INTRODUCTION
action of the Brans-Dicke type in a $\Lambda < 0$ background. The static uncharged solutions of this theory were found and analyzed by Sá, Kleber and Lemos [26] and the angular momentum has been added by Sá and Lemos [27]. The uncharged theory is specified by two fields, the graviton $g_{\mu \nu}$, the dilaton $\phi$, and two parameters, the Brans-Dicke parameter $\omega$ and the cosmological constant $\Lambda$. The pure electrically charged theory is specified by the extra electromagnetic field $F^{\mu \nu}$ and was analysed by Días and Lemos in [28]. This Brans-Dicke theory contains seven different cases and each $\omega$ can be viewed as yielding a different dilaton gravity theory. For instance, for $\omega = \pm \infty$ one gets a theory related (through dimensional reduction) to 4D General Relativity with one Killing vector [29, 30] and for $\omega = \pm \infty$ one obtains 3D General Relativity analyzed in [22, 23, 24]. For a review on other 3D theories, especially on black hole solutions, see [28, 31].

In this paper we are interested in pure magnetic solutions with $\Lambda < 0$. The static magnetic counterpart of the BTZ solution has been found by Clement [23], Hirschmann and Welch [32] and Cataldo and Salgado [33]. This spacetime generated by a static magnetic point source is horizons and reduces to the 3D BTZ black hole solution of Bañados, Teitelboim and Zanelli [22] when the magnetic field vanishes. The extension to include rotation, definition of conserved quantities, upper bounds for the conserved quantities and an interpretation for the source of the magnetic field have been made by Días and Lemos [34], Park and Kim [35] and Koikawa, Maki and Nakamura [36] have found magnetically charged solutions of Einstein-Maxwell-dilaton theories. Many authors have also found self-dual and anti-self-dual charged solutions of Einstein-Maxwell [38], Einstein-Maxwell-dilaton and Einstein-Maxwell-Chern-Simons [39] theories in 3D (for a complete review see e.g. [25, 28]).

The aim of this paper is to find and study in detail the static and rotating magnetic charged solutions generated by a magnetic point source in the Einstein-Maxwell-Dilaton action of the Brans-Dicke type mentioned above. In this sense it is a follow up of our previous paper [28] on pure electric solutions. Here, we impose that the only non-vanishing component of the vector potential is $A_\mu (x)$. For the $\omega = \pm \infty$ case, our solution reduces to the spacetime generated by a magnetic point source in 3D Einstein-Maxwell theory with $\Lambda < 0$, studied in [23], [32]-[34]. The $\omega = 0$ case is equivalent to 4D general relativity with one Killing vector analysed by Días and Lemos [40].

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when we are interested on electric solutions. Now, we focus on the well known fact that the electric field is associated with the time component, $A_t$, of the vector potential while the magnetic field is associated with the angular component $A_\varphi$. From the above facts, one can expect that a magnetic solution can be written in a metric gauge in which the components $g_{tt}$ and $g_{\varphi\varphi}$ interchange their roles relatively to those present in the “Schwarzschild” gauge used to describe electric solutions. This choice will reveal to be a good one to find solutions since the dilaton and graviton will decouple from each other on the fields equations (2) and (4). However, as we will see, for some values of the Brans-Dicke parameter $\omega$ it is not the good coordinate system to interpret the solutions.

We now assume that the only non-vanishing components of the vector potential are $A_t(r)$ and $A_\varphi(r)$, i.e.,

$$A = A_t dt + A_\varphi d\varphi.$$  

(6)

This implies that the non-vanishing components of the anti-symmetric Maxwell tensor are $F_{tr}$ and $F_{r\varphi}$. Use of metric (5) and equation (2) yields the following set of equations

$$\phi_{,rr} + 2 \phi_{,r} \mu_{,r} - (\omega + 2) (\phi_{,r})^2 - \frac{\mu_{,r}}{2r} + \frac{1}{4} \Lambda e^{-2\mu} = \frac{\pi}{2} e^{-2\phi} \frac{1}{4} T_{tr},$$  

(7)

$$\phi_{,r} - \frac{\phi_{,r}}{r} - \omega (\phi_{,r})^2 + \frac{\mu_{,r}}{2r} - \frac{1}{4} \Lambda e^{-2\mu} = \frac{T_{r\varphi}}{2},$$  

(8)

$$\phi_{,rr} + \phi_{,r} \mu_{,r} + \frac{\phi_{,r}}{r} - (\omega + 2) (\phi_{,r})^2 - \frac{\mu_{,r}}{2r} + \frac{1}{4} \Lambda e^{-2\mu} = -\frac{\pi}{2} e^{-2\phi} \frac{1}{4} T_{\varphi\varphi},$$  

(9)

$$0 = \frac{\pi}{2} T_{r\varphi} = e^{2\phi} F_{tr} F_{r\varphi},$$  

(10)

where $r$ denotes a derivative with respect to $r$. In addition, replacing the metric (5) into equations (3) and (4) yields

$$\partial_r \left[ e^{-2\phi} (F_{tr} + F_{r\varphi}) \right] = 0,$$  

(11)

$$\omega \phi_{,rr} + 2 \omega \phi_{,r} \mu_{,r} + \omega \phi_{,r} - \phi_{,r}^2 + \frac{\mu_{,r}}{2r} + \frac{1}{4} \Lambda e^{-2\mu} = \frac{1}{4} e^{-2\phi} F_{tt} + \frac{1}{4} e^{-2\phi} F_{\varphi\varphi}.$$  

(12)

B. The General static solution. Causal structure

Equations (7)-(12) are valid for a static and rotationally symmetric spacetime. One sees that equation (10) implies that the electric and magnetic fields cannot be simultaneously non-zero, i.e., there is no static dyonic solution. In this work we will consider the magnetically charged case alone ($A_t = 0, A_\varphi \neq 0$). For purely electrically charged solutions of the theory see [28]. So, assuming vanishing electric field, one has from Maxwell equation (11) that

$$F_{tr} = \frac{\chi_m}{4\alpha} e^{-2\phi},$$  

(13)

where $\chi_m$ is an integration constant which measures the intensity of the magnetic field source. To proceed we shall first consider the case $\omega \neq -1$. Adding equations (8) and (9) yields $\phi_{,rr} = 2(\omega + 1)(\phi_{,r})^2$, and so the dilaton field is given by

$$e^{-2\phi} = (\alpha r)^{\frac{1}{1+\omega}}, \quad \omega \neq -1,$$  

(14)

where $\alpha$ is a generic constant which will be appropriately defined below in equation (19). The 1-form vector potential $A = A_\mu(r) dx^\mu$ is then

$$A = -\frac{1}{4\alpha} \chi_m (\omega + 1) \left( \frac{1}{(\omega + 1)(\omega + 3)} \right) d\varphi,$$  

(15)

Replacing solutions (13)-(15) into equations (7)-(12) allows us to find the $e^{2\phi(r)}$ function for $\omega \neq -2, -3/2, -1$; $\omega = -2$ and $\omega = -3/2$, respectively

$$e^{2\phi(r)} = (\alpha r)^{\frac{1}{1+\omega}} + \frac{b}{(\alpha r)^{\frac{1}{1+\omega}}} - \frac{k \chi_m^2}{(\alpha r)^{\frac{1}{1+\omega}}},$$  

(16)

$$e^{2\phi(r)} = \left( \frac{1}{1+\omega} \ln r \right),$$  

(17)

$$e^{2\phi(r)} = r^2 \left( \left( \omega + 1 \right) \Lambda \ln (br) - \chi_m^2 r^2 \right),$$  

(18)

where $b$ is a constant of integration related with the mass of the solutions, as will be shown, and $k = (\omega + 1)^2$. For $\omega \neq -2, -3/2, -1$ $\alpha$ is defined as

$$\alpha = \sqrt{\frac{1}{(\omega + 1)\Lambda}}.$$  

(19)

For $\omega = -2, -3/2$ we set $\omega = 1$. For $\omega = -2$ equations (8) and (8) imply $\Lambda = -\chi_m^2$. Now, we consider the case $\omega = -1$. From equations (7)-(12) it follows that $\mu = C_1, \phi = C_2$, where $C_1$ and $C_2$ are constants of integration, and that the cosmological constant and magnetic source are both zero, $\Lambda = 0 = \chi_m$. So, for $\omega = -1$ the metric gives simply the 3D Minkowski spacetime and the dilaton is constant, as occurred in the uncharged case [26, 27].

Now, we must analyse carefully the radial dependence of the $e^{2\phi(r)}$ function defined in equations (16)-(18) which, recall, is related to the metric components through the relations $g_{tt} = e^{-2\phi}$ and $g_{\varphi\varphi} = e^{2\phi}/a^2$. The shape of the $e^{2\phi(r)}$ function depends on the values of the Brans-Dicke parameter $\omega$ and on the values of the parameter $b$ (where $b$ is related to the mass of the solution as we shall see in section IV B). Nevertheless, we can group the values of $\omega$ and $b$ into a small number of cases for which the $e^{2\phi(r)}$ function has the same behavior. The general shape of the $e^{2\phi(r)}$ function for these cases is drawn in the Appendix. Generally, the $e^{2\phi(r)}$ function can take positive or negative values depending on the value of the coordinate $r$. However, when $e^{2\phi(r)}$ is negative, the metric components $g_{tt} = e^{-2\phi}$ and $g_{\varphi\varphi} = e^{2\phi}/a^2$ become simultaneously negative and this leads to an apparent
change of signature from +1 to −3. This strongly indicates that we are using an incorrect extension and that we should choose a different continuation to describe the region where the change of signature occurs [32, 41]. Moreover, analysing the null and timelike geodesic motion we conclude that null and timelike particles can never pass through the zero of $e^{i2\mu(r)}$, $r_+$ (say), from the region where $g_{rr}$ is positive into the region where $g_{rr}$ is negative. This suggests that one can introduce a new coordinate system in order to obtain a spacetime which is geodesically complete for the region where both $g_{rr}$ and $g_{\varphi\varphi}$ are positive [32, 41]. That is our next step. Then, in section III C, we will check the completeness of the spacetimes. The Brans-Dicke theories can be classified into seven different cases that we display and study below.

(i) Brans-Dicke theories with \(-1 < \omega < +\infty\)

The shape of the $e^{i2\mu(r)}$ function is drawn in Fig. 1(a). We see that for $0 < r < r_+$ (where $r_+$ is the zero of $e^{i2\mu(r)}$) $g_{rr}$ and $g_{\varphi\varphi}$ become simultaneously negative and this leads to an apparent change of signature. One can however introduce a new radial coordinate $\rho$ so that the spacetime is geodesically complete for the region where both $g_{rr}$ and $g_{\varphi\varphi}$ are positive,

$$\rho^2 = r^2 - r_+^2. \tag{20}$$

With this coordinate transformation, the spacetime generated by the static magnetic point source is finally given by

$$ds^2 = -\alpha^2 r^2(\rho) dt^2 + \frac{\rho^2}{r^2(\rho)} f(\rho) \frac{1}{\alpha^2} d\rho^2 + \frac{f(\rho)}{\alpha^2} d\varphi^2, \tag{21}$$

where $0 \leq \rho < \infty$, and function $f(\rho)$, which is always positive except at $\rho = 0$ where it is zero, is given by

$$f(\rho) = \alpha^2 r^2(\rho) + \frac{b}{\alpha^2 r^2(\rho)} - \frac{k\lambda^2}{\alpha^2 r^2(\rho)} \tag{22}$$

Along this section III B, in cases (i) and (iii) we will use the definition $r^2(\rho) \equiv \rho^2 + r_+^2$ in order to shorten the formulas. This spacetime has no horizons and so there are no magnetic black hole solutions, only magnetic point sources. In three dimensions, the presence of a curvature singularity is revealed by the scalar $R_{\mu\nu}R^{\mu\nu}$

$$R_{\mu\nu}R^{\mu\nu} =$$

$$\frac{-4\omega}{(\omega + 1)^2} \frac{b\alpha^4}{[\alpha^2 r^2(\rho)]^{\frac{\omega + 1}{\omega + 2}}} + \frac{(\omega^2 + 4\omega + 3)b^2 \alpha^4}{(\omega + 1)^2} \frac{2(\omega + 1)^2[\alpha^2 r^2(\rho)]^{\frac{\omega - 1}{\omega + 2}}}{\alpha^2 r^2(\rho)}$$

$$+ \frac{(\omega - 1)}{(\omega + 1)^2} \frac{8k_{\lambda m}^2 \alpha^4}{[\alpha^2 r^2(\rho)]^{\frac{\omega - 1}{\omega + 2}}} \frac{2(\omega + 1)^2[\alpha^2 r^2(\rho)]^{\frac{\omega - 1}{\omega + 2}}}{\alpha^2 r^2(\rho)}$$

$$- \frac{\omega^2 + 2\omega + 3)}{(\omega + 1)^4} \frac{k\lambda^2 \alpha^4}{[\alpha^2 r^2(\rho)]^{\frac{\omega - 1}{\omega + 2}}} + 12\alpha^4. \tag{23}$$

This scalar does not diverge for any value of $\rho$ (if $\omega > -1$). Therefore, spacetime (21) has no curvature singularities. However, it has a conic geometry with a conical singularity at $\rho = 0$. In fact, in the vicinity of $\rho = 0$, metric (21) is written as

$$ds^2 \sim -\alpha^2 r_+^2 dt^2 + \frac{\nu}{\alpha r_+} dp^2 + (\alpha r_+ \nu)^{-1} p^2 d\varphi^2, \tag{24}$$

with $\nu$ given by

$$\nu = \left[ \alpha r_+ - \frac{b(\alpha r_+)^{\frac{\omega - 1}{\omega + 1}}}{2(\omega + 1)} + \frac{k\lambda^2}{\omega + 1}(\alpha r_+)^{\frac{\omega - 1}{\omega + 1}} \right]^{-1}. \tag{25}$$

So, there is indeed a conical singularity at $\rho = 0$ since as the radius $\rho$ tends to zero, the limit of the ratio “circumference/radius” is not $2\pi$. The period of coordinate $\varphi$ associated with this conical singularity is

$$\lim_{\rho \to 0} \frac{1}{\rho} \int_{\varphi} \frac{d\varphi}{\sqrt{g_{rr}}} \sim 2\pi \nu. \tag{26}$$

From (24)-(26) one concludes that in the vicinity of the origin, metric (21) describes a spacetime which is locally flat but has a conical singularity with an angle deficit $\delta\varphi = 2\pi(1 - \nu)$.

Before closing this case, one should mention the particular $\omega = 0$ case of Brans-Dicke theory since this theory is related (through dimensional reduction) to 4D General Relativity with one Killing vector studied in [40].

(ii) Brans-Dicke theory with $\omega = \pm \infty$

The Brans-Dicke theory defined by $\omega = \pm \infty$ reduces to the spacetime generated by a static magnetic point source in 3D Einstein-Maxwell theory with $A = 0$ studied in detail in [23], [32]-[34]. The behavior of this spacetime is quite similar to those described in case (i). It has a conical singularity at the origin and no horizons.

(iii) Brans-Dicke theories with $-\infty < \omega < -2$

For this range of the Brans-Dicke parameter we have to consider separately the case (1) $b > 0$ and (2) $b < 0$, where $b$ is the mass parameter.

(1) If $b > 0$ the shape of the $e^{i2\mu(r)}$ function is drawn in Fig. 1(b). Both $g_{rr}$ and $g_{\varphi\varphi}$ are always positive (except at $r = 0$) and there is no apparent change of signature. Hence, for this range of parameters, the spacetime is correctly described by equations (5) and (16). There are no horizons and so no magnetic black holes, but at $r = 0$ the scalar $R_{\mu\nu}R^{\mu\nu}$ diverges (in equation (23) put $r = 0$ and replace $r_+$ by $r$). Therefore, at $r = 0$ one has the presence of a naked curvature singularity.

(2) If $b < 0$ the shape of the $e^{i2\mu(r)}$ function defined in (16) is sketched in Fig. 1(c). There occurs an apparent change of signature for $0 < r < r_+$. Proceeding
exactly as we did in case (i) one can however introduce the new radial coordinate \( \rho \) defined in (24) and obtain the geodesically complete spacetime described by (21) and (22) (where now \( \omega < -2 \)). This spacetime has no horizons and the scalar \( R_{\mu\nu}R^{\mu\nu} \) given by (23) does not diverge for any value of \( \rho \) and so no curvature singularities are present. The spacetime has a conical singularity at \( \rho = 0 \) corresponding to an angle deficit \( \delta \varphi = 2\pi (1 - \nu) \), where \( \nu \) is defined in (25).

(iv) \textit{Branst-Dicke theory with} \( \omega = -2 \)

The shape of the \( e^{2\phi(r)} \) function is drawn in Fig. 2(a). There is an apparent change of signature for \( r > r_+ \), where \( r_+ \) is the zero of \( e^{2\phi(r)} \). We can however introduce a new coordinate system that will allow us to conclude that the spacetime is complete for \( 0 \leq r \leq r_+ \). First, we introduce the radial coordinate \( R = 1/r \). With this new coordinate \( 2R \bar{R} |R|^{-1} \) has a shape similar to the one shown in Fig. 1(a). Finally, we set a second coordinate transformation given by \( \rho^2 = R^2 - R^2_+ \), where \( \rho = 1/R_+ \) is the zero of \( g_{rr} |R|^{-1} \). Use of these coordinate transformations, together with equations (5) and (17), allows us to write the spacetime generated by the static magnetic point source as

\[
dS^2 = -\frac{a^2}{R^2(\rho)}dt^2 + \frac{\rho^2}{|R(\rho)|^2} \left( \frac{1}{h(\rho)} d\rho^2 + \frac{h(\rho)}{\alpha^3} d\varphi^2 \right),
\]

where \( 0 \leq \rho < \infty \) and the function \( h(\rho) \) is given by

\[
h(\rho) = \left( 1 + \frac{\lambda^2}{8} \ln |R(\rho)| \right) R^{-2}(\rho) - bR^{-1}(\rho). \tag{28}
\]

This function \( h(\rho) \) is always positive except at \( \rho = 0 \) where it is zero. Hence, the spacetime described by equation (27) and (28) has no horizon. Along this section III B, in cases (iv)-(vi) we will use the definition \( R^2(\rho) \equiv \rho^2 + R^2_+ \) in order to shorten the formulas.

The scalar \( R_{\mu\nu}R^{\mu\nu} \) is given by

\[
R_{\mu\nu}R^{\mu\nu} = \lambda^2 \left[ \frac{\lambda^2}{4} \ln^2 |R(\rho)| + \frac{\lambda^2}{4} \ln |R(\rho)| + 9 \right] +
-\frac{\lambda^2}{8} \left[ 6 \ln |R(\rho)| + \frac{4 |R(\rho)|}{R(\rho)} + \frac{3}{R(\rho)} + 5 \right] +
+8 \left[ \frac{32}{\rho} \right] \frac{R(\rho)}{R(\rho)}.
\tag{29}
\]

This scalar diverges for \( \rho = +\infty \) and so there is a curvature singularity at \( \rho = 0 \). Besides, the spacetime described by (27) and (28) has a conical singularity at \( \rho = 0 \) with coordinate \( \varphi \) having a period defined in equation (26),

\[
\text{Period}_\varphi = 2\pi \left[ \varphi + \left( \frac{\lambda^2}{8} \ln |R(\rho)| + \frac{3}{2} \ln |R(\rho)| \right) \right].
\tag{30}
\]

So, near the origin, metric (27) and (28) describe a spacetime which is locally flat but has a conical singularity at \( \rho = 0 \) with an angle deficit \( \delta \varphi = 2\pi - \text{Period}_\varphi \).

(v) \textit{Branst-Dicke theories with} \(-2 < \omega < -3/2 \)

For this range of the Branst-Dicke parameter we have again to consider separately the case (1) \( b > 0 \) and (2) \( b < 0 \), where \( b \) is the mass parameter.

(1) If \( b > 0 \) the shape of the \( e^{2\phi(r)} \) function defined in (16) is similar to the one of case (iv) and sketched in Fig. 2(a). So, proceeding as in case (iv), we find that the spacetime generated by the static magnetic point source is given by (27) with function \( h(\rho) \) defined by

\[
h(\rho) = \frac{a^2}{R^2(\rho)} + b \left( \frac{\alpha^2}{R^2(\rho)} \right) - \frac{\lambda^2}{8} \ln |R(\rho)| R^{-3}(\rho), \tag{31}
\]

which is always positive except at \( \rho = 0 \) where it is zero. Hence, the spacetime described by equations (27) and (31) has no horizons.

The scalar \( R_{\mu\nu}R^{\mu\nu} \) is given by (23) as long as we replace function \( r^{-2}(\rho) \) by \( r^{-2}(\rho) = (\rho^2 + R^2_+)^{-1} \). There is a curvature singularity at \( \rho = +\infty \).

Near the origin, equations (27) and (31) describe a spacetime which is locally flat but has a conical singularity at \( \rho = 0 \) with an angle deficit \( \delta \varphi = 2\pi - \text{Period}_\varphi \), with \( \text{Period}_\varphi \) defined in equation (26),

\[
\text{Period}_\varphi = 2\pi \left[ \varphi + \left( \frac{\lambda^2}{8} \ln |R(\rho)| + \frac{3}{2} \ln |R(\rho)| \right) \right].
\tag{32}
\]

(2) If \( b < 0 \) the \( e^{2\phi(r)} \) function can have a shape similar to the one sketched in Fig. 2(b) or similar to Fig. 2(c), depending on the values of the range. We will not proceed further with the study of this case since it has a rather exotic spacetime structure.

(vi) \textit{Branst-Dicke theory with} \( \omega = -\frac{3}{2} \)

The shape of the \( e^{2\phi(r)} \) function defined in (18) is similar to the one of case (iv) and sketched in Fig. 2(a). So, proceeding as in case (iv), we conclude that the spacetime generated by the static magnetic point source is given by (27) with function \( h(\rho) \) defined by

\[
h(\rho) = R^{-2}(\rho) \left( \frac{\lambda}{2} \ln |bR^2(\rho)| - \frac{\lambda^2}{8} R^{-2}(\rho) \right), \tag{33}
\]

which is always positive except at \( \rho = 0 \) where it is zero. Hence, the spacetime described by equations (27) and (33) has no horizons.

The scalar \( R_{\mu\nu}R^{\mu\nu} \) is

\[
R_{\mu\nu}R^{\mu\nu} = \lambda^2 \left[ 12 \ln |bR(\rho)| + 20 \ln |bR(\rho)| + 9 \right] +
-\lambda \ln^2 |R(\rho)| \left[ \frac{5}{2} \ln |bR(\rho)| + \frac{9}{2} \right] + \frac{9}{16} \lambda \ln^2 R(\rho).
\tag{34}
\]

There is a curvature singularity at \( \rho = +\infty \).
Near the origin, equations (27) and (33) describe a spacetime which is locally flat but has a conical singularity at $\rho = 0$ with an angle defect $\Delta \varphi = 2\pi - \text{Period}_{\varphi}$, with Period$_{\varphi}$ defined in equation (26),

$$\text{Period}_{\varphi} = 2\pi \left[ r_+ \left( \frac{A}{2} + \chi_m^2 r_+^2 \right) \right]^{-1}, \quad (35)$$

(vii) Brauns-Dicke theories with $-3/2 < \omega < -1$

The shape of the $e^{2\beta(r)}$ function is sketched in Fig. 2(a) and is similar to the one of case (iv). So proceeding as in case (iv), we find that the spacetime generated by the static magnetic point source is given by (27) with function $g_{\mu\nu}(r)$ defined by (31). There are no horizons and there is no curvature singularity [the scalar $R_{\mu\nu}R^{\mu\nu}$ is given by (23) if we replace function $r^2(\rho)$ by $R^{-2}(\rho)$]. The spacetime has a conical singularity at $\rho = 0$ corresponding to an angle defect $\Delta \varphi = 2\pi - \text{Period}_{\varphi}$, where Period$_{\varphi}$ is defined in (32).

### C. Geodesic structure

We want to confirm that the spacetimes described by (5), (21) and (27) are both null and timelike geodesically complete. The equations governing the geodesics can be derived from the Lagrangian

$$\mathcal{L} = \frac{1}{2}g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = -\frac{\delta}{2}, \quad (36)$$

where $\tau$ is an affine parameter along the geodesic which, for a timelike geodesic, can be identified with the proper time of the particle along the geodesic. For a null geodesic one has $\delta = 0$ and for a timelike geodesic $\delta = +1$. From the Euler-Lagrange equations one gets that the generalized momenta associated with the time coordinate and angular coordinate are constants: $p_{\tau} = E$ and $p_{\varphi} = L$. The constant $E$ is related to the timelike Killing vector $(\partial/\partial t)^{\mu}$ which reflects the time translation invariance of the metric, while the constant $L$ is associated to the spacelike Killing vector $(\partial/\partial \varphi)^{\mu}$ which reflects the invariance of the metric under rotation. Note that since the spacetime is not asymptotically flat, the constants $E$ and $L$ cannot be interpreted as the energy and angular momentum at infinity.

From the metric we can derive the radial geodesic,

$$\dot{r}^2 = -\frac{1}{g_{\rho\rho}} \frac{E^2 g_{\varphi\varphi} + L^2 g_{\varphi\varphi}}{g_{\rho\rho}} - \frac{\delta}{g_{\varphi\varphi}}. \quad (37)$$

Next, we analyse this geodesic equation for each of the seven cases defined in the last section. Cases (i), (ii) and (iii) have identical geodesic structure, and cases (iv)-(vii) also.

(i) Brauns-Dicke theories with $-1 < \omega < +\infty$

Using the two useful relations $g_{\mu\nu}g_{\varphi\varphi} = -\rho^2/g_{\rho\rho}$ and $g_{\varphi\varphi} = [\rho^2/(\rho^2 + r_+^2)](\alpha^2 g_{\varphi\varphi})^{-1}$, we can write equation (37) as

$$\rho^2 \dot{\rho}^2 = \frac{E^2}{\alpha^2 \rho^2 + r_+^2} - \frac{\delta \rho^2}{g_{\varphi\varphi}} + L^2 g_{\varphi\varphi}. \quad (38)$$

Noticing that $1/g_{\varphi\varphi}$ is always positive for $\rho > 0$ and zero for $\rho = 0$, and that $g_{\varphi\varphi} < 0$, we conclude the following about the null geodesic motion ($\delta = 0$). The first term in (38) is positive (except at $\rho = 0$ where it vanishes), while the second term is always negative. We can then conclude that spiraling ($L \neq 0$) null particles coming in from an arbitrary point are scattered at the turning point $\rho_{\text{sp}} > 0$ and spiral back to infinity. If the angular momentum $L$ of the null particle is zero it hits the origin (there is a conical singularity) with vanishing velocity.

Now we analyze the timelike geodesics ($\delta = +1$). Timelike geodesic motion is possible only if the energy of the particle satisfies $E > a r_+$. In this case, spiraling timelike particles are bounded between two turning points that satisfy $\rho^b_{\text{sp}} > 0$ and $\rho^a_{\text{sp}} < \sqrt{E^2/a^2 - r_+^2}$, with $\rho^b_{\text{sp}} > \rho^a_{\text{sp}}$. When the timelike particle has no angular momentum ($L = 0$) there is a turning point located exactly at $\rho^a_{\text{sp}} = \sqrt{E^2/a^2 - r_+^2}$ and it hits the origin $\rho = 0$. Hence, we confirm that the spacetime described by equation (21) is both timelike and null geodesically complete.

(ii) Brauns-Dicke theory with $\omega = \pm \infty$

The geodesic structure of the spacetime generated by a static magnetic point source in 3D Einstein-Maxwell theory with $A < 0$ has been studied in detail in [34]. The behavior of this geodesic structure is quite similar to the one described in case (i). In particular, the spacetime is both timelike and null geodesically complete.

(iii) Brauns-Dicke theories with $-\infty < \omega < -2$

For this range of the Brauns-Dicke parameter we have to consider separately the case (i) $b > 0$ and (ii) $b < 0$, where $b$ is the mass parameter. $b > 0$.

(1) If $b > 0$ the motion of null and timelike geodesics is correctly described by equation (38) if we replace $\rho$ by $r$ and put $r_+ = 0$. Hence, the null and timelike geodesic motion has the same feature as the one described in the above case (i). In the above statements we just have to replace $\rho$ by $r$, put $r_+ = 0$ and remember that at the origin there is now a naked curvature singularity rather than a conical singularity.
(2) If \( b < 0 \) the motion of null and timelike geodesics is exactly described by (38) and the statements presented in case (i) apply directly to this case.

(iv), (v), (vi), (vii) Brans-Dicke theories with \( -2 \leq \omega < -1 \)

In order to study the geodesic motion of spacetime described by equations (27), one first notices that

\[
g_{\mu\nu} \equiv -g_{\mu\nu} = \left\{ \frac{\rho^2}{(\rho^2 + R_+^2)^2} \right\} \left( \frac{\rho^2}{(\rho^2 + R_+^2)^2} \right)^{\omega - 1} \frac{1}{(\rho^2 + R_+^2)^{3/(\omega - 1)}}. \text{ Hence we can write equation (37) as}
\]

\[
| \rho^2 \delta_r^2 = \left[ \frac{\rho^2}{(\rho^2 + R_+^2)^2} \right] \left( \frac{\rho^2}{(\rho^2 + R_+^2)^2} \right)^{\omega - 1} \frac{1}{(\rho^2 + R_+^2)^{3/(\omega - 1)}} \left( \frac{\rho^2}{(\rho^2 + R_+^2)^2} \right)^{\omega - 1} \frac{1}{(\rho^2 + R_+^2)^{3/(\omega - 1)}} \right].
\]

One concludes that the geodesic motion of null and timelike particles has the same feature as the one described in case (i) after (38) if in the statements we replace \( r_+ \) by \( R_+ \). The only difference is that on \( \rho = +\infty \) there is now a curvature singularity for cases (iv) \( \omega = -2 \), (v) \( -2 < \omega < -3/2 \) and (vi) \( \omega = -3/2 \). In case (v) \( -2 < \omega < -3/2 \), if \( b < 0 \) (as we saw in last section) the spacetime has an exotic structure and so we do not study it.

So, we confirm that the spacetimes described by equations (27) are also both timelike and null geodesically complete.

IV. THE GENERAL ROTATING SOLUTION

A. The generating technique

Now, we want to endow our spacetime solution with a global rotation, i.e., we want to add angular momentum to the spacetime. In order to do so we perform the following rotation boost in the \( t-\varphi \) plane (see e.g. [27]-[30], [42])

\[
t \rightarrow \gamma t - \frac{\varphi}{\alpha^2} \varphi,
\]

\[
\varphi \rightarrow \gamma \varphi - \varpi t,
\]

where \( \gamma \) and \( \varpi \) are constant parameters.

(i) Brans-Dicke theory with \( -1 < \omega < +\infty \)

Use of equation (40) and (21) gives the gravitational field generated by the rotating magnetic source

\[
ds^2 = -\alpha^2 (\rho^2 + r_+^2) (\gamma dt - \frac{\varpi}{\alpha^2} d\varphi)^2 + \frac{\rho^2}{(\rho^2 + r_+^2)} \frac{1}{f(\rho)} \frac{h(\rho)}{\alpha^2} (\gamma d\varphi - \varpi dt)^2.
\]

where \( f(\rho) \) is defined in (22).

The 1-form electromagnetic vector potential, \( A = A_\mu (\rho) dx^\mu \), of the rotating solution is

\[
A = -\varpi A(\rho) dt + \gamma A(\rho) d\varphi,
\]

where

\[
A(\rho) = -\frac{1}{4\alpha^2} \chi_m(\omega + 1)[\alpha^2 (\rho^2 + r_+^2)]^{-\frac{1}{\alpha^2}}.
\]

(ii) Brans-Dicke theory with \( \omega = +\infty \)

The spacetime generated by a rotating magnetic point source in 3D Einstein-Maxwell theory with \( \Lambda < 0 \) has been obtained and studied in detail in [34].

(iii) Brans-Dicke theories with \( \omega < -2 \)

Proceeding exactly as in case (i), we conclude that the gravitational and electromagnetic fields generated by the rotating magnetic fields are also described by equations (41)-(43).

If \( b > 0 \) we have to set \( r_+ = 0 \) in equations (41) and (43).

(iv) Brans-Dicke theory with \( -\infty < \omega < -2 \)

Use of equations (40) and (27) yields the gravitational field generated by the rotating magnetic source

\[
ds^2 = -\alpha^2 (\rho^2 + r_+^2) (\gamma dt - \frac{\varpi}{\alpha^2} d\varphi)^2 + \frac{\rho^2}{(\rho^2 + r_+^2)} \frac{1}{f(\rho)} \frac{h(\rho)}{\alpha^2} (\gamma d\varphi - \varpi dt)^2,
\]

where \( h(\rho) \) is defined in (28).

The 1-form vector potential is also given by (42) but now one has

\[
A(\rho) = -\frac{1}{4\alpha^2} \chi_m(\omega + 1)[\alpha^2 (\rho^2 + r_+^2)]^{-\frac{1}{\alpha^2}}.
\]

(v) Brans-Dicke theories with \( -2 < \omega < -3/2 \)

If \( b > 0 \), the gravitational and electromagnetic fields generated by the rotating magnetic source are described by equations (44) and (45), with \( h(\rho) \) defined in (31).

(vi) Brans-Dicke theory with \( \omega = -3/2 \)

The gravitational and electromagnetic fields generated by the rotating magnetic source are described by equations (44) and (45), with \( h(\rho) \) defined in (33).
(vii) Brans-Dicke theories with $-3/2 < \omega < -1$

The gravitational and electromagnetic fields generated by the rotating magnetic sources are described by equations (44) and (47), with $h(\rho)$ defined in (31).

In equations (41), (42) and (44) we choose $\gamma^2 - (\pi^2/\alpha^2) = 1$ because in this way when the angular momentum vanishes ($\pi \equiv 0$) we have $\gamma = 1$ and so we recover the static solution.

Solutions (41)-(45) represent magnetically charged stationary spacetimes and also solve (1). Analyzing the Einstein-Rosen bridge of the static solution one concludes that spacetime is not simply connected which implies that the first Betti number of the manifold is one, i.e., closed curves encircling the horizon cannot be shrunk to a point. So, transformations (40) generate a new metric because they are not permitted global coordinate transformations [43]. Transformations (40) can be done locally, but not globally. Therefore metrics (21), (27) and (41)-(45) can be locally mapped into each other but not globally, and such they are distinct.

B. Mass, angular momentum and charge of the solutions

As we shall see the spacetime solutions describing the cases (i) $-1 < \omega < +\infty$, (ii) $\omega = \pm \infty$ and (iii) $-\infty < \omega < -2$ are asymptotically anti-de Sitter. This fact allows us to calculate the mass, angular momentum and the electric charge of the static and rotating solutions. To obtain these quantities we apply the formalism of Regge and Teitelboim [44] (see also [24], [26]-[29]).

(i) Brans-Dicke theories with $-1 < \omega < +\infty$

We first write (41) in the canonical form involving the lapse function $N^0(\rho)$ and the shift function $N^\varphi(\rho)$

$$ds^2 = -(N^0)^2 dt^2 + \frac{dp^2}{f^2} + H^2(d\varphi + N^\varphi dt)^2,$$

where $f^{-2} = g_{tt}$, $H^2 = g_{\varphi \varphi}$, $H^2N^0 = \rho g_{tt}$ and $(N^0)^2 - H^2(N^\varphi)^2 = g_{\varphi \varphi}$. Then, the action can be written in the Hamiltonian form as a function of the energy constraint $\mathcal{H}$, momentum constraint $\mathcal{H}_\varphi$ and Gauss constraint $G$

$$S = -\Delta t \int dtd^2x [\mathcal{N}^0 \mathcal{H} + N^\varphi \mathcal{H}_\varphi + A\mathcal{G}] + \mathcal{B},$$

$$= -\Delta t \int d\rho d\varphi \left[ \frac{\rho e^{-2\beta}}{H^2} - \frac{A^0}{H} \rho e^{-2\beta}, \right] +$$

$$-2\mathcal{H} \left(f^0, \varphi\right) + H^2 f^2, +2f(f H, \varphi) e^{-2\beta} + 4\omega + \frac{A^0}{H} e^{-2\beta} (E^2 + B^2) +$$

$$+ \Delta t \int d\rho d\varphi \left( 2\rho e^{-2\beta}, \right) + \frac{A^0}{H} e^{-2\beta} E^2 B,$$

$$+ \Delta t \int d\varphi \left[ \frac{2\pi e^{-2\beta}}{H^2} + \frac{A^0}{H} e^{-2\beta} E^2 B, \right].$$

In order that the Hamilton’s equations are satisfied, the boundary term $\mathcal{B}$ has to be adjusted so that it cancels the above additional surface terms. More specifically one has

$$\delta \mathcal{B} = -\Delta t N \delta M + \Delta t N \delta J + \Delta t A \delta Q_e,$$

where one identifies $M$ as the mass, $J$ as the angular momentum and $Q_e$ as the electric charge since they are the terms conjugate to the asymptotic values of $N$, $N^\varphi$ and $A\mathcal{G}$, respectively.

To determine the mass, the angular momentum and the electric charge of the solutions one must take the spacetime that we have obtained and subtract the background reference spacetime contribution, i.e., we choose the energy zero point in such a way that the mass, angular momentum and charge vanish when the matter is not present.

Now, note that for $\omega > -1$ (and $\omega < -2$), spacetime (41) has an asymptotic metric given by

$$-\left( \gamma^2 - \frac{\pi^2}{\alpha^2} \right) \rho^2 d\varphi^2 + \frac{d\rho^2}{\alpha^2 \rho^2} + \frac{\rho^2}{\alpha^2 \rho^2} (\gamma^2 - \frac{\pi^2}{\alpha^2}) \rho^2 d\varphi^2,$$
momentum and electric charge are given by
\[
M = \nu \left[ (2H_{\phi} - H_{\phi}) e^{-2\phi} (f^2 - f_{\text{red}}^2) \right] \\
+ (f_{\text{red}}^2) e^{-2\phi} (H - H_{\text{red}}) - 2f^2 e^{-2\phi} (H_{\phi}^2 - H_{\phi_{\text{red}}}^2),
\]
\[
J = -2\nu e^{-2\phi}(\pi - \pi_{\text{red}}),
\]
\[
Q_e = \frac{4\pi}{\mathcal{F}} \nu e^{-2\phi}(E^\phi - E_{\text{red}}^\phi).
\]  
(51)

Then, we finally have that the mass and angular momentum are (after taking the asymptotic limit, \(\rho \to +\infty\))
\[
M = \nu \left[ (2H_{\phi} - H_{\phi}) e^{-2\phi} \left( \frac{\gamma + \omega + 2}{\omega + 1} \right) \right] + \text{Div}_M(\chi_m, \rho),
\]
\[
J = \frac{\gamma \pi}{\alpha^2} \frac{2\omega + 3}{\omega + 1} + \text{Div}_J(\chi_m, \rho),
\]
(52)
(53)

where \(\text{Div}_M(\chi_m, \rho)\) and \(\text{Div}_J(\chi_m, \rho)\) are terms proportional to the magnetic source \(\chi_m\) that diverge as \(\rho \to +\infty\). The presence of these kind of divergences in the mass and angular momentum is a usual feature present in charged solutions. They can be found for example in the electrically charged point source solution [9], the electric counterpart of the BTZ black hole [24], in the pure electric charged black hole of 3D Brans-Dicke action [28] and in the magnetic counterpart of the BTZ solution [34]. Following [9, 24] (see also [28, 34]) the divergences on the mass can be treated as follows: one considers a boundary of large radius \(\rho_0\) involving the system. Then, one sums and subtracts \(\text{Div}_M(\chi_m, \rho_0)\) to (52) so that the mass (52) is now written as
\[
M = M(\rho_0) + [\text{Div}_M(\chi_m, \rho) - \text{Div}_M(\chi_m, \rho_0)],
\]
(54)

where \(M(\rho_0) = M_0 + \text{Div}_M(\chi_m, \rho_0)\), i.e.,
\[
M_0 = M(\rho_0) - \text{Div}_M(\chi_m, \rho_0).
\]
(55)

The term between brackets in (54) vanishes when \(\rho \to \rho_0\). Then \(M(\rho_0)\) is the energy within the radius \(\rho_0\). The difference between \(M(\rho_0)\) and \(M_0\) is then \(\text{Div}_M(\chi_m, \rho_0)\) which is interpreted as the electromagnetic energy outside \(\rho_0\) apart from an infinite constant which is absorbed in \(M(\rho_0)\). The sum (55) is then independent of \(\rho_0\), finite and equal to the total mass. In practice the treatment of the mass divergence amounts to forgetting about \(\rho_0\) and take as zero the asymptotic limit: \(\lim_{\rho \to \infty} \text{Div}_M(\chi_m, \rho) = 0\).

To handle the angular momentum divergence, one first notices that the asymptotic limit of the angular momentum per unit mass \(J/M\) is either zero or one, so the angular momentum diverges at a rate slower or equal to the rate of the mass divergence. The divergence on the angular momentum can then be treated in a similar way as the mass divergence. So, one can again consider a boundary of large radius \(\rho_0\) involving the system. Following the procedure applied for the mass divergence one concludes that the divergent term \(-\text{Div}_J(\chi_m, \rho_0)\) can be interpreted as the electromagnetic angular momentum outside \(\rho_0\) up to an infinite constant that is absorbed in \(J(\rho_0)\).

Hence, in practice the treatment of both the mass and angular divergences amounts to forgetting about \(\rho_0\) and take as zero the asymptotic limits: \(\lim_{\rho \to \infty} \text{Div}_M(\chi_m, \rho) = 0\) and \(\lim_{\rho \to \infty} \text{Div}_J(\chi_m, \rho) = 0\) in (52) and (53).

Now, we calculate the electric charge of the solutions. To determine the electric field we must consider the projections of the Maxwell field on spatial hypersurfaces. The normal to such hypersurfaces is \(n^i = (1/N^i, 0, -N^\phi/N^i)\), and the electric field is given by \(E^i = g^{ij} F_{j\mu} n^\mu\). Then, from (51), the electric charge is
\[
Q_e = \frac{4\pi}{\mathcal{F}} \nu e^{-2\phi}(E^\phi - E_{\text{red}}^\phi).
\]
(56)

Note that the electric charge is proportional to \(\pi \chi_m\). In section V we will propose a physical interpretation for the origin of the magnetic field source and discuss the result obtained in (56).

The mass, angular momentum and electric charge of the static solutions can be obtained by putting \(\gamma = 1\) and \(\pi \neq 0\) on the above expressions [see (40)].

(ii) Brans-Dicke theory with \(\omega = \pm \infty\)

The mass, angular momentum and electric charge of the spacetime generated by a magnetic point source in 3D Einstein-Maxwell theory with \(A < 0\) have been calculated in [34]. Both the static and rotating solutions have negative mass and there is an upper bound for the intensity of the magnetic source and for the value of the angular momentum. 

(iii) Brans-Dicke theories with \(-\infty < \omega < -2\)

The mass, angular momentum and electric charge of the \(\omega < -2\) solutions are also given by (52), (53) and (56), respectively. If \(b < 0\) the factor \(\nu\) is defined in (25) and if \(b > 0\) one has \(\nu = 1\).

For \(-2 < \omega < -1\) [cases (iv)-(vii)], the asymptotic and background reference spacetimes have a very peculiar form. In particular, they are not an anti-de Sitter spacetime. Therefore, there are no conserved quantities for these solutions.

C. The rotating magnetic solution in final form

For cases (i) \(-1 < \omega < +\infty\), (ii) \(\omega = \pm \infty\) and (iii) \(-\infty < \omega < -2\) we may cast the metric in terms of \(M, J\) and \(Q_e\).

(i) Brans-Dicke theories with \(-1 < \omega < +\infty\)

Use of (52) and (53) allows us to solve a quadratic equation for \(\gamma^2\) and \(\pi^2/\alpha^2\). It gives two distinct sets of
solutions

\[ \gamma^2 = \frac{M(2 - \Omega)}{2\nu b}, \quad \frac{\pi^2}{\alpha^2} = \frac{\omega + 1}{2(\omega + 2)} \frac{M\Omega}{\nu b}, \quad (57) \]

\[ \gamma^2 = \frac{M\Omega}{2\nu b}, \quad \frac{\pi^2}{\alpha^2} = \frac{\omega + 1}{2(\omega + 2)} \frac{M(2 - \Omega)}{\nu b}, \quad (58) \]

where we have defined a rotating parameter \( \Omega \) as

\[ \Omega \equiv 1 - \sqrt{1 - \frac{4(\omega + 1)(\omega + 2)J^2\alpha^2}{(2\omega + 3)^2M^2}} \quad (59) \]

When we take \( J = 0 \) (which implies \( \Omega = 0 \)), (57) gives \( \gamma \neq 0 \) and \( \pi = 0 \) while (58) gives the nonphysical solution \( \gamma = 0 \) and \( \pi \neq 0 \) which does not reduce to the static original metric. Therefore, we will study the solutions found from (57).

For \( \omega > -1 \) (and \( \omega < -2 \)), the condition that \( \Omega \) remains real imposes a restriction on the allowed values of the angular momentum

\[ |\alpha J| \leq \frac{2\omega + 3}{2\sqrt{(\omega + 1)(\omega + 2)}} M. \quad (60) \]

The parameter \( \Omega \) ranges between \( 0 \leq \Omega \leq 1 \). The condition \( \gamma^2 - \pi^2/\alpha^2 = 1 \) fixes the value of \( b \) as a function of \( M, \Omega, \chi_m \),

\[ b = \frac{M}{\nu}\left[1 - \frac{2\omega + 3}{2(\omega + 2)}\Omega\right], \quad (61) \]

where

\[ \nu = \frac{2(\omega + 1)\alpha r_+^{\frac{\omega + 3}{\omega + 2}} + M\left(1 - \frac{2\omega + 3}{2(\omega + 2)}\Omega\right)}{2(\omega + 1)\alpha r_+^{\frac{\omega + 3}{\omega + 2}} + 2\chi_m^2(\alpha r_+)^{-\frac{\omega + 3}{\omega + 2}}}. \quad (62) \]

The gravitational field (41) generated by the rotating point source may now be cast in the form

\[ ds^2 = -\left[\alpha^2(\rho^2 + r_+^2) - \frac{\omega + 1}{2(\omega + 2)} \frac{M\Omega}{\nu} + \frac{(\omega + 1)^2}{8(\omega + 2)} \frac{Q_+^2/\nu^2}{M\Omega} \right] dt^2 - \frac{\omega + 1}{2\omega + 3} \left[\alpha^2(\rho^2 + r_+^2) - \frac{\omega + 1}{4M\Omega\nu} \frac{Q_+^2/\nu^2}{M\Omega} \right] d\varphi^2 + \frac{\rho^2}{\alpha^2(\rho^2 + r_+^2)} \frac{1}{\alpha^2(\rho^2 + r_+^2) + \frac{M}{\nu} \frac{1 - 2(\omega + 3)\Omega/2(\omega + 2)}{\alpha(\rho^2 + r_+^2)^{\frac{\omega + 3}{\omega + 2}}}} \frac{k\chi_m^2}{\alpha^2(\rho^2 + r_+^2)^{\frac{\omega + 3}{\omega + 2}}} d\rho^2 + \frac{1}{\alpha^2(\rho^2 + r_+^2) + \frac{M(2 - \Omega)}{\nu} \frac{\Omega}{\alpha^2(\rho^2 + r_+^2)^{\frac{\omega + 3}{\omega + 2}}} - \frac{(\omega + 2)(2 - \Omega)}{2(\omega + 2) - (2\omega + 3)\Omega} \frac{k\chi_m^2}{\alpha^2(\rho^2 + r_+^2)^{\frac{\omega + 3}{\omega + 2}}} d\varphi^2, \quad (63) \]

with \( 0 \leq \rho < \infty \) and the electromagnetic field generated by the rotating point source can be written as

\[ \gamma^2 = \frac{A(\rho)}{\sqrt{2(\omega + 2) - (2\omega + 3)\Omega}} \left[-\alpha \sqrt{(\omega + 1)\Omega} dt + \sqrt{(\omega + 2)(2 - \Omega)} d\varphi \right], \quad (64) \]

with \( A(\rho) \) defined in (43).

The static solution can be obtained by putting \( \Omega = 0 \) (and thus \( J = 0 \) and \( Q_e = 0 \)) on the above expression [see (40)].

\[ \text{(ii) Brans-Dicke theory with } \omega = \pm \infty \]

\[ \text{(iii) Brans-Dicke theories with } -\infty < \omega < -2 \]

The spacetime generated by a rotating magnetic point source in 3D Einstein-Maxwell theory with \( A < 0 \) is written as a function of its hairs in [34].

For this range of \( \omega \), the gravitational and electromagnetic fields generated by a rotating magnetic point source are also given by (63) and (64). If \( b < 0 \) the factor \( \nu \) is defined in (62) and if \( b > 0 \) one has \( \nu = 1 \).
D. Geodesic structure

The geodesic structure of the rotating spacetime is similar to the static spacetime (see section III.C), although there are now direct (corotating with $L > 0$) and retrograde (counter-rotating with $L < 0$) orbits. The most important result that spacetime is geodesically complete still holds for the stationary spacetime.

V. Physical interpretation of the magnetic source

When we look back to the electric charge given in (56), we see that it is zero when $\varpi = 0$, i.e., when the angular momentum $J$ of the spacetime vanishes. This is expected since in the static solution we have imposed that the electric field is zero ($F_{12}$ is the only non-null component of the Maxwell tensor).

Still missing is a physical interpretation for the origin of the magnetic field source. The magnetic field source is not a Nielsen-Olesen vortex solution since we are working with the Maxwell theory and not with an Abelian-Higgs model. We might then think that the magnetic field is produced by a Dirac point-like monopole. However, this is not the case since a Dirac monopole with strength $g_m$ appears when one breaks the Bianchi identity [45], yielding $\delta_{\mu}(\sqrt{-g} F^{\mu} e^{-2\Theta}) = g_m \delta^2(x)$ (where $F^{\mu} = \epsilon^{\mu\nu\rho} F_{\nu\rho}/2$ is the dual of the Maxwell field strength), whereas in this work we have that $\delta_{\mu}(\sqrt{-g} F^{\mu} e^{-2\Theta}) = 0$. Indeed, we are clearly dealing with the Maxwell theory which satisfies Maxwell equations and the Bianchi identity

$$\frac{1}{\sqrt{-g}} \delta_{\mu}(\sqrt{-g} F^{\mu\nu} e^{-2\Theta}) = \frac{\pi}{2} \frac{1}{\sqrt{-g}} j^\mu,$$

$$\delta_{\mu}(\sqrt{-g} F^{\mu} e^{-2\Theta}) = 0,$$

respectively. In (65) we have made use of the fact that the general relativistic current density is $1/\sqrt{-g}$ times the special relativistic current density $j^\mu = \sum g_{\mu\nu} (\vec{x} - \vec{x}_I) \dot{x}^\nu$.

We then propose that the magnetic field source can be interpreted as composed by a system of two symmetric and superposed electric charges (each with strength $q$). One of the electric charges is at rest with positive charge (say), and the other is spinning with an angular velocity $\Omega$ and negative electric charge. Clearly, this system produces no electric field since the total electric charge is zero and the magnetic field is produced by the angular electric current. To confirm our interpretation, we go back to Eq. (65). In our solution, the only non-vanishing component of the Maxwell field is $F^{\phi\rho}$ which implies that only $j^\phi$ is not zero. According to our interpretation one has $j^\phi = q \delta^2(\vec{x} - \vec{x}_I) \dot{\phi}$, which one inserts in Eq. (65).

Finally, integrating over $\phi$ and $\varphi$ we have

$$\chi_m \propto q \dot{\phi} \varphi,$$

So, the magnetic source strength, $\chi_m$, can be interpreted as an electric charge times its spinning velocity.

Looking again to the electric charge given in (56), one sees that after applying the rotation boost in the $t-\phi$ plane to endow the initial static spacetime with angular momentum, there appears a net electric charge. This result was already expected since now, besides the scalar magnetic field ($F_{12} \neq 0$), there is also an electric field ($F_{12} \neq 0$) [see (64)]. A physical interpretation for the appearance of the net electric charge is now needed. To do so, we return to the static spacetime. In this static spacetime there is a static positive charge and a spinning negative charge of equal strength at the center. The net charge is then zero. Therefore, an observer at rest (S) sees a density of positive charges at rest which is equal to the density of negative charges that are spinning. Now, we perform a local rotational boost $t^\prime = \gamma t - (\Omega/\alpha^2) \varphi$ and $\varphi^\prime = \gamma \varphi - \Omega t$ to an observer (S') in the static spacetime, so that S' is moving relatively to S. This means that S' sees a different charge density since a density is a charge over an area and this area suffers a Lorentz contraction in the direction of the boost. Hence, the two sets of charge distributions that had symmetric charge densities in the frame S will not have charge densities with equal magnitude in the frame S'. Consequently, the charge densities will not cancel each other in the frame S' and a net electric charge appears. This was done locally. When we turn into the global rotational Lorentz boost of Eqs. (40) this interpretation still holds. The local analysis above is similar to the one that occurs when one has a copper wire with an electric current and we apply a translation Lorentz boost to the wire: first, there is only a magnetic field but, after the Lorentz boost, one also has an electric field. The difference is that in the present situation the Lorentz boost is a rotational one and not a translational one.

VI. Conclusions

We have added the Maxwell term to the action of a generalized 3D dilaton gravity specified by the Brans-Dicke parameter $\omega$ introduced in [26]. For the static spacetime the electric and magnetic fields cannot be simultaneously non-zero, i.e., there is no static dyonic solution. Pure electrically charged solutions of the theory have been studied in detail in [28].

In this work we have found geodesically complete spacetimes generated by static and rotating magnetic point sources. These spacetimes are horizonless and many of them have a conical singularity at the origin. These features are common in spacetimes generated by point sources in 3D gravity theories [1]-[21]. The static solution generates a scalar magnetic field while the rotating solution produces, in addition, a radial electric field. The source for the magnetic field can be interpreted as composed by a system of two symmetric and superposed electric charges. One of the electric charges is at rest and
the other is spinning. This system produces no electric field since the total electric charge is zero and the scalar magnetic field is produced by the angular electric current. When we apply a rotational Lorentz boost to add angular momentum to the spacetime, there appears a net electric charge and a radial electric field.

For $\omega = \pm \infty$ our solution reduces to the magnetic counterpart of the BTZ solution, i.e., to the spacetime generated by a magnetic point source in 3D Einstein-Maxwell theory with $\Lambda < 0$ analysed in [23], [32]-[34]. The solutions corresponding to the theories described by a Brans-Dicke parameter that belongs to the range $-1 < \omega < +\infty$ or $-\infty < \omega < -2$ have a behavior quite similar to the magnetic counterpart of the BTZ solution. For this range of the Brans-Dicke parameter, the solutions are asymptotically anti-de Sitter. This allowed us to calculate the mass, angular momentum and charge of the solutions. The properties of divergences at spatial infinity in the conserved quantities is an unusual feature in electric solutions [9, 24, 28, 34].

The relation between spacetimes generated by point sources in 3D and cylindrically symmetric 4D solutions has been noticed by many authors (see e.g., [2, 7, 25, 29, 40]). The $\omega = 0$ solution considered in this paper is the 3D counterpart of the 4D longitudinal magnetic string source found in [40]. Indeed, the dimensional reduction of 4D general relativity with one Killing vector yields the $\omega = 0$ case of Brans-Dicke theory. Further working relating 4D solutions with 3D solutions has been done in [46] where the two well-known families of 4D black holes (i.e., the toroidal charged rotating anti-de Sitter black holes, and the Kerr-Newman anti-de Sitter black holes) are analyzed and the interpretation of the fields and charges in terms of the three-dimensional point of view is given.

This paper closes our program of studying theories of the Brans-Dicke type in 3D initiated in [20] and [26] and continued in [27] and [28].

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APPENDIX A: GENERAL SHAPE OF $e^{2\rho(r)}$ FUNCTION

In this appendix we study the radial dependence of the $e^{2\rho(r)}$ function defined in equations (16)-(18). The qualitative shape of the $e^{2\rho(r)}$ function varies with the

\[ e^{2\rho} = \begin{cases} 1 & -2 < \rho < 0 \\ r & \rho = -2 \\ 1 & \rho < -2 \end{cases} \]

FIG. 1: General shape of $e^{2\rho(r)}$ for: (a) Case (i) $\omega > -1$ and case (ii) $\omega = \pm \infty$ ; (b) Case (iii) $\omega < -2$ and $b > 0$; (c) Case (iii) $\omega < -2$ and $b < 0$.

\[ e^{2\rho} = \begin{cases} 1 & -2 < \rho < 0 \\ r & \rho = -2 \\ 1 & \rho < -2 \end{cases} \]

FIG. 2: General shape of $e^{2\rho(r)}$ for the values: (a) (iv) $\omega = -2$, (v) $-2 < \omega < -3/2$, $b > 0$, (vi) $\omega = -3/2$, (vii) $-3/2 < \omega < -1$; (b) some values of the range $\omega$ $-2 < \omega < -3/2$, $b > 0$; (c) some values of the range $\omega$ $-2 < \omega < -3/2$, $b < 0$.

Brans-Dicke parameter $\omega$ and with the mass parameter $b$. However, we can form a small number of seven cases for which the $e^{2\rho(r)}$ function and the spacetime structure has the same behavior. These seven cases are analysed in the text and the corresponding figures describing the respective shape of the $e^{2\rho(r)}$ function are as follows,

(i) $-1 < \omega < +\infty$ Fig. 1(a);
(ii) $\omega = \pm \infty$ Fig. 1(a);
(iii) $-\infty < \omega < -2$; if $b > 0$ Fig. 1(b),
if $b < 0$ Fig. 1(c);
(iv) $\omega = -2$ Fig. 2(a);
(v) $-2 < \omega < -3/2$; if $b > 0$ Fig. 2(a),
if $b < 0$ Fig. 2(b) or Fig. 2(c) depending on the values;
(vi) $\omega = -3/2$ Fig. 2(a)
(vii) $-3/2 < \omega < -1$ Fig. 2(a).

From the analysis of Fig. 1 and Fig. 2, we see clearly that for the range $\omega < -2$, $b > 0$ the function $e^{2\rho(r)}$ is always positive while for some values of the range $( -2 < \omega < -3/2, b < 0 )$ the function $e^{2\rho(r)}$ is always negative. For the other ranges of $(\omega, b)$, the $e^{2\rho(r)}$ function can take positive or negative values depending on the value of the coordinate $r$. When $e^{2\rho(r)}$ is negative, the spacetimes suffer an apparent change of signature from $+1$ to $-3$ (see section III B).
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