Hecke algebraic approach to the reflection equation for spin chains

A Doikou
Mathematics, University of York, Heslington, York YO10 5DD, UK.

P P Martin
Mathematics Department, City University, Northampton Square, London EC1V 0HB, UK.

Abstract

We use the structural similarity of certain Coxeter Artin Systems to the Yang–Baxter and Reflection Equations to convert representations of these systems into new solutions of the Reflection Equation. We construct certain Bethe ansatz states for these solutions, using a parameterisation suggested by abstract representation theory.

1 Introduction and review

There has been much interest recently in the role of boundaries in integrable systems, both from the point of view of critical phenomena (see for example [?] and references therein), and integrability [?]. There has also been considerable progress in constructing representations of affine Hecke algebras [?, ?] with global (i.e. quasi–thermodynamic) limits [?, ?]. In this paper we apply this algebraic technology to the boundary $R$–matrix problem, in a way analogous to the use by many authors of the ordinary Hecke algebra in solving the Yang–Baxter equations (see [?, ?] for reviews).

We start by briefly reviewing the standard $R$–matrix formulation of the Yang–Baxter equation (YBE) in the context of spin chains, and the Hecke/Temperley–Lieb algebraic variant of this formulation. We then generalise to $K$–matrices and boundary $YBE$ — i.e. to the reflection equation (RE) [?, ?]. In §2 we discuss the algebraic structures with roles analogous to the ordinary Hecke and Temperley–Lieb algebras in the boundary case, and give a number of constructions for representations of such algebras, which representations provide candidates for solutions to RE. In §3 we show that the resultant ‘blob algebra’ $b_n$ indeed provides new (and well parameterized) solutions to RE. Finally we look at the Bethe ansatz for some intriguing ‘spin–chain–like’ representations of this algebra.

The parallels with the ordinary closed boundary $U_qsl_2$–invariant spin chain case are strong, but the symmetry algebra is not always $U_qsl_2$. This raises some very interesting questions for further study. The representation theory of $b_n$ has parallels
Define $P$ may provide a mechanism for investigating this (cf. [?, ?, ?, ?]).

Fix integers $N > 0$ and $n \gg 0$, and let $V$ be complex $N$–space. Write $V^n = \otimes_{i=1}^n V$. For $N = 2$ the Pauli $\sigma$–matrices, and indeed $U_q\mathfrak{sl}_2$, act naturally on $V$, and $V^n$ is the underlying space of the $n$–site XXZ model. Define $H_n^N(q) = \text{End}_{U_q\mathfrak{sl}_N}(V^n)$. The ordinary Temperley–Lieb algebra $T_n(q)$ (see later) is isomorphic to $H_n^2(q)$.

### 1.1 $R$–matrices

Define $\mathcal{P}$ to act on $V \otimes V$ by $\mathcal{P}x \otimes y = y \otimes x$. If $A$ is any matrix acting on $V^m = \otimes_{i=1}^m V$, and $i_1, \ldots, i_m \leq n$ distinct natural numbers, then (in ‘$R$–index notation’) $A_{i_1 \ldots i_m}$ acts on $V^n$ by embedding the $A$ action onto the $i_1^{th} \ldots i_m^{th}$ factors $V$. E.g., $\mathcal{P}_{12} = \mathcal{P}_{21}$ and $\mathcal{P}_{12} \mathcal{P}_{13} \mathcal{P}_{12} = \mathcal{P}_{23}$. Dually, if $T$ is a matrix acting on $V \otimes V^n$ (with factors indexed from $0, 1, \ldots, n$) then $T_0$ is $T$ regarded as an $N^n \times N^n$–matrix–valued $N \times N$–matrix in the obvious way. Generalising this (for a moment) so that $T_i$ is $T$ expanded with respect to the $i^{th}$ factor then $\text{tr}_i(T) = \text{tr}(T_i)$, the trace (we may also write this as $\text{tr}_i(T_i)$); and $T_i^\dagger = (T_i)^i$, the transpose.

An (adjoint) $R(\lambda)$–matrix is a matrix acting on $V^2$ which solves the Yang–Baxter equation in the ($R$–index) form [?]

$$R_{12}(\lambda - \lambda') R_{13}(\lambda) R_{23}(\lambda') = R_{23}(\lambda') R_{13}(\lambda) R_{12}(\lambda - \lambda').$$

We also require unitarity:

$$R_{12}(\lambda) R_{21}(-\lambda) \propto 1$$

(NB, $R_{21}(\lambda) = \mathcal{P}_{12} R_{12}(\lambda) \mathcal{P}_{12}$); $R_{21}(\lambda) = R_{12}(\lambda)^{-t} t_2$; and [?] that there exist $M = M^t$ and $\rho$ such that

$$R_{12}(\lambda)^{t_1} M_1 R_{12}(-\lambda - 2\rho)^{t_2} M_1^{-1} \propto 1,$$

$$[M_1 M_2, R_{12}(\lambda)] = 0.$$  

Given such an $R(\lambda)$–matrix, introduce monodromy matrix [?, ?]

$$T(\lambda) = R_{0n}(\lambda) \cdots R_{01}(\lambda).$$

NB, this acts on $V \otimes V^n = V_0 \otimes V_1 \otimes V_2 \ldots V_n$. Spaces $V_i$ ($i > 0$) are called ‘quantum’; space $V_0$ is called ‘lateral’ or ‘auxiliary’. One often makes manifest just the lateral space subscript: $T(\lambda) = T_0(\lambda)$. The YBE implies

$$R_{00'}(\lambda - \lambda') T_0(\lambda) T_{0'}(\lambda') = T_{0'}(\lambda') T_0(\lambda) R_{00'}(\lambda - \lambda').$$

It will be convenient in what follows to have in mind a pictorial realisation of the verification of equation(6). One represents the YBE itself as in figure 1. The identity then follows by repeated application of the YBE as in figure 2.

The closed chain transfer matrix is

$$t(\lambda) = \text{tr}_0 T_0(\lambda)$$

By virtue of (6) and the existence of inverse of $R(\lambda)$ this obeys

$$[t(\lambda), t(\lambda')] = 0.$$
Figure 1: Pictorial realisation of the YBE generalising the permutation diagram realisation of the symmetric group. Here a crossing labelled by 1 (resp. 2, +) represents $R_{ij}(\theta_1)$ (resp. $R_{ij}(\theta_2)$, $R_{ij}(\theta_1 + \theta_2)$), and $\theta_1 = \lambda - \lambda'$, $\theta_2 = \lambda$.

Figure 2: Application of the YBE to verify commutation.
\[ R(\lambda) = \begin{pmatrix} a(\lambda) & b(\lambda) & c_+(\lambda) \\ c_-(\lambda) & b(\lambda) & a(\lambda) \end{pmatrix} \]

where
\[ a(\lambda) = \sinh(\mu(\lambda + i)) \]
\[ b(\lambda) = \sinh(\mu \lambda) \]
\[ c_\pm(\lambda) = \sinh(i\mu)e^{\pm\mu\lambda} \]

(also known as \( A_1^{(1)} \) case, by an association with the \( A_1^{(1)} \) affine Lie algebra). This \( R \)-matrix obeys (3) and (4) with \([\, , \, ]\)
\[ M_{jk} = \delta_{jk} e^{i\mu(3-2j)}, \quad \rho = i. \]

### 1.2 \( R \)-matrices and the TL algebraic method

Given an \( R \)-matrix, set
\[ \tilde{R}_{ii+1}(\lambda) = \mathcal{P}_{ii+1} R_{ii+1}(\lambda) = R_{ii+1}(\lambda) \mathcal{P}_{ii+1}. \]

Premultiplying (1) by \( \mathcal{P}_{23} \mathcal{P}_{12} \mathcal{P}_{23} \) we get
\[ \tilde{R}_{12}(\lambda - \lambda') \tilde{R}_{23}(\lambda') \tilde{R}_{12}(\lambda') = \tilde{R}_{23}(\lambda') \tilde{R}_{12}(\lambda) \tilde{R}_{23}(\lambda - \lambda') \]

What is deep about (1) is the construction of commuting transfer matrices, and this is not restricted to, and may be abstracted away from, the \( V^n \) setting. One introduces abstract operators \( \tilde{R}_i(\lambda) \) (not in \( R \)-index notation) obeying
\[ \tilde{R}_i(\lambda - \lambda') \tilde{R}_{i+1}(\lambda') \tilde{R}_i(\lambda') = \tilde{R}_{i+1}(\lambda') \tilde{R}_i(\lambda) \tilde{R}_{i+1}(\lambda - \lambda') \]

and
\[ \tilde{R}_i(\lambda) \tilde{R}_j(\lambda') = \tilde{R}_j(\lambda') \tilde{R}_i(\lambda) \quad i - j > 1. \]

This is called the \textit{Hecke algebraic form} of the YBE. It will be evident that every \( R \)-matrix gives a solution to these equations via the substitution \( \tilde{R}_i(\lambda) \mapsto \tilde{R}_{ii+1}(\lambda) \).

The abstract Temperley–Lieb algebra \( T_n(q) \) is generated by the unit element and elements \( U_1, \ldots, U_{n-1} \) satisfying the following relations \([\, , \, ]\)
\[ U_i U_i = -(q + q^{-1}) U_i, \quad q = e^{i\mu} \]
\[ U_i U_{i+1} U_i = U_i \]
\[ [U_i, U_j] = 0 \quad i - j > 1. \]

Let \( N = 2 \), and \( V^n \) the corresponding tensor space with action of \( U_q sl_2 \) \([\, , \, ]\). Set
\[ \mathcal{R}(U_i) = \mathcal{R}_q(U_i) = \sigma_{i+1}^- \sigma_i^- + \sigma_i^+ \sigma_{i+1}^+ + \frac{q + q^{-1}}{2} \left( \sigma_i^x \sigma_{i+1}^x - \frac{1}{4} \right) + \frac{q - q^{-1}}{2} (-\sigma_i^z + \sigma_{i+1}^z) \]
\[ = 1 \otimes \ldots \otimes U \otimes \ldots \otimes 1 \]
\[ \mathcal{U} = \begin{pmatrix} 0 & -e^{i\mu} & 1 & -e^{-i\mu} \\ 1 & -e^{i\mu} \\ 0 & 1 & -e^{-i\mu} \end{pmatrix} \] (18)

(i.e. the nontrivial part is a 4 × 4 matrix acting on \( V_i \otimes V_{i+1} \), so \( \mathcal{R}(U_i) = \mathcal{U}_{ii+1} \) in \( R \)-index notation).

**Proposition 1** [?] The matrices \( \mathcal{R}(U_i) \) define a representation of \( T_n(q) \) which is (i) faithful; and (ii) commutes with the action of \( U_q sl_2 \) on \( V^n \).

For the XXZ \( R \)-matrix of equation (9) we find

\[
\tilde{R}_{ii+1}(\lambda) = \sinh(\mu(\lambda + i))1 + \sinh(\mu\lambda)\mathcal{R}(U_i).
\] (19)

Thus \( \mathcal{R} \) gives a solution to (13) and hence to (1). Since \( \mathcal{R} \) is faithful, any representation of \( T_n(q) \) would give a solution to (14). We say \( T_n(q) \) gives a meta–solution.

### 1.3 \( K \)-matrices

Given an \( R \)-matrix, a \( K(\lambda) \)-matrix acts on \( V \) and obeys the reflection equation [?]:

\[
R_{12}(\lambda_1 - \lambda_2) K_1(\lambda_1) R_{21}(\lambda_1 + \lambda_2) K_2(\lambda_2) = K_2(\lambda_2) R_{12}(\lambda_1 + \lambda_2) K_1(\lambda_1) R_{21}(\lambda_1 - \lambda_2).
\] (20)

We require \( K(0) = 1 \) and \( K(\lambda)K(-\lambda) \propto 1 \). Using this one may construct commuting open boundary transfer matrices and solve corresponding Bethe ansatz equations [?].

A suitable transfer matrix \( t(\lambda) \) for an open chain of \( n \) spins is [?, ?, ?]

\[
t(\lambda) = \text{tr}_0 M_0 K_0^+(\lambda - \rho) \mathcal{T}_0(\lambda) \mathcal{K}_0^-(\lambda) \mathcal{T}_0(\lambda),
\] (21)

where

\[
\mathcal{T}_0(\lambda) = R_{10}(\lambda) \cdots R_{n0}(\lambda),
\] (22)

\( K^-(\lambda) = K(\lambda) \) where the \( K(\lambda) \) is a solution of the reflection equation, and \( K^+ \) satisfies an equation similar to (20) [?] (we can and will set \( K^+ = 1 \) without significant loss of generality).

Following Sklyanin [?] define

\[
\mathcal{T}(\lambda) = T_0(\lambda) K_0^-(\lambda) \mathcal{T}_0(\lambda),
\] (23)

which satisfies

\[
R_{12}(\lambda_1 - \lambda_2) T_1(\lambda_1) R_{21}(\lambda_1 + \lambda_2) T_2(\lambda_2) = T_2(\lambda_2) R_{12}(\lambda_1 + \lambda_2) T_1(\lambda_1) R_{21}(\lambda_1 - \lambda_2).
\] (24)

We may again use the pictorial representation to see this. Following figure 1 the picture for the reflection equation (20) is as in figure 3. In this realisation the Sklyanin operator appears as in figure 4. The identity (24) follows in the manner of
Figure 3: Pictorial realisation of the RE

Figure 4: The Sklyanin operator $T(\lambda)$. 
Figure 5: First steps in verification of commutation. Step 1 is an application of YBE as in figure 2. Step 2 is similar. At this point the left hand side of RE has appeared in the picture. One applies RE to it and then completes the manipulation by further applications of YBE.
The transfer matrix also obeys
\[ [t(\lambda), t(\lambda')] = 0. \] (25)

Consider the XXZ/\(A_1^{(1)}\) R–matrix as before. For \(K = 1\),
\[ [t(\lambda), g] = 0 \] (26)
where \(g\) is the usual \(U_q\)sl\(2\) action \([?, ?, ?, ?]\). The symmetry for the general diagonal \(K\) is more complicated (see e.g. \([?]\)).

In the Temperley–Lieb notation the RE is
\[ \tilde{R}_1(\lambda_1 - \lambda_2) \tilde{K}(\lambda_1) \tilde{R}_1(\lambda_1 + \lambda_2) \tilde{K}(\lambda_2) = \tilde{K}(\lambda_2) \tilde{R}_1(\lambda_1 + \lambda_2) \tilde{K}(\lambda_1) \tilde{R}_1(\lambda_1 - \lambda_2). \] (27)
As we will now see, this makes it natural to seek solutions among the affine generalisations of \(T_n(q)\).

2 (Affine) braids and Hecke algebras

Recall that a Coxeter graph \(G\) is any finite undirected graph without loops (almost everybody’s attention is habitually restricted to the subset of graphs of positive type \([?, \S 2.3]\). For given \(G\) let \(m(s, s')\) denote the number of edges between vertices \(s\) and \(s'\). The Coxeter system of \(G\) is a pair \((W, S)\) consisting of a group \(W\) and a set \(S\) of generators of \(W\) labelled by the vertices of \(G\), with relations of the form
\[ g_s g_s' g_s g_s' \ldots = g_s' g_s g_s' g_s \ldots \] (28)
where the number of factors on each side is \(m(s, s') + 2\); and
\[ g_s^{-1} = g_s. \] (29)

If we relax the set of relations in (29) (and add as generators the inverse of each \(g_s \in S\)) we get a Coxeter Artin system, and \(W = A_G\) is an Artin group \([?]\). For example, let \(B_n\) denote the ordinary Artin braid group, the group of composition of finite braidings of \(n\) strings running from the northern to the southern edge of a rectangular frame. Then \(A_{A_{n-1}} \cong B_n\).

In case \(G = B_n\) the (non–commuting) relations may be written
\[ g_0 g_1 g_0 g_1 = g_1 g_0 g_1 g_0 \] (30)
\[ g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \quad n - 1 > i \geq 1. \] (31)

And here is the point of this excursion: We will use the structural similarity of these relations to the reflection equation (RE) \([?, ?]\) and Yang–Baxter equation (YBE) \([?]\) to develop various realisations of \(A_{B_n}\) into candidates for solutions to these equations. There are two parts to this task. Finding quotients of the braid group in which (30) and (31) may be deformed to solve RE and YBE respectively (see \(\S 2.3\)); and then finding realizations of these quotients suitable for Bethe ansatz formulation. Our approach to the latter problem is to borrow from what works in the ordinary case \([?]\). Thus we have to make contact with the ordinary case. We do this next.
π(1) \quad c_0 = \pi(g_0)

π(g_1) \quad π(g_2^{-1})

Figure 6: Elements of the 3 string braid group on the cylinder.

Figure 7: Example of composition $g_1g_0$, and ambient isotopy/Reidemeister move on the product in a cylinder braid group.

2.1 Boundaries, cylinder braids and $A_{B_n}$

Let $B_n^g$ denote the Artin braid group on the cylinder (or annulus — the correspondence between the cylinder and annulus versions is trivial, cf. [?, ?]), and we will use them interchangeably. Figure 6 shows some elements of $B_3^g$ (together with an assertion, to be verified later, of their preimages in $A_{B_3}$ under a certain group homomorphism). Figure 7 illustrates composition in the cylinder braid group, and the Reidemeister move [?, III§1] of type 2 in this context ([?] provides a summary of and link to Reidemeister’s original works). There is an obvious inclusion $\iota : B_n \hookrightarrow B_n^g$ got by identifying the right and left edges of the frame. There is an obvious surjective homomorphism $\sigma : B_n^g \to B_n$ got by arranging for all the string endpoints to be gathered on one side of the cylinder and then squashing the cylinder flat with this side on top. $^1$ Figure 8 shows that $\tau = g_{n-1} \ldots g_2g_1c_0$ is a useful twist element.

$^1$Most of the groups we consider here contain $B_m$ as a subgroup at least for some $m$. For example, if $A_{n-1}$ is a full subgraph of $G$ then $A_G \supset B_m$. Where it is unambiguous to do so we will refer to the elements which lie in this subgroup by their $B_m$ names (thus $g_1$ and so on).
Figure 8: Example of composition $g_2g_1c_0$, and ambient isotopy/ Reidemeister move on the product in a cylinder braid group, showing that this is a twist element $\tau$.

**Proposition 2** Each of the sets $S = \{c_0^{\pm 1}, g_1^{\pm 1}, g_2^{\pm 1}, \ldots \}$ and $S' = \{\tau^{\pm 1}, g_1^{\pm 1}, g_2^{\pm 1}, \ldots \}$ generates $\mathcal{B}^\circ_n$.

**Proof:** These sets generate each other so it is enough to prove for $S$. Let $w$ be an arbitrary cylinder braid. We may assume that it is drawn with no string tangent parallel to the top frame, and it has a finite number of crossings. Either it has no crossings, in which case it can evidently be generated by $\tau^{\pm 1}$ (figure 8), or there exist a pair of strings adjacent at the top of the diagram which cross each other before crossing any other. This crossing can be removed by multiplying by an appropriate element of $S$ (or finite product thereof). Since $w$ is finite, iterating this process produces a braid with no crossings (and hence generated by $\tau$). Thus the inverse of $w$ is generated by $S$, and so is $w$. $\square$

The interplay between $B$–type and periodic algebraic systems and boundary conditions for YBE (cf. (30), (31)) is neatly summed up by the following.

**Proposition 3** There is a group homomorphism

$$\pi : \hat{A}_{B_n+1} \to \mathcal{B}^\circ_n$$

in which the images of the set $\{g_0, g_1, g_2, \ldots \}$ of generators are (the generators) as indicated in figure 6.

Figure 9 verifies the special relation (30) in this realisation, in as much as it is manifest that the (outer) factor of $\pi(g_1)$ commutes with the rest of the diagram. Note from proposition 2 that $\pi$ is surjective. (And see [?, ?].)

It will be evident that there is a homomorphism from $\hat{A}_{\hat{A}_{n+1}}$ (with generator $\hat{g}_{n+1}$, say, where vertex $n + 1$ is adjacent to both 1 and $n$ in $\hat{A}_{n+1}$) into $\mathcal{B}^\circ_{n+1}$. This may be given in our $n = 2$ example as $\hat{g}_3 \mapsto \tau g_1 \tau^{-1}$.

### 2.2 On maps into the ordinary braid group

Recall that the pure braid group $\mathcal{B}'_n$ is normal in $\mathcal{B}_n$, and that the quotient defines a surjection onto the symmetric group $S_n$

$$P : \mathcal{B}_n \to S_n.$$
For $p$ a partition of \( \{1, 2, \ldots, n\} \), the subset of permutations which fix $p$ forms a subgroup, called the Young subgroup $S_p$ of $S_n$. We may extend this to define a subgroup $B_p$ of $B_n$ which fixes $p$ in the sense that braid $b$ fixes $p$ if $P(b)$ does.

Note that for each $p$, a part of $p$ there is a natural ‘restricting’ map from $B_p$ onto $B_{|p|}$ which simply ignores all strings not in $p$.

For $m = 1, 2, \ldots$ let $B^m_{n+m}$ denote the subgroup of $B_{n+m}$ in which the first $m$ strings are pure.

Next we establish maps between $A_{B_n}$, $B^o_n$ and $B^l_{n+1}$ which enable us to port information between them. This is useful as each brings a particular utility to the problem of their analysis ($B^o_n$ has nice diagrams, and periodicity; $B^l_{n+1}$ forms a tower of subalgebras on varying $n$, and has representations by restriction from $B_n$; and $A_{B_n}$ has the direct structural similarity with RE and the blob algebra (see later)).

There is a mapping

\[
\sigma_l : B^o_n \rightarrow B^l_{n+l}
\]

like $\sigma$, but which keeps track of which strings actually went round the back of the cylinder (i.e. it is injective). Before squashing the cylinder completely flat we slide an extra row of $l$ mutually non-crossing strings into the hole, pushing them over so that they lie at, say, the lefthand end of the row of strings in the squashed cylinder (see figure 10). For example $\sigma_1(e_0) = g_1^2$, $\sigma_1(g_i) = g_{i+1}$ ($i > 0$). To see that this map is injective note that the strings which went round the back now go round the extra strings in the appropriate sense (so the manoeuvre is reversible). The image of this map is a nonempty subgroup of $B^l_{n+l}$ which restricts, on the first $l$ strings, to
the trivial group. Note, then, that $\sigma_1$ is an isomorphism. We will again use these two realisations interchangeably where no confusion arises. (Cf. [?, ?, ?].) Indeed, for mapping the braid groups themselves the generalisation to $l > 1$ is effectively spurious. We include it because we will later want to study the maps induced by $\sigma_l$ on quotient algebras, and these maps do depend on $l$ (and even on variations like attaching an idempotent to the first $l$ strings [?]).

Next, consider the subgroup $J$ of $\mathcal{B}_{2n}$ consisting of braids which are invariant under rotation about an axis passing north to south, starting halfway between the $n^{th}$ and $n + 1^{th}$ northern endpoints, as in figure 11. There is an injective homomorphism

$$\gamma : \mathcal{B}_n \rightarrow J$$

$$\gamma : g_i \mapsto g_{n-i}g_{n+i}. \quad (32)$$

This extends to a homomorphism

$$\gamma : \mathcal{A}_{\mathcal{B}_{n+1}} \rightarrow J$$

by

$$\gamma : g_0 \mapsto g_n.$$  

Physicists will recognize an analogy in this with the method of images. There is a similar extension of the cabling map [?].

Without the extension, the map $\gamma$ is essentially the group comultiplication $\Delta : \mathcal{B}_n \rightarrow \mathcal{B}_n \times \mathcal{B}_n$ embedded, Young subgroup style, in $\mathcal{B}_{2n}$. Recall that this equips the group algebra with the property of bialgebra (indeed Hopf algebra); and implies that the category of left modules is closed under tensor products (see [?] for example). There is a generalisation of this (see [?; ?A(iii)][?]) which enables us to close the sum over $q \in \mathbb{C}$ of categories of left $T_n(q)$–modules under tensor products. It is possible to extend the representation obtained by tensoring two copies of the ordinary spin–chain representation (as in equation(17)) to a representation of $\mathcal{B}_n^\circ$ [?]. We will recall the precise construction in §5. This is in particular a faithful 2–parameter representation of the blob algebra $b_n$ [?], which is a quotient of $\mathcal{B}_n^\circ$ which explicitly solves RE — see §3. As such this representation is arguably the most interesting candidate for studying spin–chains with boundary currently available. There are other possibilities, however, as we now summarize.
The above discussion gives us a number of recipes for constructing representations of cylinder algebras from those of $\mathbb{CB}_n$. Many $\mathbb{CB}_n$ representations may be used to construct exactly solvable models, so applying the recipes to these should provide good candidates for ESMs with more general boundary conditions. Unfortunately these representations have important properties which are not necessarily preserved by passage to the cylinder. When $\mathbb{CB}_n$ is used to solve the YBE it is never, physically, a faithful representation which appears (and the vanishing of the annihilator is used in the solution). Indeed, on physical representations each $g_s$ has a finite spectrum.

If each $g_s$ has spectrum of order 2 then we are in the realm of generic algebras [?] (natural generalisations of the corresponding Coxeter systems $(W,S)$ in which, of course, $g_s^2 = 1$ for all $s \in S$). In a generic algebra $g_s$ and $g_t$ have the same spectrum if $s,t$ conjugate in $W$. Thus in the $A_n$ case each $g_s$ has the same spectrum — we write

$$ (g_i - q)(g_i + q^{-1}) = 0 \quad (33) $$

whereupon we have the ordinary Hecke algebra $H_n(q)$ [?]. Although $H_n(q)$ is a relatively tiny vestige of $\mathbb{CB}_n$, even this algebra is never faithfully represented in physical representations (and no global limit of the whole of $H_n(q)$ is known). A natural example of a quotient of $\mathbb{CB}_n$ which does have a global limit is the Temperley–Lieb algebra [?].

We may assume that a similar situation pertains in the ‘affine’ case. Applying (33) to $\mathbb{CA}_n$, we get an affine Hecke algebra [?], again too big to be physical. A number of potentially suitable quotients are discussed in [?, ?]. The $N = 2$ case (an affine equivalent of Temperley–Lieb) is the aforementioned blob algebra. It has been examined in some detail from the ordinary representation theory viewpoint [?]. On the other hand, while $\mathbb{CB}_n$ and its quotients all have a natural inclusion via $A_n \subset A_{n+1}$, and a number of physically useful representations are known, embedding cylinder algebras in towers is somewhat harder. The preceding discussion provides solutions to this problem by building cylinder algebras out of ordinary ones. The price paid is that while these constructions work at the level of braids, they do not in general factor through the quotients which we are obliged to restrict to physically. The remainder of this paper is concerned with finding cases which do factor, and using these to solve the reflection equation. We typically have some variant of the following picture:

$$ \begin{array}{c}
\mathbb{CB}_n \xrightarrow{\sigma} \mathbb{CB}_n^{t} \xrightarrow{\Psi^2} \mathbb{CB}_n^{t} \\
\downarrow q^2 \downarrow \Theta \downarrow \text{End}(V^{n+t})
\end{array} $$

Here $\sigma$ represents any of the maps constructed in §2.2; the diagonal map is defined by the commutativity of the upper triangle; $\Psi^2$ is the quotient map to the blob algebra (see §3) or some other suitable quotient; and $\Theta$ is the representation of $b_n$ we get if the diagonal map factors through $b_n$.

Solutions which do not start with XXZ, or do not end up in the blob quotient, raise rather different problems, and will be examined in a separate paper.
In this section we look for solutions to the reflection equation based on the special representations of B–braids discussed above. We show that the abstract blob algebra provides a meta–solution in the same sense as the Temperley–Lieb algebra does for the ordinary YBE.

The blob algebra $b_n = b_n(q, m)$ may be defined by generators $U_1, U_2, ... U_{n-1}$ and $e$, and relations:

$$U_iU_i = \delta U_i$$  \hspace{1cm} (34)

$$U_iU_{i\pm 1}U_i = U_i$$  \hspace{1cm} (35)

$$[U_i, U_j] = 0 \hspace{1cm} |i - j| \neq 1$$  \hspace{1cm} (36)

(so far we have the ordinary Temperley–Lieb algebra with $-\delta = q + q^{-1}$)

$$ee = \delta_e e$$  \hspace{1cm} (37)

$$U_1 e U_1 = \kappa U_1$$  \hspace{1cm} (38)

$$[U_i, e] = 0 \hspace{1cm} i \neq 1$$

Note that we are free to renormalize $e$, changing only $\delta_e$ and $\kappa$ (by the same factor), thus from $\delta, \delta_e, \kappa$ there are really only two relevant parameters. It will be natural later on to reparameterize so that they are related (they only depend on $q$ and $m$), but it will be convenient to treat them separately for the moment, and leave $m$ hidden.

Assuming for the moment that we have some viable representation of this algebra we may proceed as follows. Setting

$$R_1(\theta_1 \pm \theta_2) = a_{\pm} 1 + b_{\pm} U_1$$

the reflection equation

$$R_1(\theta_1 - \theta_2)K(\theta_1)R_1(\theta_1 + \theta_2)K(\theta_2) = K(\theta_2)R_1(\theta_1 + \theta_2)K(\theta_1)R_1(\theta_1 - \theta_2)$$

becomes

$$(a_{-1} + b_{-} U_1)(x_1 1 + y_1 e)(a_{+1} + b_{+} U_1)(x_2 1 + y_2 e)$$

$$= (x_2 1 + y_2 e)(a_{+1} + b_{+} U_1)(x_1 1 + y_1 e)(a_{-1} + b_{-} U_1)$$

and hence

$$(a_{-1} x_1 1 + a_{-} y_1 e + b_{-} x_1 U_1 + b_{-} y_1 U_1 e)(a_{+1} x_2 1 + a_{+} y_2 e + b_{+} x_2 U_1 + b_{+} y_2 U_1 e)$$

$$= (a_{+1} x_2 1 + a_{+} y_2 e + b_{+} x_2 U_1 + b_{+} y_2 U_1 e)(a_{-1} x_1 1 + a_{-} y_1 e + b_{-} x_1 U_1 + b_{-} y_1 U_1 e)$$

and hence

$$a_{-a_{+}} x_1 x_2 1$$

$$+a_{-a_{+}} (x_1 y_2 + x_2 y_1 + \delta_e y_1 y_2) e$$

$$+(a_{-b_{+}} + a_{+b_{-} + \delta b_{+} b_{-}}) x_1 x_2 U_1$$

$$+((a_{-b_{+}} + a_{+b_{-} + \delta b_{+} b_{-}}) x_1 y_2 +$$

$$a_{+b_{-}} (x_2 y_1 + \delta_e y_1 y_2)) U_1 e$$

$$= +a_{-a_{+}} x_1 x_2 1$$

$$+a_{-a_{+}} (x_1 y_2 + x_2 y_1 + \delta_e y_1 y_2) e$$

$$+(a_{-b_{+}} + a_{+b_{-} + \delta b_{+} b_{-}}) x_1 x_2 U_1$$

$$+((a_{-b_{+}} + a_{+b_{-} + \delta b_{+} b_{-}}) x_1 y_2 +$$

$$a_{+b_{-}} (x_2 y_1 + \delta_e y_1 y_2)) U_1 e$$

$$+a_{-b_{+}} x_2 y_1 U_1 e$$

$$+a_{-b_{+}} y_2 y_1 U_1 e$$

$$+a_{-b_{+}} y_1 y_2 U_1 e$$

$$+a_{-b_{+}} y_1 y_2 U_1 e$$

$$= +a_{+b_{-}} (x_2 y_1 + \delta_e y_1 y_2) e$$

$$+b_{+} y_1 x_2 U_1 e$$

$$+b_{+} y_1 y_2 U_1 e$$

$$+b_{-} b_{+} y_1 y_2 U_1 e U_1 e$$
shown to solve RE in \[ \text{parameter algebra} \] However the precise form of parameterization known from representation theory, as follows. This is even more striking when we apply the representation of \( \zeta \) (or rather a meta–solution which produces a solution for each representation of \( b_n \)). The blob algebra is a quotient of a special case of the algebras shown to solve RE in [? , ?], which guarantees that it gives a solution in principle. However the precise form of \( b_n \) leads to a significant and crucial simplification in parameterization cf. the general case. This is even more striking when we apply the parameterization known from representation theory, as follows.

Recall \([m] = \frac{\text{sh}(mi)}{\text{sh}(\mu i)}\). In the abstract form a natural parameterisation of the two parameter algebra \( b_n \) is \( \delta = -[2], \delta_e = -[m], \kappa = [m-1] \) (the two parameters are \( q \) and \( m \)), and hence

\[
\text{sh}(\mu 2\theta_j) \text{ sh}(\mu i) \ k_j = \frac{-1}{2} \left( \frac{-\text{sh}(\mu mi) \ ch(\mu(2\theta_j + i)) + \text{sh}(\mu(mi - i)) \ ch(\mu 2\theta_j)}{\text{sh}(\mu i)} + \text{ch}(\mu 2i\zeta) \right)
\]

and hence

\[
k_j = \frac{x_j}{y_j} = \frac{\text{sh}(\mu(\theta_j + i(\frac{m}{2} + \zeta))) \ text{sh}(\mu(\theta_j + i(\frac{m}{2} - \zeta)))}{\text{sh}(\mu 2\theta_j) \ text{sh}(\mu i)}.
\]

Specifically we take

\[
x_j = x(\theta_j; m) = \text{sh}(\mu(\theta_j + \frac{im}{2} + i\zeta)) \text{ sh}(\mu(\theta_j + \frac{im}{2} - i\zeta))
\]

\[
y_j = z(\theta_j) = \text{sh}(\mu i) \text{ sh}(2\mu \theta_j).
\]

(We see that \( m \) has the role of boundary parameter.)

Note that

\[ K(\theta)K(-\theta) \propto k(\theta)k(-\theta)1 + (k(\theta) + k(-\theta) + \delta_e)e = k(\theta)k(-\theta)1. \]
It remains to construct representations suitable for forming the Bethe ansatz. Our approach is to use the representations of the ordinary Temperley–Lieb algebra for which there exists a Bethe ansatz (we will concentrate on the XXZ representation) and pull them through to the blob case using the tools in §2. (Another approach would be to generalise [?], but we do not consider that here.) As noted in §2 we have to check that this procedure preserves the appropriate quotient inside $\mathbb{C}A_{B_n}$. In general it does not. The first cases we consider in which it does are the cases of $\sigma_l$ in which $l = 0, 1$. The most obvious relation obeyed by $b_n$ cf. $\mathbb{C}A_{B_n}$ is (37). The representation of $e$ will be a linear combination of that of 1 and $c_0$, so we require the representation of $c_0$ to have at most two eigenvalues. For $\sigma_l$ (and general $q$) it is easy to check that this holds for $l = 0, 1$ only. Case $l = 0$ is the trivial solution ($K \propto 1$, $m = 1$), so we will focus on $l = 1$. The XXZ representation of $T_{n}(q)$ depends only on $q$, so the representation pulled through $\sigma_l$ also depends only on $q$, thus $m$ must be fixed. Comparing (37), (38) and $\mathcal{R}(\sigma_1(c_0))$ we see that $m = 2$.

Using the XXZ representation for $T_{n+1}(q)$ as in eqn. (18) we have that $\Theta : b_n \rightarrow T_{n+1} \rightarrow \text{End}(V^{n+1})$ is given by $\Theta(e) = \mathcal{R}(U_l)$, $\Theta(U_l) = \mathcal{R}(U_{l+1})$. Then using (39), (42) the $K$–matrix becomes

$$K(\lambda) = \begin{pmatrix} x(\lambda; 2) & w^-(\lambda) & z(\lambda) & w^+(\lambda) \\ w^-(\lambda) & z(\lambda) & w^+(\lambda) & x(\lambda; 2) \end{pmatrix},$$

(43)

with $x(\lambda; 2)$, $z(\lambda)$ given by (42), and

$$w^\pm(\lambda) = \sinh \mu(\lambda + i\zeta) \sinh \mu(\lambda - i\zeta) + e^{\pm 2\mu \lambda} \sinh^2(i\mu).$$

(44)

The $K$–matrix can be written in the following $2 \times 2$ form:

$$K(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} = \begin{pmatrix} x(\lambda; 2)1 - \frac{1}{2}e^\mu z(\lambda)(1 - \sigma^z) & z(\lambda) \sigma^- \\ z(\lambda) \sigma^+ & x(\lambda; 2)1 - \frac{1}{2}e^{-\mu} z(\lambda)(1 + \sigma^z) \end{pmatrix}$$

(45)

where $\sigma^z, \sigma^\pm$ act on a two dimensional space $V_\epsilon$. NB, This means that we extend the space on which the transfer matrix acts from $2^n$–dimensional to $2^{n+1}$. This can be considered as a system with enhanced space (cf. [?]), i.e. it is as if we added an extra site, with inhomogeneity $i\zeta$, to the original spin chain. The situation is similar in quantum impurity problems (see e.g. [?]).

Suppose we are considering a system in which the underlying bulk model is a spin chain on $V^n$. Then a solution to RE is called ‘$C$–number representation’ if $K(\lambda)$ is an $N \times N$ matrix with complex entries [?]. More generally, it will be evident from figures 3, 4, 5 that given any $K(\lambda)$ which satisfies RE as in (20), the ‘factorized $K$–matrix’

$$K_f(\lambda) = R(\lambda + i\zeta)K(\lambda)R(\lambda - i\zeta)$$

(46)

where $R$ is given by (9), is also a solution of RE. It is conjectured [?] that every solution of RE is some iteration of this construction, with a $C$–number representation as base. Our solution (43) is of this factorized form with $K = 1$.

The eigenvalues of the corresponding open transfer matrix (21) can be found via the algebraic Bethe ansatz method.
Here we show explicitly how the Bethe ansatz can be applied in the case of these ‘dynamical’ [?] boundary conditions. (The analysis in this case is much closer to the usual setup than the ‘cabled’ case we will consider in §5. We include it, since it also serves the purpose of providing a preparatory review.) We define the transfer matrix as in equation (21).

The next step is to diagonalize the transfer matrix (21) using the algebraic Bethe ansatz method. The \( T \)-matrix (23) has the form

\[
T_0(\lambda) = \begin{pmatrix} A(\lambda) & B'(\lambda) \\ C'(\lambda) & D(\lambda) \end{pmatrix} \begin{pmatrix} \alpha(\lambda) & \beta(\lambda) \\ \gamma(\lambda) & \delta(\lambda) \end{pmatrix} \begin{pmatrix} A(\lambda) & B(\lambda) \\ C'(\lambda) & D(\lambda) \end{pmatrix} = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}
\]

where the matrices \( \alpha, \beta, \gamma \) and \( \delta \) are as in (45).

Define state \( |\omega_+\rangle \) to be that with all spins up (the ferromagnetic vacuum vector):

\[
|\omega_+\rangle = \left( \begin{array}{c} 1 \\ 0 \\ \vdots \\ 0 \end{array} \right) \otimes \cdots \otimes \left( \begin{array}{c} 1 \\ 0 \end{array} \right).
\]

Note that

\[
C, C' \left( \begin{array}{c} 1 \\ 0 \\ \vdots \\ 0 \end{array} \right) \otimes \cdots \otimes \left( \begin{array}{c} 1 \\ 0 \end{array} \right) = 0, \quad \gamma(\lambda) \left( \begin{array}{c} 1 \\ 0 \end{array} \right) = 0,
\]

therefore \( |\omega_+\rangle \) is annihilated by \( C(\lambda) \). The operators \( B(\lambda) \) obey

\[
[B(\lambda), B(\lambda')] = 0,
\]

and act as creation operators. The Bethe state

\[
|\psi\rangle = B(\lambda_1) \cdots B(\lambda_M) |\omega_+\rangle
\]

is an eigenstate of the transfer matrix \( t(\lambda) \), i.e.

\[
t(\lambda)|\psi\rangle = (A + D)|\psi\rangle = \Lambda(\lambda)|\psi\rangle.
\]

It is easy to determine the action of \( A \) and \( D \) on the pseudo–vacuum (see below). The action of the transfer matrix on the pseudo–vacuum, cf. (49), is given by

\[
t(\lambda)|\omega_+\rangle = \text{tr}_0 \begin{pmatrix} A(\lambda) & B'(\lambda) \\ C'(\lambda) & D(\lambda) \end{pmatrix} \begin{pmatrix} \alpha(\lambda) & \beta(\lambda) \\ \gamma(\lambda) & \delta(\lambda) \end{pmatrix} \begin{pmatrix} A(\lambda) & B(\lambda) \\ C'(\lambda) & D(\lambda) \end{pmatrix} |\omega_+\rangle
\]

\[
= \left( \alpha(\lambda)A^2 + \alpha(\lambda)CB + \delta(\lambda)D^2 \right) |\omega_+\rangle.
\]

But (45) gives

\[
\alpha(\lambda) \left( \begin{array}{c} 1 \\ 0 \end{array} \right) = x(\lambda; 2) \left( \begin{array}{c} 1 \\ 0 \end{array} \right), \quad \delta(\lambda) \left( \begin{array}{c} 1 \\ 0 \end{array} \right) = w^+(\lambda) \left( \begin{array}{c} 1 \\ 0 \end{array} \right),
\]

where \( x(\lambda; 2), w^+(\lambda) \) are given by (42), (44). We have

\[
A|\omega_+\rangle = \alpha(\lambda)A^2|\omega_+\rangle, \quad D|\omega_+\rangle = \left( \alpha(\lambda)CB + \delta(\lambda)D^2 \right) |\omega_+\rangle.
\]
\[ A|\omega_+\rangle = x(\lambda; 2)a^{2n}(\lambda)|\omega_+\rangle, \quad D|\omega_+\rangle = \left( w^+(\lambda)b^{2n}(\lambda) - x(\lambda; 2)\frac{a^{2n}(\lambda) - b^{2n}(\lambda)}{a^2(\lambda) - b^2(\lambda)} \right)|\omega_+\rangle. \tag{56} \]

Having determined the action of the transfer matrix on the pseudo–vacuum, it is easy to see via (51), (52), (56) that knowledge of the commutation relations between \( A, B \) and \( D, B \) is enough for the derivation of any eigenvalue. It is convenient \[ \] to consider instead of \( D \) the following operator

\[ \mathcal{D} = \sinh(2\mu\lambda)D - \sinh(i\mu)A. \tag{57} \]

Then from the fundamental relation for \( T \) (24) it follows that

\[ A(\lambda)B(\lambda_i) = X(\lambda, \lambda_i)B(\lambda_i)A(\lambda) + f(\lambda, \lambda_i)B(\lambda)A(\lambda_i) + g(\lambda, \lambda_i)B(\lambda)D(\lambda_i) \]
\[ D(\lambda)B(\lambda_i) = Y(\lambda, \lambda_i)B(\lambda_i)D(\lambda) + f'(\lambda, \lambda_i)B(\lambda)A(\lambda_i) + g'(\lambda, \lambda_i)B(\lambda)D(\lambda_i), \tag{58} \]

where

\[ X(\lambda, \lambda_i) = \frac{\sinh \mu(\lambda - \lambda_i - i) \sinh \mu(\lambda + \lambda_i - i)}{\sinh \mu(\lambda - \lambda_i) \sinh \mu(\lambda + \lambda_i)}, \]
\[ Y(\lambda, \lambda_i) = \frac{\sinh \mu(\lambda - \lambda_i + i) \sinh \mu(\lambda + \lambda_i + i)}{\sinh \mu(\lambda - \lambda_i) \sinh \mu(\lambda + \lambda_i)}. \tag{59} \]

The other functions \((f, g, f', g')\) are not important for our purposes since they contribute to unwanted terms, and will vanish in the final eigenvalue expression.

We can now find the eigenvalues using the above commutation relations (58), also having in mind (57) and the action of \( A \) and \( D \) on the pseudo–vacuum (56). The eigenvalue of any Bethe ansatz state is given by

\[ \Lambda(\lambda) = \frac{\sinh \mu(\lambda + i + i \zeta) \sinh \mu(\lambda + i - i \zeta)}{\sinh(\mu i)} \left( \frac{\sinh \mu(\lambda + i)}{\sinh(\mu i)} \right)^{2n} \frac{\sinh \mu(\lambda + i)}{\sinh(\mu i)} \]
\[ \prod_{\alpha=1}^{M} \frac{\sinh \mu(\lambda - \lambda_\alpha - \frac{i}{2}) \sinh \mu(\lambda + \lambda_\alpha - \frac{i}{2})}{\sinh \mu(\lambda - \lambda_\alpha + \frac{i}{2}) \sinh \mu(\lambda + \lambda_\alpha + \frac{i}{2})} \]
\[ + \frac{\sinh \mu(\lambda + i \zeta) \sinh \mu(\lambda - i \zeta)}{\sinh(\mu i)} \left( \frac{\sinh(\mu \lambda)}{\sinh(\mu i)} \right)^{2n} \frac{\sinh(\mu \lambda)}{\sinh(\mu i)} \]
\[ \prod_{\alpha=1}^{M} \frac{\sinh \mu(\lambda - \lambda_\alpha + \frac{i}{2}) \sinh \mu(\lambda + \lambda_\alpha + \frac{i}{2})}{\sinh \mu(\lambda - \lambda_\alpha - \frac{i}{2}) \sinh \mu(\lambda + \lambda_\alpha - \frac{i}{2})}. \tag{60} \]

provided that \( \{\lambda_1, \ldots, \lambda_M\} \) are distinct and obey the Bethe Ansatz equations

\[ \frac{\sinh \mu(\lambda_\alpha + i \zeta + \frac{i}{2}) \sinh \mu(\lambda_\alpha - i \zeta + \frac{i}{2})}{\sinh \mu(\lambda_\alpha + i \zeta - \frac{i}{2}) \sinh \mu(\lambda_\alpha - i \zeta - \frac{i}{2})} \left( \frac{\sinh \mu(\lambda_\alpha + \frac{i}{2})}{\sinh \mu(\lambda_\alpha - \frac{i}{2})} \right)^{2n} \]
\[ \prod_{\beta=1, \beta \neq \alpha}^{M} \frac{\sinh \mu(\lambda_\alpha - \lambda_\beta + i) \sinh \mu(\lambda_\alpha + \lambda_\beta + i)}{\sinh \mu(\lambda_\alpha - \lambda_\beta - i) \sinh \mu(\lambda_\alpha + \lambda_\beta - i)}, \quad \alpha = 1, \ldots, M. \tag{61} \]
Now consider the cabling–like representation \((\Theta : b_n \to \text{End}(V^{2n}))\) from \([?]\) discussed at the end of §2.2. There the elements of the blob algebra are represented as follows. For \(U_{n \pm i} \in T_{2n}(r)\) let \(U_{n \pm i}(r) = R_r(U_{n \pm i}) \in \text{End}(V^{2n})\), the usual XXZ representation (18). Then
\[
\Theta(U_i) = U_i(q) = U_{n-i}(r)U_{n+i}(s), \quad \Theta(e) = U_0(Q) = \frac{1}{i \sinh(i \mu)} U_n(Q) \quad (62)
\]
satisfy the relations of the blob algebra \(b_n(q, m)\) with
\[
r = i \sqrt{i q}, \quad s = \sqrt{i q}, \quad Q = ie^{i m \mu}
\]
(NB, \(rs = -q\)).

Note from (62) that the single index on a blob generator is associated to a mirror–image pair in the underlying \(V^{2n}\). The \(R\)–matrix is given by (19), with \(R(U_i) = U_i\) as defined by (62) and
\[
R_{\bar{k}l}^{\bar{i}}(\lambda) = \sinh \mu(\lambda + i)P_{k\bar{i}}P_{l\bar{i}} + \sinh \mu \lambda \tilde{U}_{k\bar{i}}(r) \tilde{U}_{l\bar{i}}(s)
\]
where we have introduced the space/mirror–space notations \(\bar{l} = (l, l')\), \(\tilde{U}_{k\bar{i}}(r) = P_{k\bar{i}}U_{k\bar{i}}(r)\), \(\bar{R}_{\bar{k}l}^{\bar{i}}(\lambda) = P_{k\bar{i}}R_{k\bar{i}}^{l\bar{i}}(\lambda)\), \(P_{k\bar{i}} = P_{k\bar{i}}P_{k\bar{i}}\). In the \(R\)–index form here, any operator \(O_l = O_{l\bar{i}}\) acts on \(V_l \otimes V_{l'}\), where the \(V_{l'}\) space can be considered as the ‘mirror' space of \(V_l\) in the sense of figure 11.

This \(R\)–matrix satisfies the unitarity and crossing properties
\[
R_{\bar{k}l}^{\bar{i}}(\lambda) R_{ik}^{\bar{i}}(-\lambda) \propto 1, \quad R_{\bar{k}l}^{\bar{i}}(\lambda) = V_k R_{\bar{i}l}^{\bar{i}}(-\lambda - i) V_k
\]
where
\[
V_k = V_{kk'} = V_k(r) V_k(s), \quad (66)
\]
and e.g.
\[
V_k(r) = 1 \otimes \ldots \otimes \left( \begin{array}{cc} 0 & -ir^\frac{1}{2} \\ ir^{-\frac{1}{2}} & 0 \end{array} \right) \otimes \ldots \otimes 1. \quad (67)
\]
This \(R\)–matrix is a \(16 \times 16\) matrix,
\[
R(\lambda) = \begin{pmatrix} A(\lambda) & B_1(\lambda) & B_2(\lambda) & B(\lambda) \\ C_1(\lambda) & A_1(\lambda) & B_3(\lambda) & B_3(\lambda) \\ C_2(\lambda) & C_5(\lambda) & A_2(\lambda) & B_4(\lambda) \\ C(\lambda) & C_3(\lambda) & C_4(\lambda) & D(\lambda) \end{pmatrix},
\]
where the entries shown are \(4 \times 4\) matrices acting on \(V \otimes V\) (see the Appendix for the explicit form of the \(R\)–matrix).

The corresponding \(K\)–matrix (39), (42) is given in matrix form by the following expression (recall that \(U_0\) is given by (62))
\[
K(\lambda) = \begin{pmatrix} x(\lambda; m) & w^-(\lambda) & z(\lambda) & w^+(\lambda) \\ w^+(\lambda) & z(\lambda) & w^{-}(\lambda) & x(\lambda; m) \end{pmatrix},
\]
(69)
\[ w^{\pm}(\lambda) = x(\lambda; m) - \frac{1}{2} e^{\mp im\mu} \sinh(2\mu\lambda). \] (70)

The monodromy matrix has the following structure.
\[
\begin{pmatrix}
\mathcal{A}(\lambda) & B_1(\lambda) & B_2(\lambda) & B(\lambda) \\
C_1(\lambda) & A_1(\lambda) & B_5(\lambda) & B_7(\lambda) \\
C_2(\lambda) & C_5(\lambda) & A_2(\lambda) & B_4(\lambda) \\
C(\lambda) & C_3(\lambda) & C_4(\lambda) & D(\lambda)
\end{pmatrix}
\] (71)

We define a reference state
\[
|\omega_+\rangle = \left(\begin{array}{c} 1 \\ 0 \\ \vdots \\ 1 \\ 0 \end{array}\right) \otimes \cdots \otimes \left(\begin{array}{c} 1 \\ 0 \end{array}\right) = \otimes_{i=1}^{2n} |+\rangle_i,
\] (72)

\[
|+\rangle = \left(\begin{array}{c} 1 \\ 0 \end{array}\right).
\] (73)

Then \( C_i, B_5|+\rangle = 0 \), i.e. \( |\omega_+\rangle \) is annihilated by the operators \( C_i, B_5 \). Therefore, the action of the monodromy matrix on the reference state produces an upper triangular matrix,
\[
\begin{pmatrix}
\mathcal{A}(\lambda) & B_1(\lambda) & B_2(\lambda) & B(\lambda) \\
0 & A_1(\lambda) & 0 & B_3(\lambda) \\
0 & 0 & A_2(\lambda) & B_4(\lambda) \\
0 & 0 & 0 & D(\lambda)
\end{pmatrix}
\] (74)

Thus for the bulk case (71), (7) the pseudo-vacuum eigenvalue is given by
\[
 t(\lambda)|\omega_+\rangle = \left(\begin{array}{c} \mathcal{A}(\lambda) + A_1 + A_2 + D \end{array}\right) |\omega_+\rangle = \left(\begin{array}{c} a^n(\lambda) + b^n(\lambda) \end{array}\right) |\omega_+\rangle
\] (75)

where
\[
\mathcal{A}(\lambda) = \prod_{l=1}^{n} A^l, \quad A_1(\lambda) = \prod_{l=1}^{n} A^l_1, \quad A_2(\lambda) = \prod_{l=1}^{n} A^l_2, \quad D(\lambda) = \prod_{l=1}^{n} D^l
\] (76)

(see also Appendix).

Now consider the open transfer matrix (21),
\[
 t(\lambda) = tr_{\bar{0}} M_{\bar{0}} T_{\bar{0}}(\lambda) K_{\bar{0}}(\lambda) T_{\bar{0}}^{-1}(-\lambda),
\] (77)

where \( K_{\bar{0}} = K_{00'} \) given by (69) and
\[
 M_{\bar{0}} = V_{\bar{0}} V_{\bar{0}}^\dagger.
\] (78)

Then the pseudo–vacuum eigenvalue will be
\[
\Lambda^0(\lambda) = \langle \omega_+ | \left( qx(\lambda; m) A^2 + q^{-1} x(\lambda; m) D^2 + q^{-1} x(\lambda; m) CB + i x(\lambda; m) C_1 B_1 \\
-ix(\lambda; m) C_2 B_2 + q^{-1} w^+(\lambda) C_3 B_3 + q^{-1} w^+(\lambda) C_4 B_4 \right) |\omega_+\rangle.
\] (79)
\[ C_{1,2}(\lambda) = \prod_{l=1}^{n-1} A_i^l \tilde{C}_{1,2}^l, \quad B_{1,2}(\lambda) = \prod_{l=1}^{n-1} A_i^l B_{1,2}^l \]
\[ C_{3,4}(\lambda) = \prod_{l=2}^{n} D_i^l \tilde{C}_{3,4}^l, \quad B_{3,4}(\lambda) = \prod_{l=2}^{n} D_i^l B_{3,4}^l \] (80)

and
\[ C(\lambda) = \sum_{l=1}^{n} D_i^l \cdots D_i^{l+1} C_i^l A_i^{l-1} \cdots A_i^1 + \sum_{l=1}^{n-1} D_i^l \cdots D_i^{l+2} C_i^{l+1} C_i^l A_i^{l-1} \cdots A_i^1 \]
\[ + \sum_{l=1}^{n-1} D_i^l \cdots D_i^{l+2} C_i^{l+1} C_i^l A_i^{l-1} \cdots A_i^1 \] (81)
\[ B(\lambda) = \sum_{l=1}^{n} D_i^l \cdots D_i^{l+1} B_i^l A_i^{l-1} \cdots A_i^1 + \sum_{l=1}^{n-1} D_i^l \cdots D_i^{l+2} B_i^{l+1} B_i^l A_i^{l-1} \cdots A_i^1 \]
\[ + \sum_{l=1}^{n-1} D_i^l \cdots D_i^{l+2} B_i^{l+1} B_i^l A_i^{l-1} \cdots A_i^1 \] (82)

It is also useful to derive the action of the following operators on the \(|+\rangle\) state:
\[ A^2|+\rangle = a^2(\lambda)|+\rangle, \quad B^2|+\rangle = b^2(\lambda)|+\rangle, \]
\[ C_1 B_1|+\rangle = a^2(\lambda)|+\rangle, \quad C_2 B_2|+\rangle = a^2(\lambda)|+\rangle, \]
\[ C B|+\rangle = (a(\lambda) - q b(\lambda)) (a(\lambda) - q^{-1} b(\lambda)) |+\rangle, \]
\[ C_3 B_3|+\rangle = b^2(\lambda)|+\rangle, \quad C_4 B_4|+\rangle = b^2(\lambda)|+\rangle. \] (83)

Taking into account equations (79)–(83) we conclude that the pseudo−vacuum eigenvalue has the form
\[ \Lambda^0(\lambda) = f_1(\lambda)a(\lambda)^{2n} + f_2(\lambda)b(\lambda)^{2n} \] (84)
where the functions \(f_1(\lambda), f_2(\lambda)\) are due to the boundary, and are determined explicitly by (79)–(83)\(^2\). The important observation here is that we are able to derive the pseudo−vacuum eigenvalue explicitly. Furthermore, we note that it has the expected form, compared to the corresponding bulk eigenvalue (75), in as much as the powers of \(a\) and \(b\) are doubled in the open chain, and the functions \(f_1\) and \(f_2\) appear as a result of the presence of the boundary. The next step is the derivation of the general Bethe ansatz state and the corresponding eigenvalue. Here, we do not give the details of this derivation. However we conjecture that the general eigenvalue will have the following form
\[ \Lambda(\lambda) = f_1(\lambda)a(\lambda)^{2n} \mathfrak{A}_1(\lambda) + f_2(\lambda)b(\lambda)^{2n} \mathfrak{A}_2(\lambda), \] (85)
\[ 2f_1(\lambda) = \frac{x(\lambda)}{a^2(\lambda) - b^2(\lambda)} \left\{ q \left[ a^2(\lambda) - b^2(\lambda) \right] + q^{-1} \left[ a^2(\lambda) + 3 b^2(\lambda) - (q + q^{-1}) a(\lambda) b(\lambda) \right] \right\}, \]
\[ f_2(\lambda) = \frac{q^{\frac{1}{2}}}{a(\lambda) - b(\lambda)} \left\{ \left[ x(\lambda) + w^r(\lambda) + w^l(\lambda) \right] \left[ a^2(\lambda) - b^2(\lambda) \right] - x(\lambda) \left[ 3 a^2(\lambda) + b^2(\lambda) - (q + q^{-1}) a(\lambda) b(\lambda) \right] \right\}. \]
We have arrived at this solution from abstract considerations, however, it clearly describes a spin chain model and it does not coincide with any known solution. Furthermore we have retained complete freedom of choice of the boundary parameter $m$. This model also has interesting symmetry properties which appear to significantly generalize the role of $U_qsl_2$ for ordinary XXZ. This makes the model a very interesting candidate for study, and a full spectrum analysis. From the representation theory of $b_n$ we know that $T_n(q)$ appears in $b_n$ in two different ways — as a subalgebra on dropping the boundary generator $e$, and as a quotient for the special boundary parameter choice $m=1$. We also know that the structure of $b_n$ depends profoundly on the boundary parameter $m$. It will be interesting to see how the spectrum of $t(\lambda)$ depends on $m$, and also how the connections with $T_n(q)$ relate the spectrum of $t(\lambda)$ here to that in the ordinary XXZ case. Indeed it is an interesting (and hopefully simpler) preliminary question to ask what is the spectrum of $t(\lambda)$ in this ‘representation’ without the boundary term. (For example, does this spectrum still depend on $r$ and $s$ separately?) This should give an insight into the spectrum with boundary.

6 The Hamiltonian

Here we derive the Hamiltonians of the auxiliary string and cabling realizations.

6.1 The auxiliary string realization

The open spin chain Hamiltonian $\mathcal{H}$ is related to the derivative of the transfer matrix at $\lambda = 0$:

$$\mathcal{H} = \sum_{m=1}^{n-1} \mathcal{H}_{mm+1} + \frac{1}{4\mu x(\lambda; 2)} \frac{d}{d\lambda} K_1(\lambda) \bigg|_{\lambda=0} + \frac{\text{tr}_0 M_0 \mathcal{H}_{n0}}{\mu \text{tr} M},$$

(86)

where $x(\lambda; m)$ is given by (42), and the two–site Hamiltonian $\mathcal{H}_{jk}$ is given by

$$\mathcal{H}_{jk} = \frac{1}{2\mu} \mathcal{P}_{jk} \frac{d}{d\lambda} R_{jk}(\lambda) \bigg|_{\lambda=0} - \frac{1}{4} \cosh(i\mu),$$

(87)

where the $R$–matrix is given by (19). This Hamiltonian is Hermitian.

Consider the model defined by the Hamiltonian in (87), (86):

$$\mathcal{H} = \frac{1}{4} \sum_{i=1}^{n-1} \left( \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \cosh(i\mu) \sigma_i^z \sigma_{i+1}^z \right) + \frac{\sinh(i\mu)}{4} \left( \sigma_n^z - \sigma_1^z \right) + \frac{\sinh(i\mu)}{4x(0; 2)} \left( \sigma_e^x \sigma_1^x + \sigma_e^y \sigma_1^y + \cosh(i\mu) \sigma_e^z \sigma_1^z \right) + \frac{\sinh^2(i\mu)}{4x(0; 2)} \left( \sigma_e^z - \sigma_1^z \right),$$

(88)
spin chain with first neighbour interaction. The last two terms describe the boundary interaction and come from the derivative of the $K$-matrix. This Hamiltonian describes a model which is coupled to a quantum mechanical (spin) system at the boundaries. Note that it is nothing more than an $n + 1$-site Hamiltonian with an inhomogeneity at the end.

Consider the Hamiltonian $H_f$ obtained when we take boundaries of the form (46), where $K$ is the diagonal matrix $[?, ?, ?]$

$$K(\lambda) = \text{diag} \left( \sinh \mu (-\lambda + i\xi) e^{\mu\lambda}, \sinh \mu (\lambda + i\xi) e^{-\mu\lambda} \right).$$  \hspace{1cm} (89)

By direct computation we find here

$$H_f = H + \frac{\coth (i\mu\xi) - 1}{4x(0; 2)} \left( \sinh^2 (i\mu\xi) \sigma^x_e - \sinh^2 (i\mu) \sigma^x_i \right) + \left( \coth (i\mu\xi) - 1 \right) \frac{\sinh (i\mu) \sinh (i\mu\xi)}{2x(0; 2)} F_\xi(\xi) G_1 (-\xi)$$  \hspace{1cm} (90)

where $H$ is from (88) and

$$F(\xi) = \begin{pmatrix} 0 & e^{i\mu\xi} \\ e^{-i\mu\xi} & 0 \end{pmatrix} \hspace{0.5cm} G(\xi) = \begin{pmatrix} 0 & e^{i\mu\xi} \\ -e^{-i\mu\xi} & 0 \end{pmatrix}.$$  \hspace{1cm} (91)

For $\xi = 0$ the above matrices become proportional to $\sigma^x$ and $\sigma^y$ respectively. For $i\xi \to \infty$ we see that $H_f$ coincides with $H$. Interestingly, the Hamiltonian $H_f$ does not appear to have been written down explicitly before.

6.2 The cabling representation

Note from (64) that for $\lambda = 0$ the $R$-matrix reduces to a product of two permutation operators. Therefore the corresponding local Hamiltonian is defined:

$$H_{\text{open}} = \sum_{l=1}^{n-1} H_{\tilde{l}\tilde{l+1}} + \frac{1}{4\mu x(\lambda; m)} \frac{d}{d\lambda} K_{\tilde{1}\tilde{1}}(\lambda) \bigg|_{\lambda=0}$$

$$+ \frac{\text{tr}_0 M_0 H_{\tilde{0}\tilde{0}}}{\mu \text{tr} M},$$  \hspace{1cm} (92)

where the two-site Hamiltonian $H_{\tilde{i}\tilde{j}}$ is given by

$$H_{\tilde{i}\tilde{j}} = \frac{1}{2\mu} P_{\tilde{i}\tilde{j}} \frac{d}{d\lambda} R_{\tilde{i}\tilde{j}}(\lambda) \bigg|_{\lambda=0} - \frac{1}{4} \cosh (i\mu),$$  \hspace{1cm} (93)

and the $R$-matrix is given by (64). Unlike (88) this is completely new. It will be studied in detail elsewhere.

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In this section we write explicitly the $16 \times 16 \ R_{kl}$ matrix. In particular we write down the $4 \times 4$ entries of the matrix,

\[
A(\lambda) = \begin{pmatrix}
  a(\lambda) & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & b(\lambda) \\
  0 & 0 & 0 & 0
\end{pmatrix}, \quad D(\lambda) = \begin{pmatrix}
  b(\lambda) & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & a(\lambda) \\
  0 & 0 & 0 & 0
\end{pmatrix}, \quad (94)
\]

\[
A_1(\lambda) = \begin{pmatrix}
  0 & 0 & 0 & 0 \\
  0 & a(\lambda) & 0 & 0 \\
  0 & 0 & b(\lambda) & 0 \\
  0 & 0 & 0 & 0
\end{pmatrix}, \quad A_2(\lambda) = \begin{pmatrix}
  0 & 0 & 0 & 0 \\
  0 & b(\lambda) & 0 & 0 \\
  0 & 0 & a(\lambda) & 0 \\
  0 & 0 & 0 & 0
\end{pmatrix}, \quad (95)
\]

\[
B_1(\lambda) = \begin{pmatrix}
  0 & 0 & 0 & 0 \\
  a(\lambda) & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & -sb(\lambda) & 0
\end{pmatrix}, \quad B_2(\lambda) = \begin{pmatrix}
  0 & 0 & 0 & 0 \\
  0 & a(\lambda) & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & -rb(\lambda) & 0
\end{pmatrix}, \quad (96)
\]

\[
B_5(\lambda) = \begin{pmatrix}
  0 & 0 & 0 & 0 \\
  0 & a(\lambda) - rs^{-1}b(\lambda) & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0
\end{pmatrix}, \quad B(\lambda) = \begin{pmatrix}
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  a(\lambda) - qb(\lambda) & 0 & 0 & 0
\end{pmatrix}, \quad (97)
\]

$B_3, B_4$ have the same structure as $B_2, B_1$ respectively, with the matrix elements interchanged. Also, $C_i(p) = B_i(p^{-1})^t$, where $p$ is in general the anisotropy parameter, it can be $r, s, q$. 
