) plus deviation,

$$g_{\mu\nu} \equiv \bar{g}_{\mu\nu} + h_{\mu\nu}. \quad (3)$$

As originally explained in [2], which we follow here, we linearize (2) about $\bar{g}_{\mu\nu}$ and move all terms nonlinear in $O(h)$ into the right hand side as sources, in order to attain an equation of the form

$$O(\bar{g})_{\mu\nu\alpha\beta}h^{\alpha\beta} = T_{\mu\nu}, \quad (4)$$

where $O(\bar{g})$ is a hermitian operator, depending only on the background. It inherits a background Bianchi identity and background gauge invariance from the full theory, namely

$$\nabla_{\mu}O(\bar{g})_{\nu\alpha\beta} = O(\bar{g})_{\nu\alpha\beta}\nabla_{\alpha}V_{\beta} = 0.$$ As a consequence of these invariances, it is guaranteed that if the background $\bar{g}_{\mu\nu}$ is a vacuum that admits Killing vectors $\bar{\xi}_{\mu}$: $\nabla_{\mu}\bar{\xi}_{\nu} + \nabla_{\nu}\bar{\xi}_{\mu} = 0$, then there are associated conserved charges, and they are expressible as 2-surface integrals,

$$Q^\mu(\bar{\xi}) = \int dS_i \mathcal{F}^{\mu i} \quad (5)$$

where $\mathcal{F}^{\mu i}$, an anti-symmetric tensor derived from $O(\bar{g})$, depends on the specific model. The energy is simply the charge corresponding to a time-like Killing vector.

It is here that the first departure from the Einstein framework occurs: the theories of (1) are scale-invariant and have no unique vacuum: Any constant (or zero) curvature space provides a candidate background. Let us linearize (2) about (A)dS background $\bar{g}_{\mu\nu}$, which moves all indices, to obtain

$$T_{\mu\nu} = (2\alpha + \beta)(\bar{g}_{\mu\nu}\Box - \nabla_{\mu}\nabla_{\nu} + \Lambda \bar{g}_{\mu\nu})R_{L} + 4\Lambda(2\alpha + \frac{\beta}{3})\mathcal{G}^{L}_{\mu\nu} + \beta\Box \mathcal{G}^{L}_{\mu\nu} - \frac{2\beta\Lambda}{3}\bar{g}_{\mu\nu}R_{L}, \quad (6)$$

where $\mathcal{G}^{L}_{\mu\nu} \equiv R^{L}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R_{L} - \Lambda h_{\mu\nu}$ with $\nabla^{\nu}\mathcal{G}^{L}_{\mu\nu} = 0$; we define $\bar{R}_{\mu\nu\rho\sigma} = \frac{1}{3}(\bar{g}_{\mu\nu}\bar{g}_{\rho\sigma} - \bar{g}_{\mu\rho}\bar{g}_{\sigma\nu})$, so that $\bar{R}_{\mu\nu} = \Lambda \bar{g}_{\mu\nu}$. The degenerate, $\Lambda = 0$, case $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$, just leads to $T_{\mu\nu} \to (\partial\partial R^{L})_{\mu\nu}$, which necessarily implies that the energy vanishes: this is an obvious aspect of the fact that equations of the form $\nabla^{4}\phi = \rho$ are solved by $\phi \to r[\int d^{3}x \rho]$: energy and source are
obviously not related by a Poisson operator. [This remark directly accounts for the well-known result [4] that energy in Weyl gravity vanishes for asymptotically flat metrics, just as a particular case of \((\alpha, \beta)\) system.] We emphasize that while energy is too degenerate to be meaningful here, this in no way excludes asymptotically flat solutions, nor the usefulness of Hamiltonian methods.

We come next to the generic case of \(\bar{g}_{\mu\nu}\) with \(\Lambda \neq 0\). Here the linearization produces a universal effect: We find that

\[
8\pi\kappa Q^\mu(\bar{\xi}) = 2\Lambda(4\alpha + \beta) \int d^4 x \sqrt{-\bar{g}} \bar{\xi}_\nu \bar{G}^{\mu\nu}_L \\
+ (2\alpha + \beta) \int dS_i \sqrt{-g} \left\{ \bar{\xi}^\mu \bar{\nabla}^i R_L + R_L \bar{\nabla}^\mu \bar{\xi}^i - \bar{\xi}^i \bar{\nabla}^\mu R_L \right\} \\
+ \beta \int dS_i \sqrt{-g} \left\{ \bar{\xi}_\nu \bar{\nabla}^i \bar{G}^{\mu\nu}_L - \bar{\xi}_\nu \bar{\nabla}^\mu \bar{G}^{i\nu}_L - \bar{G}^{i\nu}_L \bar{\nabla}^\mu \bar{\xi}_\nu + \bar{G}^{\mu\nu}_L \bar{\nabla}^\nu \bar{\xi}_\nu \right\}.
\]

(7)

Here the integral in the first line is the standard (AD) charge of cosmological Einstein gravity [2], itself a 2-surface integral of course. In obtaining the above, gauge-invariant, surface form of the charge, one organizes the terms to exhibit antisymmetry in \(\mu\) and \(i\). Simple as this result is, it becomes even nicer when we turn to the evaluation of the relevant asymptotic solutions, namely the Schwarzschild-(A)dS-(SdS)- metrics. [ Here, a major difference between dS and AdS, that the former has an intrinsic horizon, enters. As explained in [2], the dS energy definition is strictly valid only inside the horizon, where the relevant Killing vector stays time-like: This restriction also logically entails that the black hole horizon be small compared to the cosmological horizon. We do not discuss the question of global definability or usefulness of dS energy [6], as it is really separate from the choice of dynamical model. No such problem affects the AdS case, where the surface integrals may be taken at spatial infinity.] In cosmological Einstein gravity, \(Q^0 = \int \bar{\xi}_\nu \bar{G}_L^{0\nu}\)

indeed gives the desired value \(8\pi MG\), where \(M\) is the “Schwarzschild” mass. Here, we find that the extra, second and third, lines of (7) all vanish for SdS spaces, so generically the energy is proportional to that of cosmological Einstein gravity:

\[
E = \frac{4\Lambda r_0}{\kappa}(4\alpha + \beta),
\]

(8)
where $r_0$ is the coefficient of $1/r$ in the usual static form of SdS, i.e. the monopole moment of the total source, $\rho$, including (as always) gravitational contributions \(^2\); it becomes proportional to the source mass, $m = \int d^3x \tau_{00}$, for weak fields and sources, just as in Einstein theory. Note that Weyl theory has non-vanishing energy unlike the special case $\beta = -4\alpha$, whose action reduces to the square of the traceless Ricci tensor $\tilde{R}_{\mu\nu} = R_{\mu\nu} - \frac{1}{4}R$, and vanishes in (A)dS.

We consider now the combined Einstein plus quadratic curvature theories. If there is no explicit $\Lambda$ term, then constant curvature spaces are not solutions of the combined equations, and we are forced to flat background: consequently, the Einstein term’s energy expression is the whole story (which does not mean that the quadratic terms do not contribute, as sources, to its value!). If instead, a $\Lambda$-term is also present, then constant curvature with precisely that $\Lambda$ value is not only an allowed, but the unique vacuum; the scale is now fixed by the Einstein part, and the energy is

$$E = \frac{r_0}{G} + \frac{4\Lambda r_0}{\kappa}(4\alpha + \beta).$$

Here, $r_0$ is the “Schwarzschild mass” that solves the Poisson’s equation with contributions from both $R$ and $R^2$ parts. For weak fields and sources, of course, $E$ reduces to $m$.

Thus far, we have worked in $D = 4$ and considered only models with at most quadratic terms. Depending on their physical origins, higher powers may be viewed as a part of the fundamental action (this is obvious if there is no Einstein term, as in Weyl gravity), or as small corrections that should not be part of the “kinetic” term and also not parts of the energy expressions though they still affect their values. There is a non-trivial point in the latter case, if these terms do not fall off sufficiently rapidly as sources of the Gauss equation. Take, for example, a generic higher curvature invariant $\int R^n$, $n > 2$, $R$ representing Riemann, Ricci or scalar curvatures, possibly also acted on by (an even number of) covariant derivatives. Schematically, this gives rise to a field equation contribution of the form $(R^n)_{\mu\nu} + (\nabla \nabla R^{n-1})_{\mu\nu}$. Just as for $n = 2$, its linearization about flat space does

\(^2\)More precisely the Gauss equation is of the form $\Lambda \nabla^2 \phi = \kappa \rho$, so that $r_0 = (\kappa/\Lambda)M$; the effective gravitational constant is $(\kappa/\Lambda)$ here.
not affect the energy. However generically these terms do not allow constant curvature solutions, since they are not homogeneous of order zero in the metric: $\bar{R}_{\mu\nu}$ does not vanish even though $\nabla\nabla(\bar{R}^{n-1})_{\mu\nu} = 0$ does. While the linearization, $\Lambda^{n-2}[\Lambda + \nabla\nabla]R_{L}$, does resemble that of $n = 2$, the background part $(\bar{R}^{n})_{\mu\nu}$, acts as a constant source in the Gauss equation. This story is similar to the situation for higher dimensions, where even quadratic corrections are no longer homogeneous of degree zero and so forbid (A)dS backgrounds. Consider in fact the generic quadratic action

$$I = \int d^{D}x \sqrt{-g}(\alpha R^{2} + \beta R_{\mu\nu}^{2} + \gamma R_{\mu\nu\sigma\rho}^{2}); \quad (10)$$

the condition [see also [5]] for (A)dS to be a solution is $(D - 4)[2\gamma + (D - 1)(D\alpha + \beta)] = 0$. For $D \neq 4$, this is a constraint on the allowed mixture in (10). Not in particular that $\gamma = 0$ requires the traceless combination that led to $E = 0$ in four dimensions; while this also allows Schwarzschild-dS, the energy again vanishes when $\beta = -D\alpha$. Generically, when $\gamma \neq 0$, even if the equality is satisfied so (A)dS is permitted, SdS is not; it seems difficult to extract a useful energy under these conditions.

In summary, we have defined energy for arbitrary general covariant gravitational models, particularly the simplest, quadratic curvature, systems. In $D = 4$, pure quadratic actions have non-vanishing energy in vacua with constant but non-zero curvature. While these are infinitely degenerate, their value for any chosen $\Lambda$ is quite physical, being proportional to that of the Schwarzschild mass in the relevant SdS metric. Einstein plus quadratic models differ in imposing a unique background; here the total energy is the sum of contributions proportional to the cosmological Einstein mass, if there is a $\Lambda$-term, and just equal to the ADM mass if $\Lambda = 0$. We also noted the problematical aspects of energy usefulness under rather general circumstances: First, if higher powers of $R$ are present, (A)dS is no longer a permitted background-only flat vacuum is. Second, terms involving powers of the full Riemann tensor generally forbid SdS external metrics. In either of these cases, the energy seems to lose much of its usefulness. Details will be presented elsewhere.

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References


