In their constructions of system of quantum stochastic differential equations, mathematicians and/or several physicists interpret that the function of random force operator is to preserve the canonical commutation relation in time, i.e., to secure the unitarity of time evolution generator even for dissipative systems. If this is the case, it means physically that the origin of dissipation is attributed to quantum non-commutativity (quantumness). The mechanism that the mathematician’s approaches rest on will be investigated from the unified view point of Non-Equilibrium Thermo Field Dynamics (NETFD) which is a canonical operator formalism of quantum systems in far-from-equilibrium state including the system of quantum stochastic equations.

1 Introduction

There are arguments\(^1,2,3,4,5,6\) that the function of random force operator is to preserve the canonical commutation relation in time. The contents of issue are the following. The time-evolution of free Boson operators is given by

\[
\frac{da(t)}{dt} = -i\omega a(t), \quad \frac{da^\dagger(t)}{dt} = i\omega a^\dagger(t).
\]

The canonical commutation relation \([a, a^\dagger] = 1\) at time \(t = 0\) preserves in time, i.e., \([a(t), a^\dagger(t)] = 1\). If a relaxation is introduced simply by

\[
\frac{da(t)}{dt} = -i\omega a(t) - \kappa a(t), \quad \frac{da^\dagger(t)}{dt} = i\omega a^\dagger(t) - \kappa a^\dagger(t),
\]

the canonical commutation relation decays as \([a(t), a^\dagger(t)] = e^{-2\kappa t}\). This inconvenience is secured by introducing random operators \(F(t)\) and \(F^\dagger(t)\) which are assumed to satisfy \([a, F^\dagger(t)] = 0, \quad [a^\dagger, F(t)] = 0\) etc. for \(t > 0\). The solutions of Langevin equations\(^a\)

\[
\frac{da(t)}{dt} = -i\omega a(t) - \kappa a(t) + F(t), \quad \frac{da^\dagger(t)}{dt} = i\omega a^\dagger(t) - \kappa a^\dagger(t) + F^\dagger(t),
\]

are given by

\[
a(t) = ae^{-\omega t - \kappa t} + \int_0^t dt' F(t')e^{-i\omega(t-t') + \kappa(t+t')}, \quad a^\dagger(t) = a^\dagger e^{i\omega t - \kappa t} + \int_0^t dt' F^\dagger(t')e^{i\omega(t-t') + \kappa(t+t')}.\]

Then, one knows that the canonical commutation relation preserves in time if the condition

\[
1 = [a(t), a^\dagger(t)] = e^{-2\kappa t} \left\{ 1 + \int_0^t dt_1 \int_0^t dt_2 [F(t_1), F^\dagger(t_2)]e^{i\omega(t_1-t_2) + \kappa(t_1+t_2)} \right\}
\]

is satisfied. It is realized by the commutation relation among the random force operators:

\[
[F(t), F^\dagger(t')] = 2\kappa \delta(t - t').
\]

\(^a\)These stochastic differential equations should be interpreted as those of the Stratonovich type\(^{16}\), since we perform calculations as if they were ordinary differential equations.
The above argument seems to show us that the function of random force operators is to recover the unitarity of time-evolution generator rather than to represent dissipative thermal effects. If it is correct, doesn’t it mean, physically, that the origin of dissipation can be attributed to quantum non-commutativity \((\text{quantumness})\)? There are, however, physical systems described by classical mechanics, or quantum systems with commutative random force operators.

In this paper, with the help of the framework of Non-Equilibrium Thermo Field Dynamics (NETFD) \(^7,^8,^9,^{10,11,12,13,14}\), we will investigate the above argument in a systematic manner by means of martingale operator paying attention to the non-commutativity among random force operators. In section 2, the structure of the system of stochastic differential equations in classical mechanics is reviewed. In section 3, the system of quantum stochastic differential equations within NETFD is introduced. In section 4, Boson system will be treated, which has a linear dissipative coupling with environment system within the rotating wave approximation. The mathematician’s arguments will be studied from the unified viewpoint based on the canonical operator formalism of NETFD by changing the intensity of non-commutativity parameter \(\lambda\) among a martingale operator. In section 5, Boson system having a linear dissipative interaction between environment without the rotating wave approximation will be investigated with the help of NETFD. This is the case where the system has commutative random force operators. Section 6 will be devoted to some remarks.

2 System of Stochastic Differential Equations in Classical Mechanics

2.1 Stochastic Liouville Equation

We will show the structure of the system of classical stochastic differential equations \(^{15}\) starting with the stochastic Liouville equation

\[
df(u, t) = \Omega(u, t)dt \odot f(u, t), \tag{5}
\]

of the Stratonovich type with \(\Omega(u, t)dt = -\langle \partial/\partial u \rangle du\) where the flow \(du\) in the velocity space is defined by \(du = -\gamma u dt + m^{-1}dR(t)\). Here, the circle \(\odot\) represents the Stratonovich stochastic product \(^{16}\) which is defined in appendix A together with the definition of the Ito stochastic product \(^{17}\). The increment of random force \(dR(t)\) is a Gaussian white stochastic process defined by the fluctuation-dissipation theorem of the second kind:

\[
\langle dR(t) \rangle = 0, \quad \langle dR(t)dR(t) \rangle = 2m\gamma T dt, \tag{6}
\]

where \(\gamma (> 0)\) is a relaxation constant, and \(T\) a temperature of the environment represented by the random force \(dR(t)\). Note that within the Stratonovich calculus \(dR(t)\) and \(f(u, t)\) are not stochastically independent: \(\langle dR(t) \odot f(u, t) \rangle \neq 0\). The average \(\langle \cdot \cdot \cdot \rangle\) is taken over all the possibility of the stochastic process \(\{dR(t)\}\).

By making use of the relation between the Ito and Stratonovich products (62), the stochastic Liouville equation (5) can be rewritten as the one of the Ito type:

\[
df(u, t) = \Omega(u, t)dt f(u, t), \tag{7}
\]
with $\Omega(u,t)dt = -(\partial/\partial u)du$ where $du$ is the flow in the Ito calculus given by $du = -\gamma (u + m^{-1}T(\partial/\partial u)) dt + m^{-1}dR(t)$. Note that there appears temperature $T$ in the flow, and that within the Ito calculus $dR(t)$ and $f(u,t)$ are stochastically independent: $\langle dR(t)f(u,t) \rangle = 0$.

The initial condition for the stochastic distribution function $f(u,t)$ is given by $f(u,0) = P(u,0)$, where $P(u,t)$ is the velocity distribution function defined below in subsection 2.3.

Note that the stochastic distribution function conserves its probability within the relevant velocity phase-space: $\int du f(u,t) = 1$

2.2 Langevin Equation

The system described by the stochastic Liouville equation (5) can be treated by the Langevin equation:

$$du(t) = -\gamma u(t)dt + m^{-1}dR(t). \tag{8}$$

This Stratonovich type stochastic differential equation does not contain the diffusion term, which is very much related to the way how physicists originally introduced the Langevin equation.\footnote{The Langevin equation was introduced by adding a random force term, such as $m^{-1}dR(t)/dt$, to a macroscopic phenomenological equation, for example, like $du(t)/dt = -\gamma u(t)$.}

It is worthwhile to note here that one could have introduced the Langevin equation within the Ito calculus of the form

$$du(t) = -\gamma [u(t) + m^{-1}T(\delta/\delta u(t))] dt + m^{-1}dR(t), \tag{9}$$

which has a term with the functional derivative operator $\delta/\delta u(t)$. In the system of quantum stochastic differential equations within NETFD, the Langevin equation of the Ito type, such as (9), can be introduced on the equal footing as the one of the Stratonovich type (8), although this was not the original motivation for the invention of NETFD (see section 3).

2.3 Fokker-Planck Equation

In precise, the stochastic distribution function is given by $f(u,t) = f(\Omega(u,t), P(u,0))$. Taking the random average $\langle \cdots \rangle$, we have an ordinary velocity distribution function $P(u,t) = \langle f(\Omega(u,t), P(u,0)) \rangle$ which satisfies the Fokker-Planck equation

$$\partial P(u,t)/\partial t = (\partial/\partial u)[u + m^{-1}T(\partial/\partial u)]P(u,t). \tag{10}$$

This can be derived most conveniently from the stochastic Liouville equation (7) of the Ito type because of the orthogonal property mentioned in subsection 2.1.

The fluctuation-dissipation theorem (6) of the second kind is introduced in order that the stochastic Liouville equation (5) and the Langevin equation (8) are consistent with the Fokker-Planck equation (10).
3 System of Quantum Stochastic Differential Equations

3.1 Non-Equilibrium Thermo Field Dynamics

In order to treat dissipative quantum systems dynamically, we constructed the framework of NETFD \cite{7,8,9,10,11,12,13,14}. It is a canonical operator formalism of quantum systems in far-from-equilibrium state which enables us to treat dissipative quantum systems by a method similar to the usual quantum field theory that accommodates the concept of the dual structure in the interpretation of nature, i.e. in terms of the operator algebra and the representation space. In NETFD, the time evolution of the vacuum is realized by a condensation of $\gamma^+\tilde{\gamma}^-$-pairs into vacuum, and that the amount how many pairs are condensed is described by the one-particle distribution function $n(t)$ whose time-dependence is given by corresponding kinetic equation (see appendix B).

We further succeeded to construct a unified framework of the canonical operator formalism for quantum stochastic differential equations with the help of NETFD. To the author's knowledge, it was not realized, until the formalism of NETFD had been constructed, that one can put all the stochastic differential equations for quantum systems into a unified method of canonical operator formalism; the stochastic Liouville equation \cite{15} and the Langevin equation within NETFD are, respectively, equivalent to the Schrödinger equation and the Heisenberg equation in quantum mechanics. These stochastic equations are consistent with the quantum master equation which can be derived by taking random average of the stochastic Liouville equation.

3.2 Quantum Stochastic Liouville Equation

Let us start the consideration with the stochastic Liouville equation of the Ito type:

$$d|0_f(t)\rangle = -i\hat{H}_{f,t} dt |0_f(t)\rangle.$$  \hspace{1cm} (11)

The generator $\hat{V}_f(t)$, defined by $|0_f(t)\rangle = \hat{V}_f(t)|0\rangle$, satisfies $d\hat{V}_f(t) = -i\hat{H}_{f,t} dt \hat{V}_f(t)$ with $\hat{V}_f(0) = 1$. The stochastic hat-Hamiltonian $\hat{H}_{f,t} dt$ is a tildian operator satisfying $(i\hat{H}_{f,t} dt)^\sim = i\hat{H}_{f,t} dt$. Any operator $A$ of NETFD is accompanied by its partner (tilde) operator $\tilde{A}$, which enables us treat non-equilibrium and dissipative systems by the method similar to usual quantum mechanics and/or quantum field theory. Here, the tilde conjugation $\sim$ is defined by $(A_1A_2)^\sim = A_1\tilde{A}_2$, $(c_1A_1 + c_2A_2)^\sim = c_1^*\tilde{A}_1 + c_2^*\tilde{A}_2$, $(\tilde{A})^\sim = A$, and $(A^\dagger)^\sim = \tilde{A}^\dagger$ with $A$'s and $c$'s being operators and c-numbers, respectively. The thermal ket-vacuum is tilde invariant: $|0_f(t)\rangle^\sim = |0_f(t)\rangle$.

From the knowledge of the stochastic integral, we know that the required form of the hat-Hamiltonian should be

$$\hat{H}_{f,t} dt = \tilde{H} dt + :d\tilde{M}_t:$$  \hspace{1cm} (12)

where $\tilde{H}$ is given by $\tilde{H} = \tilde{H}_S + i\tilde{H}$ with $\tilde{H}_S = H_S - \tilde{H}_S$, and $\tilde{H} = \tilde{H}_R + \tilde{H}_D$ where $\tilde{H}_R$ and $\tilde{H}_D$ are, respectively, the relaxational and the diffusive parts of the damping operator $\tilde{H}$. The martingale $d\tilde{M}_t$ is the term containing the operators
representing the quantum Brownian motion $dB_t$, $d\hat{B}_t^\dagger$ and their tilde conjugates, and satisfies $\langle|d\hat{M}_t|\rangle = 0$. The symbol $:d\hat{M}_t:$ indicates to take the normal ordering with respect to the annihilation and the creation operators both in the relevant and the irrelevant systems (see (23)).

The operators of the quantum Brownian motion are introduced in appendix C, and satisfy the weak relations:

\begin{align*}
    dB_t^\dagger dB_t &= \hat{n}dt, & dB_t dB_t^\dagger &= (\hat{n} + 1)dt, \\
    d\hat{B}_t dB_t &= \hat{n}dt, & d\hat{B}_t dB_t^\dagger &= (\hat{n} + 1)dt,
\end{align*}

and their tilde conjugates, with $\hat{n}$ being the Planck distribution function given in appendix B. $\langle|$ and $|$ denote the vacuum states representing the quantum Brownian motion. They are tilde invariant: $\langle|\sim \langle| = \langle|$. It is assumed that, at $t = 0$, a relevant system starts to contact with the irrelevant system representing the stochastic process included in the martingale $d\hat{M}_t$.

### 3.3 Quantum Langevin Equation

The dynamical quantity $A(t)$ of the relevant system is defined by the operator in the Heisenberg representation: $A(t) = \hat{V}^{-1}_f(t) A \hat{V}_f(t)$ where $\hat{V}^{-1}_f(t)$ satisfies $d\hat{V}^{-1}_f(t) = \hat{V}^{-1}_f(t) i\hat{H}^{[f]}_t dt$ with $\hat{H}^{[f]}_t dt = \hat{H}^{[f]}_t dt + id\hat{M}_t d\hat{M}_t$.

In NETFD, the Heisenberg equation for $A(t)$ within the Ito calculus is the quantum Langevin equation of the form

\[ dA(t) = i[\hat{H}^{[f]}_t dt, A(t)] - d\hat{M}(t) [d\hat{M}(t), A(t)], \]

with $\hat{H}^{[f]}_t dt = \hat{V}^{-1}_f(t) \hat{H}^{[f]}_t dt \hat{V}_f(t)$, and

\[ d\hat{M}(t) = \hat{V}^{-1}_f(t) d\hat{M}_t \hat{V}_f(t). \]

Since $A(t)$ is an arbitrary observable operator in the relevant system, (15) can be the Ito’s formula generalized to quantum systems.

Applying the bra-vacuum $\langle|1\rangle = \langle|1\rangle$ to (15) from the left, we obtain the Langevin equation for the bra-vector $\langle|A(t)\rangle$ in the form

\[ d\langle|A(t)\rangle = i\langle|[H_S(t), A(t)]dt + \langle|A(t)\hat{H}(t)dt - i\langle|A(t) d\hat{M}(t). \]

In the derivation, use had been made of the properties $\langle|\hat{A}^\dagger(t) = \langle|A(t), \langle|d\hat{B}^\dagger(t) = \langle|d\hat{B}^\dagger(t), \text{ and } \langle|d\hat{M}(t) = 0.$

### 3.4 Quantum Master Equation

Taking the random average by applying the bra-vacuum $\langle|$ of the irrelevant subsystem to the stochastic Liouville equation (11), we can obtain the quantum master equation as

\[ \frac{\partial}{\partial t} |0(t)\rangle = -i\hat{H}|0(t)\rangle, \]

with $\hat{H}dt = \langle|\hat{H}^{[f]}_t dt\rangle$ and $|0(t)\rangle = \langle|0_f(t)\rangle$.

Within the formalism, the random force operators $dB_t$ and $dB_t^\dagger$ are assumed to commute with any relevant system operator $A$ in the Schrödinger representation: $[A, dB_t] = [A, dB_t^\dagger] = 0$ for $t \geq 0$. 

5
3.5 Stratonovich-Type Stochastic Equations

By making use of the relation between the Ito and Stratonovich stochastic calculuses, we can rewrite the Ito stochastic Liouville equation (11) and the Ito Langevin equation (15) into the Stratonovich ones, respectively, i.e.,

\[ d\langle 0_f(t) \rangle = -i\hat{H}_{f,t}dt \circ \langle 0_f(t) \rangle, \]

with \( \hat{H}_{f,t}dt = \hat{H}_Sdt + i(\hat{H}dt + \frac{1}{2}d\hat{M}_t d\hat{M}_t) + d\hat{M}_t, \) and

\[ dA(t) = i[\hat{H}_f(t)dt \circ A(t)], \]

with \( \hat{H}_f(t)dt = \hat{H}_S(t)dt + i(\hat{H}(t)dt + \frac{1}{2}d'\hat{M}(t)d'\hat{M}(t)) + :d'\hat{M}(t):. \)

3.6 Fluctuation-Dissipation Relation

The fluctuation-dissipation theorem of the second kind for the multiple of martingales, \( d\hat{M}_t \) \( d\hat{M}_t \), is determined by the criterion that there is no diffusive term comes out in the terms \( \hat{H}dt + \frac{1}{2}d\hat{M}_t d\hat{M}_t \) appeared in \( \hat{H}_{f,t}dt \) in subsection 3.5:

\[ d\hat{M}_t \hat{M}_t = -2\hat{\Pi}_D dt. \]

The origin of this criterion is attributed to the way how the Langevin equation was introduced in physics, as explained before, i.e., relaxation term and random force term were introduced in mechanical equation within the Stratonovich calculus. Therefore, there is no dissipative terms in stochastic equations of the Stratonovich type. We adopted this criterion in quantum cases.

The operator relation (21) may be called a generalized fluctuation-dissipation theorem of the second kind, which should be interpreted within the weak relation.

4 A System in the Rotating Wave Approximation

4.1 Model

We will apply the above formalism to the model of a harmonic oscillator embedded in an environment with temperature \( T \). The Hamiltonian \( H_S \) of the relevant system is given by \( H_S = \omega a a^\dagger \) where \( a, a^\dagger \) and their tilde conjugates are stochastic operators of the relevant system satisfying the canonical commutation relation \([a, a^\dagger] = 1, \) and \([\tilde{a}, \tilde{a}^\dagger] = 1. \) The tilde and non-tilde operators are related with each other by the relation \( (1|a^\dagger = (1|\tilde{a} \) where \( (1| \) is the thermal bra-vacuum of the relevant system.

Since we are interested in the system in the rotating wave approximation, we will confine ourselves to the case where the stochastic hat-Hamiltonian \( \hat{H}_t \) is bilinear in \( a, a^\dagger, dB_t, dB_t^\dagger \) and their tilde conjugates, and is invariant under the phase transformation \( a \rightarrow ae^{i\theta}, \) and \( dB_t \rightarrow dB_t e^{i\theta}. \) This gives us the system of linear-dissipative coupling.

Then, \( \hat{\Pi}_R \) and \( \hat{\Pi}_D \) consisting of \( \hat{\Pi} \) introduced in subsection 3.2 become

\[ \hat{\Pi}_R = -\kappa(\gamma^2 \gamma_{t\nu} + \tilde{\gamma}^2 \tilde{\gamma}_{t\nu}), \quad \hat{\Pi}_D = 2\kappa(\tilde{a} + \nu)\gamma^2 \tilde{\gamma}, \]

(22)
respectively, where we introduced a set of canonical stochastic operators 
\[ \gamma_\nu = \mu a + \nu \bar{a}^\dagger, \quad \gamma^\dagger = a^\dagger - \bar{a} \] with \( \mu + \nu = 1 \), which satisfy the commutation relation 
\[ [\gamma_\nu, \gamma^\dagger] = 1 \]. The parameter \( \nu \) (or \( \mu \)) is closely related to the ordering of operators when they are mapped to c-number function space with the help of the coherent state representation \(^{11}\), i.e., \( \nu = 1 \) for the normal ordering, \( \nu = 0 \) for the anti-normal ordering, and \( \nu = 1/2 \) for the Weyl ordering. The new operators \( \gamma^\dagger \) and \( \gamma^\dagger \) annihilate the relevant bra-vacuum: \( \langle 1|\gamma^\dagger = 0, \quad \langle 1|\gamma^\dagger = 0 \).

### 4.2 Martingale Operator

Let us adopt the martingale operator:

\[
: d\hat{M}_t := : d\hat{M}_t^{(-)} : + \lambda : d\hat{M}_t^{(+)} :
\]

with \( : d\hat{M}_t^{(-)} := i(\gamma^\dagger dW_t + \gamma dW_t) \) and \( : d\hat{M}_t^{(+)} := -i(dW_t^\dagger \gamma_\nu + d\bar{W}_t^\dagger \gamma_\nu) \). Here, the annihilation and the creation random force operators \( dW_t \) and \( d\bar{W}_t^\dagger \) are defined, respectively, by \( dW_t = \sqrt{2\kappa}(\mu dB_t + \nu dB_t^\dagger) \), \( d\bar{W}_t^\dagger = \sqrt{2\kappa}(dB_t^\dagger - dB_t) \). The latter annihilates the bra-vacuum \( \langle 1 \) of the irrelevant system: \( \langle 1|dW_t^\dagger = 0, \langle 1|d\bar{W}_t^\dagger = 0 \). Note that the normal ordering \( \cdots \) is defined with respect to \( \gamma \)'s and \( dW \)'s.

The real parameter \( \lambda \) measures the degree of non-commutativity among the martingale operators: \( [: d\hat{M}_t^{(-)} : , : d\hat{M}_t^{(+)} :] = -2\hat{H}_R dt \). In deriving this, we used the facts that

\[
dW_t \ d\bar{W}_t = d\bar{W}_t \ dW_t = 2\kappa (\bar{n} + \nu) dt, \quad dW_t \ dW_t^\dagger = d\bar{W}_t \ d\bar{W}_t^\dagger = 2\kappa dt,
\]

and that the other combinations are equal to zero. Note that \( [dW_t, dW_t^\dagger] = 2\kappa dt \) should be compared with (4). There exist at least two physically attractive cases \(^{11,14}\), i.e., one is the case for \( \lambda = 0 \) giving non-Hermitian martingale:

\[
\hat{M}_t = i\sqrt{2\kappa} \left[ (a^\dagger - \bar{a}) \ d\bar{B}_t^\dagger + \text{t.c.} \right],
\]

and the other for \( \lambda = 1 \) giving Hermitian martingale:

\[
\hat{M}_t = i\sqrt{2\kappa} \left[ (a^\dagger dB_t - dB_t^\dagger a) + \text{t.c.} \right],
\]

where t.c. stands for tilde conjugation. The former follows the characteristics of the classical stochastic Liouville equation where the stochastic distribution function satisfies the conservation of probability within the phase-space of a relevant system (see section 2). Whereas the latter employed the characteristics of the Schrödinger equation where the norm of the stochastic wave function preserves itself. In this case, the consistency with the structure of classical system is destroyed \(^{11,14}\).

The fluctuation-dissipation theorem of the system is given by

\[
: d\hat{M}_t : = d\hat{M}_t := -2(\lambda\hat{H}_R + \hat{H}_D) dt,
\]

where we used the relations \( : d\hat{M}_t^{(-)} : = d\hat{M}_t^{(-)} := -2\hat{H}_D dt, \quad : d\hat{M}_t^{(+)} : = -2\hat{H}_R dt \) and \( : d\hat{M}_t^{(-)} : = d\hat{M}_t^{(+)} := 0 \), which can be derived by making use of (24).
The hat-Hamiltonians of the model are given by
\[ \hat{H}_{f,t} dt = \hat{H}_S dt + i(1 - \lambda)\hat{H}_R dt + d\hat{M}_t, \]  
\[ \hat{H}_{f,t}^- dt = \hat{H}_S dt + i((1 - 2\lambda)\hat{H}_R - \hat{H}_D) dt + d\hat{M}_t, \]  
\[ \hat{H}_f(t) dt = \hat{H}_S(t) dt + i(1 - \lambda)\hat{H}_R(t) dt + d'\hat{M}(t). \]  

4.3 Heisenberg Operators of the Quantum Brownian Motion

The Heisenberg operators of the Quantum Brownian motion are defined by
\[ B(t) = \hat{V}_f^{-1}(t) B_t \hat{V}_f(t), \quad B^\dagger(t) = \hat{V}_f^{-1}(t) B^\dagger_t \hat{V}_f(t), \]  
and their tilde conjugates. Their derivatives \( dB(t) = d(\hat{V}_f^{-1}(t) B^\#_t \hat{V}_f(t)) \), \( \# : \text{nul, dagger and/or tilde} \) with respect to time in the Ito calculus are given, respectively, by
\[ dB(t) = dB_t + \sqrt{2\kappa} \left[(1 - \lambda) \nu(\tilde{a}_t(t) - a(t)) - \lambda a(t)\right] dt, \]  
\[ dB^\dagger(t) = dB^\dagger_t - \sqrt{2\kappa} \left[(1 - \lambda) \mu(a^\dagger_t(t) - \tilde{a}_t(t)) + \lambda a_t(t)\right] dt, \]  
and their tilde conjugates. Then, we have
\[ dW(t) = dW_t - 2\kappa\gamma_\nu(t) dt, \quad dW^\pm(t) = dW^\pm_t - 2\kappa\gamma^\pm(t) dt. \]

Since, by making use of (34), we see that
\[ d\hat{M}(t) = d'\hat{M}(t) = i[\gamma^\pm(t) dW_t + \tilde{\gamma}^\pm(t) dW^\pm_t - i\lambda[dW^\pm_t\gamma_t(t) + d\tilde{W}^\pm_t\tilde{\gamma}_t(t)], \]  
we know that the martingale operator in the Heisenberg representation keeps the property: \( \langle d\hat{M}(t) \rangle = 0. \)

4.4 Quantum Langevin Equations

The quantum Langevin equation is given by
\[ dA(t) = i[\hat{H}_S(t), A(t)] dt \]
\[ + \kappa\{(1 - 2\lambda)(\gamma^\mp(t)\gamma_\nu(t), A(t)) + \tilde{\gamma}^\pm(t)\tilde{\gamma}_\nu(t), A(t))\} \]
\[ + \gamma^\pm(t), A(t))\gamma_\nu(t) + [\tilde{\gamma}^\pm(t), A(t)]\tilde{\gamma}_\nu(t)\} dt \]
\[ + 2\kappa(n + \nu)[\tilde{\gamma}^\pm(t), [\gamma^\pm(t), A(t)]\} dt \]
\[ - \{[\gamma^\pm(t), A(t)]dW_t + [\tilde{\gamma}^\pm(t), A(t)]d\tilde{W}_t \}
\[ + \lambda[dW^\pm_t\gamma_t(t), A(t)] + d\tilde{W}^\pm_t[\tilde{\gamma}_t(t), A(t)]\} \]
\[ = i[\hat{H}_S(t), A(t)] dt \]
\[ + \kappa\{(\gamma^\pm(t)\gamma_\nu(t), A(t)) + \gamma^\mp(t)\gamma_\nu(t), A(t))\}
\[ + (1 - 2\lambda)((\gamma^\pm(t), A(t))\gamma_\nu(t) + [\tilde{\gamma}^\pm(t), A(t)]\tilde{\gamma}_\nu(t))\} dt \]
\[ + 2\kappa(n + \nu)[\tilde{\gamma}^\pm(t), [\gamma^\pm(t), A(t)]\} dt \]
\[ - \{[\gamma^\pm(t), A(t)]dW(t) + [\tilde{\gamma}^\pm(t), A(t)]d\tilde{W}(t) \}
\[ + \lambda[dW^\pm(t)\gamma(t), A(t)] + d\tilde{W}^\pm(t)[\tilde{\gamma}(t), A(t)]\}. \]
with \( \hat{H}_S(t) = \hat{V}^{-1}_f(t)\hat{H}_S \hat{V}_f(t) = H_S(t) - \hat{H}_S(t) \). Note that the Langevin equation is written by means of the quantum Brownian motion in the Schrödinger (the interaction) representation (the input field \(^{18}\)) in (36), and by means of that in the Heisenberg representation (the output field \(^{18}\)) in (37).

The Langevin equation for the bra-vector state, \( \langle 1|A(t) \rangle \), reduces to

\[
d\langle 1|A(t) \rangle = i\langle 1| [H_S(t), A(t)] dt \]
\[
-\kappa \{ \langle 1| [A(t), a(t)] a(t^\dagger) + \langle 1| a(t^\dagger) [a(t), A(t)] \} dt \\
+2\kappa \bar{n} \langle 1|[a(t), [A(t), a(t^\dagger)]] dt \\
+\langle 1|[A(t), a(t^\dagger)]\sqrt{2\kappa} dB_t + \langle 1|\sqrt{2\kappa} B_t^1 [a(t), A(t)] \\
= i\langle 1| [H_S(t), A(t)] dt \\
-\kappa (1 - 2\nu) \{ \langle 1|[A(t), a^\dagger(t)] a(t) + \langle 1| a^\dagger(t) [a(t), A(t)] \} dt \\
+2\kappa \bar{n} \langle 1|[a(t), [A(t), a^\dagger(t)]] dt \\
+\langle 1|[A(t), a^\dagger(t)]\sqrt{2\kappa} dB(t) + \langle 1|\sqrt{2\kappa} B(t) [a(t), A(t)].
\]

(38)

(39)

The relation between the expression (38) and (39) can be interpreted as follows. Substituting the solution of the Heisenberg random force operators (32) and (33) for \( dB(t) \) and \( dB^1(t) \), respectively, into (39), we obtain the quantum Langevin equation (38) which does not depend on the non-commutativity parameter \( \lambda \).

The Langevin equations for \( a(t) \) and \( a^\dagger(t) \) of the system reduce to

\[
da(t) = (-i\omega - \kappa) a(t) dt + dW_t - 2(1 - \lambda) \nu \kappa \left[ \bar{a}^\dagger(t) - a(t) \right] dt - \lambda \nu d\bar{W}_t^2, \quad (40)
\]
\[
da^\dagger(t) = (i\omega - \kappa) a^\dagger(t) dt + d\bar{W}_t + 2(1 - \lambda) \mu \kappa \left[ a^\dagger(t) - \bar{a}(t) \right] dt + \lambda \mu dW_t^2. \quad (41)
\]

Note that the last two terms in the above equations disappear when one applies \( \langle 1| \) to them. For \( \lambda = 0 \), (40) and (41) become, respectively, to

\[
da(t) = -i\omega a(t) dt - \kappa \bar{a}^\dagger(t) dt + dW_t, \quad (42)
\]
\[
da^\dagger(t) = i\omega a^\dagger(t) dt - \kappa \bar{a}(t) dt + d\bar{W}_t, \quad (43)
\]

where we put \( \mu = \nu = 1/2 \), for simplicity. For \( \lambda = 1 \), we get

\[
da(t) = -i\omega a(t) dt - \kappa a(t) dt + \sqrt{2\nu} dB_t, \quad (44)
\]
\[
da^\dagger(t) = i\omega a^\dagger(t) dt - \kappa a^\dagger(t) dt + \sqrt{2\nu} B_t^1, \quad (45)
\]

which may correspond to (2). Applying \( \langle 1| = \langle 1| \) to (40) and (41), we obtain, for any values of \( \lambda, \mu \) and \( \nu \), the Langevin equations of the vectors \( \langle 1|a(t) \rangle \) and \( \langle 1|a^\dagger(t) \rangle \) in the forms

\[
d\langle 1|a(t) \rangle = -i\omega \langle 1|a(t) \rangle dt - \kappa \langle 1|a(t) \rangle dt + \sqrt{2\kappa} \langle 1|dB_t, \quad (46)
\]
\[
d\langle 1|a^\dagger(t) \rangle = i\omega \langle 1|a^\dagger(t) \rangle dt - \kappa \langle 1|a^\dagger(t) \rangle dt + \sqrt{2\kappa} \langle 1|dB_t^1. \quad (47)
\]

Note that these have the same structure as those in (2).
5 A System with Commutative Random Force Operators

Let us investigate Boson system having $x$-$X$ type interaction between environment, i.e., a system without the rotating wave approximation. The Hamiltonian of a harmonic oscillator can be written in the form $H_S = \frac{p^2}{2m} + m\omega^2 x^2/2$ with $x = \sqrt{1/2m\omega}(a + a^\dagger)$, $p = -i\sqrt{m\omega/2}(a - a^\dagger)$ where $x$ and $p$ satisfy the canonical commutation relation $[x, p] = i$. The normal ordering : $\cdots ;$, here, is taken with respect to $a$ and $a^\dagger$. The relaxational and the diffusive parts in $\hat{H} = \hat{H}_R + \hat{H}_D$ are given, respectively, as

$$\hat{H}_R = -i\kappa(x - \tilde{x})(p + \tilde{p})$$
$$\hat{H}_D = -2\kappa m\omega(\bar{n} + 1/2)(x - \tilde{x})^2.$$  \hspace{1cm} (48)

The martingale operator corresponding to $x$-$X$ type interaction may have the form

$$d\hat{M}_t = 2\sqrt{\kappa m\omega}(xdX_t - \tilde{x}d\tilde{X}_t),$$  \hspace{1cm} (49)

with $dX_t = (dB_t + dB_t^\dagger)/\sqrt{2}$ where $dB_t$ and $dB_t^\dagger$ are the quantum Brownian motion defined in appendix C. Then, we have $dX_t dX_t = dX_t d\tilde{X}_t = (\bar{n} + 1/2)dt$ which gives us the fluctuation-dissipation theorem

$$d\hat{M}_t d\hat{M}_t = -2\hat{H}_D dt.$$  \hspace{1cm} (50)

The form of the martingale (49) was adopted by following the structure of microscopic interaction Hamiltonian of the $x$-$X$ type.

The stochastic hat-Hamiltonian $\hat{H}_{f,t} dt$ for the stochastic Liouville equation (11) of the Ito type is given by

$$\hat{H}_{f,t} dt = \hat{H}_S + i(\hat{H}_R + \hat{H}_D) + d\hat{M}_t.$$  \hspace{1cm} (51)

Then, the stochastic hat-Hamiltonian of the Stratonovich type becomes

$$\hat{H}_{f,t} dt = \hat{H}_S dt + i\hat{H}_R dt + d\hat{M}_t,$$  \hspace{1cm} (52)

where one does not see $\hat{H}_D$ thanks to the fluctuation-dissipation theorem (50). We can also check that

$$\hat{H}_{f,t} dt = \hat{H}_S dt + i(\hat{H}_R - \hat{H}_D) dt + d\hat{M}_t.$$  \hspace{1cm} (53)

The Langevin equation has the forms

$$dx(t) = \frac{1}{m}p(t)dt + \kappa(x(t) - \tilde{x}(t))dt,$$  \hspace{1cm} (54)
$$dp(t) = -m\omega^2 x(t)dt - \kappa(p(t) + \tilde{p}(t))dt$$
$$+ 4i\kappa m\omega(\bar{n} + 1/2)(x(t) - \tilde{x}(t))dt - 2\sqrt{\kappa m\omega} dX_t.$$  \hspace{1cm} (55)

Applying (1) to (54) and (55), we have the Langevin equation for $\langle 1|x(t) \rangle$ and $\langle 1|p(t) \rangle$ in the forms

$$d\langle 1|x(t) \rangle = m^{-1}\langle 1|p(t) \rangle dt,$$  \hspace{1cm} (56)
$$d\langle 1|p(t) \rangle = -m\omega^2 \langle 1|x(t) \rangle dt - 2\kappa \langle 1|p(t) \rangle dt - 2\sqrt{\kappa m\omega} \langle 1|dX_t \rangle,$$  \hspace{1cm} (57)
respectively. They can be written in terms of $a$ and $a^\dag$ as

\[ d\langle 1 | a(t) = -i\omega \langle 1 | a(t)dt - \kappa \langle 1 | [a(t) - a^\dag(t)] dt - i\sqrt{\kappa} \langle 1 | \left( d B_t + d B_t^\dag \right) , \tag{58} \]

\[ d\langle 1 | a^\dag(t) = i\omega \langle 1 | a^\dag(t)dt - \kappa \langle 1 | [a^\dag(t) - a(t)] dt - i\sqrt{\kappa} \langle 1 | \left( d B_t^\dag + d B_t \right) . \tag{59} \]

If we take the rotating wave approximation at this stage the coefficients in front of the quantum Brownian motion are not equal to those appeared in (44) and (45). It may indicate that a naive procedure of taking the rotating wave approximation will not give us correct results. It might also be related to the renormalization procedure needed to derive stochastic differential equations for the system with $x$-$X$ type interaction from a microscopic Heisenberg equation 19.

6 Concluding Remarks

We have revealed that the non-commutativity among $d \hat{M}_t^{(-)}$ and $d \hat{M}_t^{(+)}$ appeared in the martingale operator of the model within the rotating wave approximation affect the relaxation part of the stochastic hat-Hamiltonian. When the measure $\lambda$ of the non-commutativity has the value $\lambda = 1$, the hat-Hamiltonian becomes Hermite, and therefore, it looks like being related to a microscopic description. On the other hand, for $\lambda = 0$, the system of the quantum stochastic differential equations has the same structure as that of classical mechanics, and it is related to a semi-macroscopic description. As has been shown in this paper, Hermiticity of the hat-Hamiltonian is realized thanks to the non-commutativity between $d \hat{M}_t^{(-)}$ and $d \hat{M}_t^{(+)}$.

Does this mean that the system with commutative martingale does not have any microscopic realization? As an example of system with commutative martingale, we studied the system corresponding to the quantum Kramers equation which has $x$-$X$ type interaction Hamiltonian between the relevant system and environment system. Since, there is no non-commutative parts in the martingale operator, this system cannot have an Hermitian stochastic hat-Hamiltonian. The non-commutative parts appears when one takes the rotating wave approximation to the interaction Hamiltonian. Does dissipation originate in the approximation causing non-commutative character in martingale? Can quantumness, appeared in this way, be the origin of dissipation? On the contrary, the following question arises naturally. Is it always possible to put all random force operators to be commutative?

There are still a lot of problems to be resolved before we know the origin of dissipation. However, with the help of NETFD, we can see the problems from a unified viewpoint which may provide us with good prospects for further developments. Introducing the parameter $\lambda$ in the martingale term as given by (23), we can transform the equation to the non-Hermitian version by shifting $\lambda \rightarrow 0$ (see (37)). In other words, it seems that the non-commutativity is renormalized into the relaxational and diffusive terms.

Substituting the solution of the random force operators (32) and (33) in the Heisenberg representation (the output field) into (37), we have the Langevin equation (36) expressed by means of those in the Schrödinger (or, more properly, the
interaction) representation (the input field). Note that the Langevin equation (38) for the bra-vector state \(|1/\rangle\langle A(t)|\) does not depend on \(\lambda\) when it is represented by the random force operator in the Schrödinger representation (the input field).

We are intensively investigating what is the physical meaning of the renormalization of non-commutativity by changing the parameter \(\lambda\). The relation between the present argument and the procedure of the coarse graining is under investigation. Related to the system with commutative random force operators, a microscopic derivation of quantum stochastic equations corresponding to the quantum Kramers equation are in progress \(^{19}\). There, an appropriate renormalization is required in accordance with the separation of two time-scales, i.e., microscopic and macroscopic time-scales. Without the renormalization, one gets quantum stochastic equations in the rotating wave approximation, which do not correspond to the system described by the Kramers equation. Including these studies, the further progress will be reported elsewhere.

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**Appendix A**

The definitions of the Ito \(^{17}\) and the Stratonovich \(^{16}\) stochastic products are given, respectively, by

\[
X_t \, dY_t = X_t \left( Y_{t+dt} - Y_t \right), \quad dX_t \, Y_t = (X_{t+dt} - X_t) \, Y_t \tag{60}
\]

and

\[
X_t \circ dY_t = \frac{X_t + X_{t+dt}}{2} \left( Y_{t+dt} - Y_t \right), \quad dX_t \circ Y_t = (X_{t+dt} - X_t) \, \frac{Y_{t+dt} + Y_t}{2} \tag{61}
\]

for arbitrary stochastic operators \(X_t\) and \(Y_t\). From (60) and (61), we have the formulae which connect the Ito and the Stratonovich products in the differential form

\[
X_t \circ dY_t = X_t \, dY_t + (1/2)dX_t \, dY_t, \quad dX_t \circ Y_t = dX_t \, Y_t + (1/2)dX_t \, dY_t. \tag{62}
\]

**Appendix B**

The time-evolution of the thermal vacuum \(|0(t)\rangle\), satisfying the quantum master equation (18) with the hat-Hamiltonian for the semi-free system specified by \(H_S = \)
\( \omega a^\dagger a \) and (22), is given by

\[
|0(t)\rangle = \exp \left\{ [n(t) - n(0)] \gamma^\dagger \gamma \right\} |0\rangle, \tag{63}
\]

where the one-particle distribution function, \( n(t) = \langle 1|a^\dagger(t)a(t)|0\rangle \), satisfies the kinetic (Boltzmann) equation of the model: \( dn(t)/dt = -2\kappa [n(t) - \bar{n}] \) with the Planck distribution function \( \bar{n} = (e^{\omega/T} - 1)^{-1} \). Here, \( T \) is the temperature of environment system.

### Appendix C

Let us introduce the annihilation and creation operators \( b_t, b_t^\dagger \) and their tilde conjugates satisfying the canonical commutation relation:

\[
[b_t, b_t^\dagger] = \delta(t - t') , \quad [\tilde{b}_t, \tilde{b}_t^\dagger] = \delta(t - t'). \tag{64}
\]

The vacuums (0) and |0\rangle are defined by \( b_t|0\rangle = 0, \tilde{b}_t|0\rangle = 0 \) and \( (0)b_t^\dagger = 0, (0)\tilde{b}_t^\dagger = 0 \). The subscript or the argument \( t \) represents time.

Introducing the operators \( B_t = \int_0^t dt' dB_{t'}, B_t^\dagger = \int_0^t dt' b_{t'} \), and their tilde conjugates for \( t \geq 0 \), we see that they satisfy \( B(0) = 0, B^\dagger(0) = 0, [B_s, B_t^\dagger] = \min(s, t), \) and their tilde conjugates, and that they annihilate the vacuums \( |0\rangle \) and \( (0) : d|0\rangle = 0, d\tilde{B}|0\rangle = 0, (0)dB^\dagger = 0, (0)d\tilde{B}^\dagger = 0 \). These operators represent the quantum Brownian motion.

Let us introduce a set of new operators by the relation \( dC_\mu^\nu = B_\mu^\nu dB_\nu^\dagger \) with the Bogoliubov transformation defined by

\[
B_\mu^\nu = \begin{pmatrix} 1 + \bar{n} & -\bar{n} \\ -1 & 1 \end{pmatrix}, \tag{65}
\]

where \( \bar{n} \) is the Planck distribution function. We introduced the thermal doublet:

\[
dB_\mu^1 = dB_t, \quad dB_\mu^2 = dB_t^\dagger, \quad d\tilde{B}_\mu^1 = d\tilde{B}_t, \quad d\tilde{B}_\mu^2 = -d\tilde{B}_t, \tag{66}
\]

and the similar doublet notations for \( dC_\mu^\nu \) and \( d\tilde{C}_\mu^\nu \). The new operators annihilate the new vacuum \( |\rangle \) and \( |\rangle : d|\rangle = 0, d\tilde{C}|\rangle = 0, (0)d|\rangle = 0, (0)d\tilde{C}|\rangle = 0 \).

We will use the representation space constructed on the vacuums \( |\rangle \) and \( |\rangle \). Then, we have, for example,

\[
\langle |dB_t\rangle = \langle |dB_t^\dagger\rangle = 0, \quad \langle dB_t dB_t^\dagger\rangle = \bar{n}dt, \quad \langle dB_t dB_t^\dagger dB_t^\dagger\rangle = (\bar{n} + 1)dt. \tag{67}
\]

### References