A number of interesting investigations have been reported on wave-packet dynamics [1] including in particular recent studies on issues such as the observation of revivals of wave packets [2]. Of late, we [3] pointed out a hitherto unexplored effect by considering the time dependent reflection probability of a Gaussian wave packet reflected from a perturbed potential barrier. By reducing the height of the barrier to zero in a short span of time during which there is a significant overlap with the wave packet, we observed that the reflection probability is larger compared to the case of reflection from a static barrier for a small but finite interval of time. This phenomenon is what we have called “Quantum Superarrivals”. The speed with which the effect due to reducing the barrier height propagates across the wavefunction was noticed to be depending on the rate at which the barrier height is reduced. We also found that the magnitude of superarrivals is proportional to the the rate of reduction of the barrier height propagating across the wavefunction (perturbation of the barrier) across it.

The aim of this paper is to further generalize the phenomenon of superarrivals and understand how superarrivals occur. We begin by first showing that superarrivals also indeed occur in the transmission probability when the barrier height is raised from zero to some value (this is complementary to the superarrival phenomena occurring in the reflected wave packet). We then compute particle trajectories using the Bohm model. We derive a quantitative estimate of the magnitude of superarrivals using Bohmian trajectories. We show that it is possible to obtain a deeper insight into the nature of superarrivals using the computed trajectories of individual particles in the case of a wave packet which is reflected from the perturbed barrier.

Let us first briefly recapitulate the essential features of quantum superarrivals. Consider a Gaussian wave packet originating around \( x_c \) with half width \( \sigma \). It moves to the right a strikes a potential barrier centred at a point \( x_0 \). A detector placed at a point \( x' \) left of the initial position of the wave packet measures the time-dependent reflection probability by counting the reflected particles arriving there up to various instants for both the case of a static barrier, and also when the barrier is perturbed by reducing its height to zero linearly in time. At any instant before the asymptotic value of the reflection probability is attained, the time evolving reflection probability in the region \(-\infty < x \leq x'\) is given by

\[
|R(t)|^2 = \int_{-\infty}^{x'} |\psi(x, t)|^2 \, dx
\]  

We denote the reflected probability for the static and the perturbed cases as \( R_s(t) \) and \( R_p(t) \) respectively. In [3] we computed these probabilities versus time for various values of \( \epsilon \), the time span over which the barrier height goes to zero (implying different rates of reduction of the potential barrier). We observed that \( R_p(t) > R_s(t) \) during the time interval \( t_d < t < t_c \). If \( t_p \) is the instant at which the perturbation starts, \( t_c \) the instant when the static and the perturbed curves cross each other, and \( t_d \) the time from which the curve corresponding to the perturbed case starts deviating from that in the unperturbed case, we found that \( t_c > t_d > t_p \).

Let us now consider the case when initially there is no barrier, and the wave packet is allowed to propagate freely towards the right. A second detector placed far away at \( x'' \) records the time-dependent transmission probability \( T_s(t) \) (counting the transmitted particles up
where the quantities \( I, T \) and \( \Delta t \) are defined with respect to various instants of time. If a barrier is raised in the path of the wave packet, a portion of it will be reflected back. We denote by \( T_p(t) \) the transmitted probability in this case. At any instant before the asymptotic value of the transmission probability (= 1 since there is no dissipation) is attained, the time evolving transmission probability in the region \( x'' \geq x \geq \infty \) is given by

\[
|T(t)|^2 = \int_{x''}^{\infty} \psi(x,t)^2 \, dx 
\]

In Figure 1 we plot the computed values of \( T_s(t) \) and \( T_p(t) \) for different values of \( \epsilon \), the time taken for raising the barrier from 0 to \( V \). It is seen that superarrivals are exhibited in the transmitted wave packet.

Superarrivals can be quantitatively defined by a parameter \( \eta \) given by

\[
\eta = \frac{I_p - I_s}{I_s} 
\]

where the quantities \( I_p \) and \( I_s \) are defined with respect to \( \Delta t = t_c - t_d \) during which superarrivals occur. For the case of superarrivals in the reflected probability,

\[
I_p = \int_{\Delta t} |R_p(t)|^2 \, dt 
\]

\[
I_s = \int_{\Delta t} |R_s(t)|^2 \, dt 
\]

Replacing the static and perturbed reflected probabilities by \( T_s(t) \) and \( T_p(t) \) respectively, one obtains the corresponding expression of \( \eta \) for the case of the transmitted wave packet.

It has been observed [3] that both \( \Delta t \) and superarrivals given by \( \eta \) depend on the instant \( t_p \) around which the barrier is perturbed. The magnitude of superarrivals is appreciable only in cases where the wave packet has some significant overlap with the barrier while it is being perturbed. The magnitude of superarrivals falls off with increasing \( \epsilon \), for the reflected as well as the transmitted wave packets. Another interesting observation is about information transfer from the perturbing barrier to the detector. We defined signal velocity

\[
v_c = \frac{D}{t_d - (t_p - \frac{\epsilon}{2})} 
\]

measuring how fast the influence of barrier perturbation travels across the wave packet. We found that \( v_c \) is again proportional to \( \epsilon \) as was the case with \( \eta \). These features leads one to argue that the wave packet acts as a carrier (objective physical field-like behaviour) through which information about barrier perturbation propagates with a velocity that is proportional to the “disturbance” (reflected in terms of the rate of barrier reduction) imparted on the packet by the barrier.

Now, in order to understand how superarrivals originate, we use the concept of particle trajectories in terms of the Bohm model (BM). We recall that BM provides an ontological and a self-consistent interpretation of the formalism of quantum mechanics [5,6]. Predictions of BM are in agreement with that of standard quantum mechanics. In BM a wave function \( \psi \) is taken to be an incomplete specification of the state of an individual particle. An objectively real “position” coordinate (“position” existing irrespective of any external observation) is ascribed to a particle apart from the wave function. Its “position” evolves with time obeying an equation that can be “derived” in some sense from the Schroedinger equation (considering the one dimensional case)

\[
i\hbar \frac{\partial \psi}{\partial t} = H\psi = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x)\psi 
\]

by writing

\[
\psi = Re^{iS/\hbar} 
\]

and using the continuity equation

\[
\frac{\partial}{\partial x}(\rho v) + \frac{\partial \rho}{\partial t} = 0 
\]

with the probability distribution \( \rho(x,t) \) being given by

\[
\rho = |\psi|^2. 
\]

It is important to note that \( \rho \) in BM is ascribed an ontological significance by regarding it as representing the
probability density of “particles” occupying actual positions and the velocity $v$ is interpreted as an ontological (premeasurement) velocity. On the other hand, in the standard interpretation, $\rho$ is interpreted as the probability density of finding particles around certain positions and there is no concept of an ontological velocity. Integrating Eq.(9) and using Eqs.(7), (8) and (10) and requiring that $v$ should vanish when $\rho$ vanishes leads to the Bohmian equation of motion where the particle velocity $v(x, t)$ is given by [6]

$$v \equiv \frac{dx}{dt} = \frac{1}{m} \frac{\partial S}{\partial x}$$  \hspace{1cm} (11)$$

The particle trajectory is thus deterministic and is obtained by integrating the velocity equation for a given initial position.

Another perspective on the notion of particle trajectories in BM is obtained by decomposing the Schrödinger equation in terms of two real equations for the modulus $R$ and the phase $S$ of the wave function $\psi$ [5]

$$\frac{\partial S}{\partial t} + \frac{\nabla S}{2m} - \frac{\hbar^2}{2m} \nabla^2 R - V = 0$$  \hspace{1cm} (12)$$
$$\frac{\partial R^2}{\partial t} + \nabla \cdot \left( \frac{R^2 \nabla S}{m} \right) = 0$$  \hspace{1cm} (13)$$

and by indentifying

$$Q(x, t) = -\frac{\hbar^2}{2m} \frac{\nabla^2 R}{R}$$  \hspace{1cm} (14)$$

as the “quantum potential” [5]. The equation of motion of a particle along its trajectory can now be written in a form analogous to Newton’s second law

$$\frac{d}{dt}(m \dot{X}) = -\nabla (V + Q)|_{x=X(t)}$$  \hspace{1cm} (15)$$

(with $d/dt = \partial / \partial t + \dot{X} \cdot \nabla$) where the particle is subjected to a quantum force $-\nabla Q$ in addition to the classical force $-\nabla V$. The effective potential on the particle is $(Q + V)$. We plot the profile of $Q$ versus $x$ at various instants of time near the potential barrier (when its height is reduced) in Figure 2. It is transparent how perturbing the classical potential $V$ affects $Q$ away from the immediate vicinity of the boundary of $V$. This accounts for the sharp turn experienced by those particles which contribute towards superarrivals (as we shall see explicitly later).

We compute Bohmian trajectories for a given set of initial positions with a Gaussian distribution corresponding to the initial wave packet. This procedure is carried out for both the cases of lowering and raising the barrier. Since our purpose is to obtain conceptual clarity of the phenomenon of superarrivals, it suffices to illustrate our scheme through the example of superarrivals in the reflection probability when the barrier is reduced. Furthermore, all the qualitative as well as quantitative features of superarrivals are similar in the case where one observes the transmitted probability from a rising barrier. Thus, henceforth we shall consider only the former case in this paper.

The following procedure is used to investigate superarrivals through the Bohmian trajectories. First, a particular value of the barrier reduction rate, or $\epsilon$ is chosen. We then choose a range of initial position for which the trajectory arrival time at the detector lies between $t_d$ and $t_c$ (i.e., corresponding to only those trajectories which contribute to superarrivals). We consider $N$ such trajectories whose initial positions form a Gaussian distribution. Let us denote one such trajectory by $S_{ip}$ having the initial position $x_{ip}$ and the arrival time $t_{ip}$. Turning to the static case, the trajectory $S_i$ for the same initial position $x_i$ is computed. Let the corresponding arrival time be $t_i$. A superarrival parameter $\beta_i$ for the $i$-th Bohmian trajectory is then defined as

FIG. 2. Snapshots of the quantum potential $Q(x, t)$ are plotted versus $x$ at various instants of time. The potential barrier is located in the region $-0.008 < x < 0.008$. The black curve represents $Q$ at $t = 350$. Barrier perturbation is from $t = 400$ to $t = 410$. The black, red and green curves represent $Q$ at times $t = 420, 425$ and $430$ respectively. The gorges in the quantum potential move towards the left with time and reflect incoming particles away from the vicinity of the classical barrier. This explains why certain particles arrive at the detector earlier than they would have done if reflected from a static barrier.
\[ \beta_i = \frac{t_i - t_{ip}}{t_i} \]  

(16)

which provides a quantitative measure of superarrivals for a particular value of initial position. Next we define an average value

\[ \tilde{\beta} = \frac{\sum \beta_i}{N} \]  

(17)

which provides a quantitative estimate of superarrivals obtained through Bohmian trajectories.

Our results show that the arrival time \( t_{ip} \) for the perturbed case is sensitive to the value of initial position\(^1\). We have checked that \( t_i \) exceeds \( t_{ip} \) for only those trajectories which contribute to superarrivals. This is a distinct feature associated with the superarrivals that can be identified in terms of the Bohmian trajectories. We plot a set of Bohmian trajectories in Figure 3. Note that the trajectories of the particles corresponding to the perturbed case take a sharp turn and arrive at the detector earlier than they would have for a static barrier. Any abrupt perturbation of the potential barrier has a global effect on the wave function (solution of the Schrodinger equation) and thus affects the values of the quantum potential \( Q(x,t) \) at various points. Then, through the Bohmian equation of motion the velocities of the incident particles get correspondingly affected much before reaching the vicinity of the potential barrier. Superarrivals originate from those particles in the perturbed case which reach the detector earlier than those corresponding to the same initial positions in the static case. This accounts for why the detector records more counts in the perturbed case during a particular time interval compared to that in a static situation. The origin of superarrivals can thus be understood in terms of the Bohmian trajectories.

The effect of altering the barrier perturbation time \( \epsilon \) on the magnitude of superarrivals \( \tilde{\beta} \) can be studied by computing \( \tilde{\beta} \) for various values of \( \epsilon \). We display the results of this study in Figure 4. Note that the the magnitude of superarrivals decreases monotonically with increasing \( \epsilon \), or decreasing rate of perturbation. This effect was also observed in [3] where we obtained a similar behaviour for the superarrival parameter \( \eta \). The similarity of these two results obtained through entirely different techniques reinforces our contention about the dynamical nature of superarrivals originating from a “kick” provided by the lowering potential barrier, which propagates across the wave function with a definite speed.

\[ x(t) \]

FIG. 3. Bohmian trajectories for particles originating from the same initial positions get reflected differently from the static and the perturbed barriers. The trajectories undergo sharper turns when the barrier is perturbed and arrive the the detector earlier than they would have done for the static barrier case. The barrier is placed at \( x = -0.008 \) to \( x = 0.008 \). Perturbation takes place from \( t = 400 \) to \( t = 410 \).

\[ \text{FIG. 4. The Bohmian superarrival parameter } \tilde{\beta} \text{ is plotted versus } \epsilon. \]

\[ \text{FIG. 3. Bohmian trajectories for particles originating from the same initial positions get reflected differently from the static and the perturbed barriers. The trajectories undergo sharper turns when the barrier is perturbed and arrive the the detector earlier than they would have done for the static barrier case. The barrier is placed at } x = -0.008 \text{ to } x = 0.008. \text{ Perturbation takes place from } t = 400 \text{ to } t = 410. \]

To conclude, in this paper we have explored further the nature and origin of quantum superarrivals manifested in terms of enhanced reflection and transmission probabilities of wave packets from a perturbed potential barrier which is respectively lowered or raised. We have

\(^1\)For conceptual subtleties concerning arrival time in the bohm model, and in general its definition in quantum theory, see [7]
shown how the concept of particle trajectories obtained from the Bohm model enables one to have an insight into the phenomenon of quantum superarrivals. This analysis substantiates our earlier contention [3] that superarrivals arise from a dynamical disturbance provided by the perturbed barrier to the wave packet which propagates across the wave function (acting like a “physically real field”) with a definite speed and affects the “particles”. Such kind of time dependent quantum phenomenon could be useful in furnishing examples of the utility of the Bohm model in supplementing calculations within standard quantum mechanics from a perspective different from other such examples recently studied [8].

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