Prompt Quark Production by exploding Sphalerons

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(June 5, 2002)

Following recent works on production and subsequent explosive decay of QCD sphaleron-like clusters, we discuss the mechanism of quark pair production in this process. We first show how the gauge field explosive solution of Luscher and Schechter can be achieved by non-central conformal mapping from the O(4)-symmetric solution. Our main result is a new solution to the Dirac equation in real time in this configuration, obtained by the same inversion of the fermion O(4) zero mode. It explicitly shows how the quark acceleration occurs, starting from the spherically O(3) symmetric zero energy chiral quark state to the final spectrum of non-zero energies. The sphaleron-like clusters with any Chern-Simons number always produce \( N_f \) fermions, and the antischphaleron-like clusters the chirality opposite. The result are relevant for hadron-hadron and nucleus-nucleus collisions at large \( \sqrt{s} \), wherein such clusters can be produced.

I. INTRODUCTION

Although it would hardly ever be possible to observe instanton-induced events in electroweak theory, intense studies of such processes have started in this context [1]. Even prior to these studies, the decay of the sphaleron solution [2] has been studied in great details numerically [3].

It was found that the Chern-Simons number at the exit state is frozen at some non-integer value. The question then is what happens to the relationship between the anomaly and the fermion chirality flow? Naively, the two concepts are related, yet the fermion spectral flow can only accommodate integer numbers. Should there be a generalization of index theorem for non-zero fields radiated to infinity? This question remained unanswered in general, although several points of view and partial results have been expressed in the literature. In this paper we would not address these questions in general, working out the solution explicitly, by finding exact solutions to the Dirac equation in exploding background fields, which demonstrates in detail how quark production and subsequent acceleration occurs.

Recently the whole discussion of such questions has shifted into the QCD domain. In refs [4,5] it was suggested that nonperturbative configurations composed of instanton/antistanton play an important role in parton-parton scattering amplitudes at high-energy, and may account for some part of the soft pomeron: the estimates resulted in the slope and intercept in approximate agreement with data. Detailed discussion of a specific gluonic clusters which are produced have been made [11], with some proposed applications to high energy heavy ion collisions [16].

In the calculation of the cross section both the action of the gauge field corresponding to emitted gluons, or instanton-instanton interaction, can be naturally combined with the overlap factor with the incoming high energy partons in the so called Landau method, in which the semiclassical approximation with singular gauge configurations is used [7,8]. Pair singularities of the gauge field on the Euclidean time axis play the role of coordinate infinity in the original Landau work on quantum mechanics. Those singularities are key to interpolating between a vacuum configuration with zero energy, and the escape configuration with finite energy \( Q(T) \), where \( T \) measures the finite tunneling time. In a recent paper [9] we have shown that the gluon multiplicity associated to the escaping configurations at finite \( Q \) follows from the configuration at the sphaleron point \( M_S \) through a pertinent rescaling of the sphaleron size and energy density.

In the present work we continue along the same line, focusing now on the quark production enforced by such exploding sphalerons configurations. In section 2, we show how a conformal mapping from the O(4) conformally symmetric solutions of the Euclidean Yang-Mills equations leads to the O(3) symmetric solutions by Luscher and Schechter [10] which describes the sphaleron explosion [11]. In section 3, we construct the normalizable zero mode solutions to the O(4) Yang-Mills background, which includes the instanton zero modes as a special case. In the central sections 4,5 we show that the solution to the massless Dirac equation in the Minkowski background field of exploding sphalerons can also be obtained by the same conformal mapping, from the O(4) Euclidean zero modes. We explicitly construct these states and show that at the initial Minkowski time \( t=0 \) those are zero energy states, while at asymptotically large time they reduce to a free quark or free antiquark of specific chirality. We also calculate the spectrum of the produced fermions. In section 6, we argue that pair production in the sphaleron (antischphaleron) background is through pair level crossing at \( t = 0 \). Our conclusions are in section 7.

II. LS SOLUTION FROM O(4) BY INVERSION

In this section we show that the spherically symmetric O(3) Luscher-Schechter (LS) solution [10] describing the explosion of the sphaleron-type clusters [11] can be obtained from the O(4) symmetric solution through a pertinent inversion that breaks O(4). Similar considerations for the \( \phi^4 \) model were originally discussed in [12]. In our case, consider the O(4) symmetric ansatz for the SU(2) gauge configuration.
The gauge configuration at $t = 0$ satisfies $A_0^a = 0$ and $A_i^a = 0$ which corresponds to zero chromoelectric field (zero momentum) and finite chromomagnetic field. The $t = 0$ point is a classical turning (or escape) point for the gauge configuration, where Euclidean and Minkowski parts of the path join together, see more on that in [11].

### III. THE O(4) SYMMETRIC ZERO MODES

In this section we will derive the generic zero mode solution to the Dirac equation in the O(4) background (1). Our discussion in this section is limited to just one massless quark flavor, and will parallel the discussion in [13] for the instanton. For that, we introduce the conventions

$$\sigma^\pm_\mu = (1, \mp i\sigma)$$

which are related to the t’Hooft symbols through

$$\sigma^\pm_\mu \sigma^\pm_\nu = 1_{\mu\nu} + i \eta_{\mu\nu} \sigma_\alpha = \eta_{\mu\nu} \sigma^\pm_\alpha$$

$$\sigma^\pm_\mu \sigma^\mp_\nu = 1_{\mu\nu} - i \eta_{\mu\nu} \sigma_\alpha = \eta_{\mu\nu} \sigma^\mp_\alpha$$

In terms of (10-11) the Dirac operator in the background (1) reads

$$D^- = \left( \partial_\mu + \frac{1}{2} \partial_\nu F_{\mu c} \sigma^+_{\mu c} \right) \sigma^+_{\mu c}$$

while in the dual background it reads

$$D^+ = \left( \partial_\mu + \frac{1}{2} \partial_\nu F_{\mu c} \sigma^-_{\mu c} \right) \sigma^-_{\mu c}$$

where the subscripts $c, s$ refer to the color, spin SU(2) matrices. The coupling $g$ drops in the spectrum of $D_\pm$ making the quark states of order $g^0$, while the gauge fields are of order $1/g$. In the strict semiclassical approximation, the quark effects on the semiclassical gauge configurations can be ignored. We note that the combination

$$P_+ = \frac{1}{4} \sigma^-_{\mu s} \sigma^+_{\mu c} = \frac{1}{4} \sigma^+_{\mu s} \sigma^-_{\mu c}$$

projects onto a color-spin singlet. In particular $(U_+ = \sigma_2)_{aa}$ is an eigenstate of $P_+$, i.e. $P_+ U_+ = U_+$. Finally, let $\pm L, R$ chiralities for which $\gamma_5 \Psi_\pm = \pm \Psi_\pm$. In the $r$-field ($D_+$) or $\eta$-field ($D_-$) the zero modes carry specific chirality $D_\pm \Psi_\pm = 0$.

To construct the normalizable zero mode state in the configuration (1), we note that only the positive chirality state is ‘bound’

$$\left( \sigma^\pm_\mu \partial_\mu + 2 P_+ \sigma^-_\mu \partial_\mu F \right) \Psi_+ = 0$$

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1Here $\eta_{\mu\nu} = -\eta_{\mu\nu}$. 

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The instanton and sphaleron configurations follow from a similar construction with a dual background $\eta \rightarrow \bar{\eta}$. The O(4) solutions to the Yang-Mills equations with a turning point at $\xi = 0$ relates to the LS solution [10] with a turning point at $t = 0$ by the conformal transformation

$$(x + a)_\mu = \frac{2\rho^2}{(y + a)^2} (y + a)_\mu$$

with $a = (\rho, \bar{0})$, which maps the sphere $y^2 = \rho^2$ onto the upper-half of the x-space. Indeed, under the conformal transformation (7), (1) transforms as

$$A_{a\mu}(x) = \frac{(y + a)^2}{2\rho^2} \left( g_{\mu\nu} - 2 \frac{(y + a)_\mu (y + a)_\nu}{(y + a)^2} \right) A_{a\nu}(y),$$

with $y$ solving (7). The LS solution in Minkowski space follows by an analytical continuation $x_4 \rightarrow it$,

$$A_0^a = \frac{f(\xi)}{g} \, 8 \rho \, tr \, n_a \frac{\sqrt{2}}{(t - i\rho)^2 - r^2)(t + i\rho)^2 - r^2)}$$

$$A_i^a = \frac{f(\xi)}{g} \left( \frac{4 \rho (\delta_{ai} (t^2 - r^2 + \rho^2) + 2 \rho \epsilon_{aij} n_j + 2 n_a n_j)}{(t - i\rho)^2 - r^2)(t + i\rho)^2 - r^2)} \right).$$
while the negative chirality state remains free. The general solution to (15) can be sought in the form

$$\Psi_+(y) = C \sigma_{\mu}(\partial_{\mu} \Phi) U_+$$

(16)

up to an overall rigid color rotation and normalization (see below). Inserting (16) into (15) and using the identity

$$P_+ \sigma_{\alpha c} \sigma_{\beta s} U_+ = \delta_{\alpha \beta} U_+$$

(17)

it follows that

$$\Box \Phi = -2 (\partial_{\mu} F)(\partial_{\mu} \Phi)$$

(18)

which is solved by

$$\frac{d\Phi}{dy} = \frac{1}{y^3} e^{-2 F(\xi(y))}$$

(19)

where $F(\xi)$ is given by (2). Using the identity (17) for the norm of (16) it follows that the overall constant is fixed by

$$C = \left(2 \int d^4 y \Phi^2(y)\right)^{-1/2}.$$  

(20)

The expression (16) thus defines the Euclidean normalized positive chirality ($R$) quark state in the O(4) background (1). Since the O(4) Dirac spectrum is charge self-conjugate, the O(4) background admits also a normalized negative chirality ($L$) anti-quark state in the O(4) background, i.e. $\Psi_+^\dagger$. The O(4) background admits a pair of zero modes with opposite chirality.

IV. SOLVING THE DIRAC EQUATION BY INVERSION

In this section we use the conformal transformation (7) to map the pair of O(4) zero mode states with turning point at $\xi = 0$, onto a pair of O(3)-symmetric solutions which solves the Dirac equation in the background LS configuration described in section 2. They are localized (normalizable) in 3-space at all times. They describe propagation of the produced quarks in Minkowski space-time into asymptotically free outgoing chiral quarks, with computable spectra. We will also show that those solutions in fact start with zero energy at the turning time $t = 0$: this implies that the sea contribution for positive and negative energy states is symmetric and cancels in the chiral charge.

The solution itself is obtained by the following inversion formula

$$Q_+(x) = \gamma_4 \frac{\gamma_\mu(y+a)_{\mu}}{1/(y+a)^2} \Psi_+(y)$$

(21)

and it solves the $(\gamma_4 \times)$ Dirac equation in the (Euclidean) LS gauge configuration (9). Under the conformal transformation (7) followed by $\gamma_4$ multiplication, the right-handed zero mode state (16) remain right-handed. Unwinding $y$ in terms of $x$ and analytically continuing to Minkowski space yield

$$Q_+(t,r) = \frac{C}{(t+i)^2 - r^2} \times \left((\rho - it) + i \vec{\sigma} \cdot \vec{x}\right) e^{-2 F(\xi)} U_+,$$

(22)

with

$$\xi = \frac{1}{2} \ln \left(\frac{(t-\rho)^2 - r^2}{(t+\rho)^2 - r^2}\right).$$

(23)

The normalization is fixed at $t = 0$ by noting that $F(0) = 0$, i.e.

$$Q_+(0,r) = \frac{C}{(\rho^2 + r^2)^2} (\rho + i \vec{\sigma} \cdot \vec{x}) U_+,$$

(24)

and demanding that in 3-space,

$$2 |C|^2 \int d^3 x \frac{1}{(\rho^2 + r^2)^3} = 1,$$

(25)

which gives $|C| = \sqrt{2/\rho^4/\pi}$. We note that the norm is independent of which LS configuration is chosen, making all mapped O(3) states normalizable.

The next question we address is whether the solution for the exploding sphaleron indeed starts with the zero energy state. Naively, at small $t$ it does not look as a static solution at all. However, one can observe that the initial time derivative of (22) fulfills

$$-i \dot{Q}_+(0,r) = \frac{(3 - 8 f(0)) \rho - i \vec{\sigma} \cdot \vec{x}}{\rho^2 + r^2} Q_+(0,r).$$

(26)

This may be rewritten as a gauge phase

$$-i \dot{\hat{Q}}_+(0,r) = \Lambda(0,r) Q_+(0,r),$$

(27)

with SU(N) and U(1) gauge phases. The former is

$$\ln \Lambda = -it \frac{\vec{\sigma} \cdot \vec{x}}{\rho^2 + r^2},$$

(28)

while the latter, $\hat{\Theta} = e^{i\Theta} Q_+$, is

$$\Theta = \frac{(3 - 8 f(0)) \rho}{\rho^2 + r^2}.$$  

(29)

Since $A_0 = 0$ at $t = 0$, the configurations (22) have zero energy modulo an SU(2)×U(1) gauge transformation provided that $f(0)$ is real. The SU(2) is related to

2Note that we are still using the notation $C$ for the normalization of the zero energy state.
the color gauge group, and the U(1) to an additional local symmetry of the Dirac Hamiltonian $H(t) = \gamma_4 \gamma_i \cdot \nabla$ at $t = 0$, namely

$$e^{it \Theta} H(t) e^{-it \Theta} = H(t) - it \gamma_4 \gamma_i \cdot \nabla \Theta$$  \hspace{1cm} (30)

The solution thus looks static for the modified Hamiltonian, and the difference between the transformed and original solution is absent at $t = 0$. We thus conclude that we do start from the zero energy state, masked by gauge transformations. All the O(3) right-handed quarks at and above the sphaleron point are normalizable zero energy states to the Minkowski Dirac equation with the LS background.

At large times $t \sim r \gg \rho$ with $v = r - t$, (22) simplifies to

$$Q_+(t, r) = \frac{-i C}{4t(v - i \rho)^2} (1 - \vec{\sigma} \cdot \hat{r}) e^{2F(\xi) \vec{\sigma} \cdot \hat{r}} U_+ , \hspace{1cm} (31)$$

with

$$\xi = \frac{1}{2} \ln \left( \frac{v + i \rho}{v - i \rho} \right) . \hspace{1cm} (32)$$

We have used the relation

$$e^{-2F(\xi) (1 - \vec{\sigma} \cdot \hat{r})} = (1 - \vec{\sigma} \cdot \hat{r}) e^{2F(\xi) \vec{\sigma} \cdot \hat{r}} , \hspace{1cm} (33)$$

to display the exponential as an SU(2) hedgehog gauge rotation, since $F(\xi)$ is purely imaginary asymptotically. The quark field is localized and normalized at $t = 0$ and weaken asymptotically as $1/t$ up to a gauge transformation. This is a hallmark of a radiation field which admits a normal mode decomposition as we will show below. The normalization is preserved by the time-dependent evolution since

$$\int d^3x |Q_+(t, r)|^2 = \pi |C|^2 \int_{-\infty}^{+\infty} \frac{dv}{(v^2 + \rho^2)^2} = 1 . \hspace{1cm} (34)$$

The radial density is then given by

$$\rho_+(t, r) = 4\pi r^2 \text{Tr} \left( Q_+^\dagger Q_+ \right)(t, r) \hspace{1cm} (35)$$

that is

$$\rho_+(t, r) = \frac{16}{\pi} e^{-4\Re F(\xi)} \frac{\rho^3 r^2 (\sqrt{t^2 + \rho^2} + r^2)}{(t^2 + \rho^2 - r^2)^2 + 4r^2 \rho^2)^2} \hspace{1cm} (36)$$

at the (anti)sphaleron point as a function of the evolution time $t/\rho = 0, 1, 3, 6$. At the escape point $t = 0$, the zero energy state is fully localized in space. At large times $t > \rho$ the energy density rapidly obtains a frozen form, and eventually the final quarks move luminally as free waves. A sample of density profiles is shown in Fig.1, where it is easily seen that the time needed to reach the asymptotic form is $t_* \sim \rho$.

V. SPECTRAL ANALYSIS OF THE OUTGOING QUARKS

A normal mode decomposition allows for a spectral analysis of the final quark states released by the anti-sphaleron (sphaleron). Using the free field decomposition asymptotically,

$$Q_+(t, k) = (2\pi)^2 \frac{1}{\sqrt{2k}} \left( q_R(k) e^{-ikt} + q_L^\dagger(-k) e^{ikt} \right) , \hspace{1cm} (37)$$

allows the identification of the 1+1 right-mover $q_R$ with the right-handed asymptotic chiral quark and the 1+1 left-mover $q_L^\dagger$ with the left-handed asymptotic chiral anti-quark. In the antispaleron background we have

$$\frac{\sqrt{2k}}{(2\pi)^{3/2}} Q_+(t, k) = C \sqrt{\pi} k \theta(k) e^{-k \rho - ikt} (1 - \vec{\sigma} \cdot \hat{k}) U_+ \hspace{1cm} (38)$$

so that

$$q_L^\dagger(k) = 2\rho^3 \sqrt{\frac{\pi}{k}} e^{-k \rho} (1 - \vec{\sigma} \cdot \hat{k}) U_+ \hspace{1cm} (39)$$

$q_R$ follows from (39) by charge conjugation after mapping the O(4) conjugate zero mode to the O(3) zero energy state,

$$q_R(k) = 2\rho^3 \sqrt{\frac{\pi}{k}} e^{-k \rho} (1 + \vec{\sigma} \cdot \hat{k}) U_+ \hspace{1cm} (40)$$

The (chiral) density of left antiquarks and right quarks are opposite asymptotically $n_L = -n_R$, with
Fig. 2 shows the phase space distribution $n_R/\rho$ of the right quarks released at the sphaleron point as a function of $k\rho$. The distribution integrates exactly to one produced quark,

$$n_R = \int_0^\infty dk n_R(k) = 1.$$  \hspace{1cm} (42)

The quark spectrum is close to Planckian with an effective temperature $T = 2/\rho$ of about 300 MeV for a standard $\rho = 1/3$ fm. Incidentally, this is close to the initial temperature of a quark-gluon plasma in the RHIC energy domain. The released quarks carry asymptotically a total energy

$$M_F = 2 \int_0^\infty dk k n_R(k) = \frac{3}{\rho},$$ \hspace{1cm} (43)

which is small in the weak coupling limit under consideration, i.e. $M_F/M_S = \alpha/\pi \ll 1$. We recall that in [9] the number of prompt gluons released was evaluated at $1.1/\alpha_s(\rho)$, which is parametrically large in this limit. For this reason, back reaction of fermions onto gluons can be neglected.

The situation is however quite different for ‘typical’ QCD instantons and sphalerons, with $\rho \sim 1/3$ fm. In this case the produced $2N_F = 6$ quarks\(^3\) should be compared to $\sim 3$ gluons released classically from the explosion at the sphaleron point. The total energy carried by one flavor species would be $3/\rho \sim 1.8$ GeV, while the whole available energy (the sphaleron mass) is only about 3 GeV. Needless to say that such results follow from a wrong assumption: in this regime the fermions are not a small perturbation riding on the back of the gluonic wave. Their back reaction on the YM fields cannot be ignored. This implies a joint solution of the coupled Yang-Mills and Dirac equations is needed, better yet with vacuum polarization effects included. Although such analysis is still missing, we may speculate that the spectra would approximately look the same, with a rescaled energy per quark appropriate for total energy conservation.

VI. PAIR PRODUCTION BY SPECTRAL FLOW

We have established that the release of an anti-sphaleron liberates $N_F \overline{LR}$ pairs, while the release of a sphaleron liberates $N_F \overline{RL}$ pairs. This mechanism operates for all energies $Q$ above and including the sphaleron point, and is signaled by a pair of energy levels crossing at $t = 0$ (pair production) in the form of zero energy states, with their subsequent acceleration by pertinent gluoelectric fields.

Compare this to what happens at low energies, when the gauge field configurations are well-separated instanton and anti-instanton, and the production of $2N_f$ fermions is described by the well-known ’t Hooft vertex. Although we have not shown it explicitly, we conjecture that the same number of fermions is produced at any energy, by continuity.

On the other hand, the escaping gauge configurations above the sphaleron point were shown to carry fractional Chern-Simons number [9]

$$N(0) = \frac{1}{2} \left( \frac{Q}{M_S} \right)^{2/5}. \hspace{1cm} (44)$$

The non-integer character of (44) calls for a key question: How does the conventional Adler-Bell-Jackiw (ABJ) anomaly with integer fermion number work? The answer is as follows: The net chirality in the Dirac spectrum is carried by both the valence and sea parts,

$$\Delta n(t) = (n_R - n_L)(t) = \Delta n_V(t) + \Delta n_S(t). \hspace{1cm} (45)$$

At $t = 0$ the Dirac spectrum associated to $H(t)$ is C-conjugate in the O(3) LS configuration except for the two zero energy states,

$$\Delta n(0) = \Delta n_S(0) = 2N_F N(0). \hspace{1cm} (46)$$

For $t > 0$, one level dives in and one-level pulls out. The chirality count changes discretely by 2. In particular,

$$\Delta n_V(t) = 2N_F$$

$$\Delta n_S(t) = 2N_F(N(t) - 1) \hspace{1cm} (47)$$

\(^3\) We have ignored strange quark mass here, because the corresponding scale at which the development occurs is $\sim 1/\rho \sim 1$ GeV.
weaken the valence part is t-independent. Asymptotically, the chiral polarization in the Dirac spectrum carried by the sea quarks vanishes $\Delta n_S(\infty) = 0$, since the $O(3)$ asymptotic configuration is commensurate with plane waves, thus C-conjugate. The net result is

$$\Delta n(\infty) = \Delta n_V(\infty) = 2N_F,$$

which is the result explicitly obtained from the zero energy states released.

Some unsatisfied readers may ask how one can generalize index theorems, so that the number of level crossing would be calculable directly from the gauge field itself, without the explicit solution of the Dirac eqn. Or, in a more practical form of the same, Does the conformal mapping exhausts all level crossings of the time-dependent Hamiltonian $H(t)$ for all $Q \geq M_S$?

Interesting work toward answering these questions can be found in recent work [14], where one can also find earlier references. In particular, it was found that the answer to the last one is negative. In a numerical analysis presented there it was found that (in our units) for $Q > 19.45M_S$ other crossings in the LS background do occur. (However, these crossings are for practical purposes irrelevant since the cross section there is clearly too small.)

To summarize: Sphaleron (antisphaleron) production leads to pair production of chiral quarks by spectral flow, and it produces one pair of quarks for each light flavor with unit probability. This nonperturbative mechanism for prompt chiral production is not suppressed by powers of $\alpha_s$ as in pQCD, or exponentially hampered by the constituent quark masses as in the string production mechanism. The qualitative effects of the current quark masses is to cause a power suppression in the production mechanism as opposed to the exponential suppression in the string.

VII. CONCLUSIONS

We have started with zero mode solutions to the $O(4)$ configurations in Euclidean space that solves exactly the Yang-Mills equations. We then have shown that through a pertinent conformal mapping and analytical continuation they map exactly on the spherically $O(3)$-symmetric LS solutions [10] with a turning point at $t = 0$ describing explosion of the YM sphaleron-like clusters [11]. We then have used the same conformal mapping to construct a class of normalizable solutions to the Dirac equation, in the LS background. We have shown that these solutions do correspond to the zero energy states at $t = 0$ for all LS configurations with energies larger or equal to the sphaleron mass.

By following their evolution in time, we have explicitly shown that the zero energy states becomes at large time free Dirac outgoing waves with fixed chirality $\pm 1$ and exactly $1$ quantum. Thus sphaleron (antisphalerons) production liberates $2N_F$ chiral quarks. We have identified this liberation by a pair level crossing at $t = 0$ in the Dirac spectrum. We have used the asymptotic zero energy states to construct the pertinent quark spectra produced.

Many more things can be done based on the results reported here. One is an account of back reaction of fermions on the YM fields, as discussed in the previous section. Another is a projection of the outgoing waves to the wave function of outgoing hadrons, with many detailed predictions about exclusive and inclusive hadronic production to follow as we will report elsewhere.

The present results are of interest to both hadron-hadron and nucleus-nucleus collisions at large $\sqrt{s}$. Simple arguments indicate that while in hadron-hadron scattering production via sphalerons, it is only a small part of the multiplicity, in nucleus-nucleus collisions it is not so and hundreds of sphalerons may be released, contributing substantially to the total entropy produced [16]. In view of that, the production of $2N_F$ chiral quarks per cluster may make the quark-gluon plasma produced in heavy-ion collisions quark rich, contrary to expectations based on perturbation theory or the color glass description [15].

Acknowledgments

This work was supported in parts by the US-DOE grant DE-FG-88ER40388.