A model field theory, in which the interaction between quarks is mediated by dressed vector boson exchange, is used to analyse the pionic sector of QCD. It is shown that this model, which incorporates dynamical chiral symmetry breaking, asymptotic freedom and quark confinement, allows one to calculate $f_\pi$, $m_\pi$, $r_\pi$ and the partial wave amplitudes in $\pi-\pi$ scattering and obtain good agreement with the experimental data, with the latter being well described up to energies $E \simeq 700$ MeV.
I. INTRODUCTION

Chiral Perturbation Theory (ChPT) has been applied extensively and successfully to a broad range of low energy strong interaction phenomena; both purely mesonic [1] and with the inclusion of nucleons [2]. A fundamental ingredient of ChPT, and, indeed, what it emphatically establishes, is the notion that the theory underlying the strong interaction must manifest dynamically broken chiral symmetry.

Fundamental to obtaining physical information from ChPT, in the sense of calculating experimental observables, is the fact that at first nonleading order \(O(E^4)\); i.e., to fourth order in derivatives of the chiral field, one obtains a complete effective Lagrangian by including one-pion-loop graphs which introduce ten counterterms (whose coefficients are infinite). Higher order loops do not modify the Lagrangian at this order [3] but the ChPT Lagrangian is only completely defined upon the inclusion of these one-loop contributions.

This framework can be used to analyse the pionic sector of the strong interaction: in particular, \(\pi-\pi\) scattering [4,5]; and it is quite successful. One can choose the coefficients of the counterterms in such a way as to obtain rather good agreement with the partial wave amplitudes. The approach of Ref. [5] augments standard ChPT by the inclusion of scalar and vector resonances and obtains good agreement with the experimental S-, P-, D- and F-wave scattering lengths and reasonable agreement with the phase shifts up to \(E \simeq 800\) MeV.

There is another approach to modelling QCD which we wish to employ to study the pionic sector of QCD, in general, and \(\pi-\pi\) scattering in particular. This approach, as will be seen, actually contains ChPT in the sense that it generates an effective Lagrangian for the chiral field with the same structure as that employed in ChPT but with the difference that the coefficients of each term are finite with values determined by three parameters associated with the quark-gluon substructure of the model. Our approach is based on a model field theory, with elementary quark fields interacting via dressed vector boson exchange, that has been referred to as the Global Colour-symmetry Model (GCM) and which was first described in Ref. [6] and extended to the \(\pi\)-sector in Ref. [7]. The extension to other mesons and to the baryon is reviewed in Ref. [8].

Including quarks explicitly, in the sense of actually carrying out the operations on the generating functional for the model field theory, \(Z\), that lead to an expression of \(Z\) in terms of the chiral field so that the relation between the coefficients in the effective action and those describing the model quark-gluon dynamics is made explicit, is one property that sets this approach apart from ChPT. Quark confinement is incorporated by ensuring that the quark propagator obtained in the model has no singularities that can give rise to absorptive parts associated with free-quark production in the Feynman diagrams for any given process. (This is a sufficient but not necessary condition for quark confinement, as discussed in Ref. [6].)

In the GCM the effective action for mesonic degrees of freedom is generated by a derivative expansion of the fermion determinant, that arises through an Hubbard-Stratonovich transformation
and subsequent integration over the quark fields, in a manner analogous to that employed in the Nambu–Jona-Lasinio model (NJL) [10]. In contrast to the NJL studies, however, each fermion loop that arises is finite because the meson fields are not pointlike but, rather, they are extended objects whose internal structure is given by Bethe-Salpeter amplitudes obtained via solution of the ladder Bethe-Salpeter equation. [11] These Bethe-Salpeter amplitudes play the same role as the phenomenological form factors employed in the calculations of intermediate energy nuclear physics but, it should be stressed, they are calculated quantities in the GCM.

Another property that sets the GCM apart from ChPT is the fact that meson loops are also finite. This is because the meson-meson vertices are not pointlike; i.e., there are no contact $\pi(x)^4$ interactions, but instead each interaction has a momentum dependence determined by an underlying quark loop structure. This quark substructure generates “form factors” at the vertices and therefore the tree level effective action only receives finite corrections from one-loop diagrams, a fact which is illustrated in Ref. [13]. This is an important difference from ChPT. The studies of Ref. [13] also suggest that, in our model, the the loop corrections are small near threshold.

The GCM is a model which microscopically manifests dynamical chiral symmetry breaking (DCSB) via the mechanism well know from phenomenological Schwinger-Dyson equation (SDE) studies [14] and hence it will have in common with ChPT all of the consequences entailed by this. Herein we use the phenomenon of $\pi-\pi$ scattering to illustrate this. Of course, in specifying a model one has lost the “model independence” which is a beauty of ChPT. However, one might argue that a stronger connection with QCD has been obtained since, in addition to DCSB, the GCM incorporates the ultraviolet renormalisation group features of QCD by construction. This enables one to connect observable quantities with the dynamical mass scale of QCD, $\Lambda_{QCD}$, for example, and also to study their dependence on the qualitative and quantitative features of the propagators of the elementary excitations in QCD.

In Sec. II we briefly summarise the GCM and the procedure that allows one to proceed to an effective action for the Goldstone modes in the model. The action one obtains is of the same structural form as would be obtained in the NJL model but the calculation of physical observables is quite different due to the internal structure of the mesons. In this section we also present the expressions for $f_\pi$, $m_\pi$ and $r_\pi$ in the GCM. In Sec. III we study $\pi-\pi$ scattering at tree level. We calculate the scattering amplitude, $A(s, t, u)$, and the five lowest order partial wave amplitudes. In addition, through exploratory parameter fitting, we demonstrate that the GCM can reasonably be expected to describe this sector of QCD rather well. In Sec. IV we proceed to use the GCM to calculate the quantities that characterise the pion sector. We present a three parameter version of the model, the parameters characterising the nature of the quark-quark interaction at small $k^2$, and show that a rather good description of the scattering lengths and partial wave amplitudes is easy to obtain. We summarise our results and present our conclusions in Sec. V.
II. EFFECTIVE ACTION FOR PIONS

The action for the GCM is, in Euclidean metric (with \(\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}\) and \(\delta_{\mu\nu}\) the Kronecker delta):

\[
S[q, q] = \int d^4x \bar{q}(x) [\gamma \cdot \partial + M] q(x) + \frac{1}{2} \int d^4x \, d^4y \, j_\mu^a(x) g^2 D_{\mu\nu}(x - y) j_\nu^a(y). \tag{1}
\]

Here \(M\) is the quark mass matrix, \(j_\mu^a(x) = \bar{q}(x) \gamma_\mu \gamma_5 q(x)\) and

\[
g^2 D_{\mu\nu}(x - y) = \int \frac{d^4k}{(2\pi)^4} e^{ik \cdot x} \frac{1}{k^2} \left( \left[ \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right] F_T(k^2) + \xi \frac{k_\mu \gamma_5 k_\nu}{k^2} \right) \tag{2}
\]

is the dressed gluon propagator in the model. The specification of \(F_T(k^2)\) completes the definition of the model. (It should be noted that the action for an Abelian theory can be expressed exactly in this form, with \(F_T(k^2) = 1\), after functional integration over the gauge field. [15])

The generating functional for the field theory associated with this model action can be written:

\[
Z[\bar{\eta}, \eta] = \int D\bar{q} Dq \exp \left[ -S[q, q] + \int d^4x \left( \bar{q}(x) q(x) + \bar{q}(x) \eta(x) \right) \right] \tag{3}
\]

where \(\bar{\eta}\) and \(\eta\) are Grassmannian sources for the fermion fields. Following the bilocal bosonisation procedure described in detail in Ref. [6], a crucial step in which is the application of the bilocal analogue of the Hubbard-Stratonovich transformation familiar from statistical mechanics, this generating functional can be rewritten as:

\[
Z[\bar{\eta}, \eta] = \int DU \exp \left[ -S[U] + \text{source terms} \right] \tag{4}
\]

which now involves the local chiral field

\[
U(x) = \exp \left( i \frac{T^i \phi^i(x)}{\frac{1}{f_\phi}} \right), \tag{5}
\]

where \(\{T^i; i = 1, \ldots, N^2_f - 1\}\) are the analogue of the Pauli matrices in \(SU_f(N_f)\).

The action in Eq. (4) is

\[
S[U] = \Omega[U] + i \Gamma[U] = -\text{Tr} \ln \left[ G^{-1}(x, y; [U]) \right] + I[U] \tag{6}
\]

where

\[
G^{-1}(x, y; [U]) = \tilde{G}^{-1}(x - y; \tilde{U}) + \left\{ P_R \left[ U \left( \frac{x + y}{2} \right) - \tilde{U} \right] + P_L \left[ U^\dagger \left( \frac{x + y}{2} \right) - \tilde{U}^\dagger \right] \right\} B(x - y), \tag{7}
\]

\[
\tilde{G}^{-1}(x - y; \tilde{U}) = \gamma \cdot \partial A(x - y) + \left\{ P_R \tilde{U} + P_L \tilde{U}^\dagger \right\} B(x - y), \tag{8}
\]

and
\[ I[U] = \left( \int d^4w \right) \left( \frac{1}{8N_c} \int d^4z \, \text{tr} \left[ \left( P_R \tilde{U} + P_L \tilde{U}^\dagger \right) B(-z)T^i \left( \text{tr} \left[ T^i \tilde{G}(z; \tilde{U}) \right] + \gamma_5 \text{tr} \left[ \gamma_5 T^i \tilde{G}(z; \tilde{U}) \right] \right) \right] + \frac{1}{2} \int d^4z \, \text{tr} \left[ \gamma_\mu \tilde{G}(-z; \tilde{U}) \right] \partial_\mu \left[ A(z) - \delta^4(z) \right] \right) \] (9)

is the term necessary to complete the square in the Hubbard transformation. In Eq. (9),

\[ P_R = \frac{1}{2} (I + \gamma_5) \quad P_L = \frac{1}{2} (I - \gamma_5) \] (10)

are the right and left handed helicity projection operators.

The vacuum or ground state configuration of the chiral field is, as usual, defined as that configuration \( \hat{U} \) for which

\[ \frac{\delta}{\delta U} S[U] \bigg|_{U=\hat{U}} = 0 \] (11)

In the present case Eq. (11) entails that \( \hat{G}^{-1}(z; \hat{U}) \) is the solution of the ladder Schwinger-Dyson equation which has the following form in momentum space:

\[ \hat{G}^{-1}(p; \hat{U}) = i \gamma \cdot p + M + \frac{4}{3} \int \frac{d^4k}{(2\pi)^4} g^2 D_{\mu\nu}(p - k) \gamma_\mu \hat{G}(k; \hat{U}) \gamma_\nu . \] (12)

If the fermions are massless; i.e., \( M \equiv 0 \), then this (vacuum) equation admits the same solution for every spacetime independent, unitary flavour matrix \( \hat{U} \). This signals that chiral symmetry is dynamically broken if the ladder SDE admits a nonzero solution for \( B(p^2) \); a well known result seen here from another perspective.

We analyse the action in Eq. (4) using a derivative expansion of the fermion determinant (TrLn). In this process we expand about the vacuum configuration \( \hat{U} \) and, with the form of \( I[U] \) in Eq. (9), our “bare” pion fields are not pointlike but actually have an internal structure described by the ladder Bethe-Salpeter equation (11): the Bethe-Salpeter amplitude and fermion mass function, \( B(p^2) \), are identical because of the identity between the ladder SDE and Bethe-Salpeter equations in vector exchange theories with dynamically broken chiral symmetry. (12) We have already made use of this result in writing Eqs. (7-9).

The expansion of the TrLn generates terms whose coefficients are constructed from quark loops coupling to varying numbers of external pion fields; for example, the diagram in Fig. 1 is associated with the expression for \( f_\pi \). In this, and the diagrams associated with the other coefficients, the quark lines represent dressed quark propagators obtained from the solution of Eq. (12) and the pion Bethe-Salpeter amplitude provides an intrinsic cutoff on the momentum integration. As a consequence, each of the coefficients is finite and the action is completely determined, with no arbitrary finite or infinite constants being necessary, once the dressed gluon propagator is specified. [The only exception to this is the fermion condensate which, in the more sophisticated versions of the GCM (and QCD itself), diverges with the upper bound on the momentum integral in a manner governed by the anomalous dimension of the fermion propagator so that \( m^2_{\mu^2} \langle \overline{q}q \rangle_{\mu^2} \) is a renormalisation point invariant. This will become clear below.]
The real part of the action in Eq. (6) involves even numbers of pions and, to $O(E^4)$, it is [7]:
\[
\Omega[U] = \int d^4x \left\{ \frac{f_\pi^2}{4} \text{tr} \left[ \partial_\mu U \partial_\mu U^\dagger \right] + \frac{\rho_{\mu_2}}{4} \text{tr} \left[ \left( 2I_f - U - U^\dagger \right) M_{\mu_2} \right] 
- N_c K_1 \text{tr} \left[ \partial^2 U \partial^2 U^\dagger \right] + N_c K_2 \text{tr} \left[ \partial_\mu U \partial_\mu U^\dagger \right]^2 
- N_c K_3 \frac{1}{2} \text{tr} \left[ \partial_\mu U \partial_\nu U^\dagger \partial_\mu U \partial_\nu U^\dagger \right] \right\}
\]
where $M_{\mu_2} = \text{diag}(m_{\mu_2}^u, m_{\mu_2}^d)$ is the quark mass matrix ($\rho_{\mu_2} M_{\mu_2}$ is renormalisation point invariant in QCD and in the more sophisticated versions of the GCM) and
\[
U(x) = e^{i f_\pi \tau \cdot \vec{\pi}(x)}.
\]

At this point we have specialised to the case of $SU_f(2)$ [$\tau$ are the Pauli matrices] and we have chosen the vacuum configuration $\tilde{U} = I_f$ which preserves $SU_V(2)$ symmetry and necessarily follows when $M \neq 0$. The structural form of this part of the action is the same as that of Ref. [10].

The imaginary part of the action involves odd numbers of pions and the leading order term is $O(E^5)$:
\[
\imath \Gamma[U] = \int d^4x \frac{N_c}{240 \pi^2} \epsilon_{\mu\nu\lambda\rho} \frac{1}{2} \text{tr} \left[ \left( \tilde{U}^\dagger U(x) - U^\dagger(x) \tilde{U} \right) \partial_\mu U(x) \partial_\nu U(x) \partial_\lambda U(x) \partial_\rho U(x) \right].
\]

At lowest order in the pion field [$O(\pi^5)$] this term yields the Wess-Zumino term [16] and when vector mesons are included this imaginary part of the action contains all of the “anomalous” interactions. [17]

In the GCM, as mentioned above, the coefficients in Eq. (??) ($f_\pi$, $\rho$, etc.) are completely determined once the fermion propagator is known and this follows as the solution of the fermion SDE, Eq. (12), once the model gluon propagator, Eq. (2), is specified. Writing the fermion propagator in the form
\[
\tilde{G}(p) = -i \gamma \cdot p \sigma_V(p) + \sigma_S(p)
\]
then: [7]
\[
f_\pi^2 = \frac{N_c}{8 \pi^2} \int_0^\infty dx \ B^2 \left( \sigma_V^2 - 2 \left[ \sigma_S \sigma'_S + x \sigma_V \sigma'_V \right] - x \left[ \sigma_S \sigma''_S - (\sigma'_S)^2 - x^2 \left[ \sigma_V \sigma''_V - (\sigma'_V)^2 \right] \right) \right),
\]
\[
\rho_{\mu_2} = \frac{N_c}{4 \pi^2} \int_0^{\mu_2^2} dx \ x \sigma_S
\]
and $K_1$, $K_2$, $K_3$ are given in the appendix.

One can also use this approach to study the electromagnetic pion form factor, $F_\pi(q^2)$, and this will be described elsewhere. [18] From the generalised impulse approximation diagram, Fig. (2), one obtains the piece of the charge radius that is regular in the chiral limit, $m_\pi \to 0$, and only weakly dependent on $m_\pi$. Preliminary studies suggest that this is the dominant piece at the physical value of $m_\pi$ [19], a result supported by the lattice QCD studies of Ref. [20]. [In ChPT $r_\pi \sim \ln m_\pi^2$ as
\[ m_\pi \rightarrow 0. \] Since the pion is a point particle in ChPT it derives its form factor from self dressing; the lowest order process being interaction with the photon via a pion loop arising from a \( \pi \pi \rightarrow \pi \pi \) interaction. The loop will, of course, have a branch cut at the mass of the loop particle - in this case \( m_\pi \) - and this is the origin of the logarithmic divergence of \( r_\pi \) in ChPT. \[ \text{[21]} \] The preliminary studies of Ref. \[ \text{[19]} \] suggest, however, that at the physical value of \( m_\pi \) the charge radius is dominated by the regular contribution from Fig. \[ \text{[2]} \] which is not present in ChPT.

III. \( \pi \pi \) SCATTERING AT TREE LEVEL

A. Tree level \( \pi \pi \) scattering amplitude: \( A(s,t,u) \)

In considering \( \pi \pi \) scattering we need only consider the real part of the action in Eq. (23). It will be observed that this differs from that of Eq. (5) in Ref. \[ \text{[4]} \]: in \( SU_f(2) \) the \( \alpha_1 \) and \( \alpha_2 \) terms therein are contained within the \( K_{1,2,3} \) terms in our action which actually has a richer structure. This is easily seen using the relations:

\[
\text{tr} \left[ \left( \partial_\mu U \partial_\mu U^\dagger \right)^2 \right] = \frac{1}{2} \left[ \text{tr} \left( \partial_\mu U \partial_\mu U^\dagger \right) \right]^2 \tag{19}
\]

and

\[
\text{tr} \left[ \partial_\mu U \partial_\nu U^\dagger \partial_\nu U \right] = \frac{1}{2} \text{tr} \left[ \partial_\mu U \partial_\nu U^\dagger \right] \text{tr} \left[ \partial_\mu U \partial_\nu U^\dagger \right] - 2 \left[ 2 \partial_\mu S \partial_\nu S \partial_\nu V^i \partial_\nu V^i - 2 \partial_\mu S \partial_\nu S \partial_\nu V^i \partial_\nu V^i + \partial_\mu V^i \partial_\nu V^i \partial_\nu V^i \partial_\nu V^i - \partial_\mu V^i \partial_\nu V^i \partial_\mu V^i \partial_\nu V^i \right] \tag{20}
\]

where

\[
S(x) = \frac{1}{4} \text{tr} \left[ U(x) + U^\dagger(x) \right] \tag{21}
\]

\[
V(x) = \frac{1}{4} \text{tr} \left[ \tau^i \left( U(x) - U^\dagger(x) \right) \right] \tag{22}
\]

As usual, the \( \pi \pi \) scattering kernel can be written as follows (with \( s \), \( t \), and \( u \) the usual Mandelstam variables):

\[
T_{\alpha \beta \gamma \delta} = A(s,t,u) \delta_{\alpha \beta} \delta_{\gamma \delta} + A(t,s,u) \delta_{\alpha \gamma} \delta_{\beta \delta} + A(u,t,s) \delta_{\alpha \delta} \delta_{\beta \gamma} \tag{23}
\]

which can be decomposed into amplitudes with definite isospin as follows:

\[
T^0(s,t,u) = 2 T_{++:+} - 2 T_{+0:+0} + T_{00:00} = 3 A(s,t,u) + A(t,s,u) + A(u,t,s) \tag{24}
\]

\[
T^1(s,t,u) = 2 T_{+0:+0} - T_{++:++} = A(t,s,u) - A(u,t,s) \tag{25}
\]

\[
T^2(s,t,u) = 2 T_{00:00} + 2 T_{00:+} + T_{++:++} = A(t,s,u) + A(u,t,s) \tag{26}
\]

The partial wave amplitudes are obtained from these expressions as follows:
\[ T_I^f(s) = \frac{1}{64\pi} \int_{-1}^{1} dy P_I(y)T_I^f(s,t,u) . \] 

(When the pions are on mass shell one has, of course, \( s + t + u = 4m^2 \) and, in evaluating the partial wave amplitudes it is simplest to work in the centre of mass frame where \( t = -(s - 4m^2)(1-y)/2 \).)

To calculate \( A(s,t,u) \) it is necessary to expand the action of Eq. (??) to \( O(\pi(x)^4) \):

\[
\Omega[\pi] = \int d^4x \left\{ \frac{1}{2} \left[ \partial_\mu \pi^i \partial_\mu \pi^i + m_\pi^2 \pi^i \pi^i \right] - \frac{1}{24} \frac{m_\pi^2}{f_\pi^2} (\pi^i \pi^i)^2 \right. \\
+ \frac{1}{6f_\pi^2} \left[ (\pi^i \partial_\mu \pi^i)^2 - \pi^i \pi^j \partial_\mu \pi^i \partial_\mu \pi^j \right] - \frac{m^2_\pi}{f^2_\pi} \left[ \alpha_1 (\partial_\mu \pi^i \partial_\mu \pi^i)^2 + \alpha_2 \partial_\mu \pi^i \partial_\nu \pi^j \partial_\mu \pi^j \partial_\nu \pi^i \right] \\
+ \frac{2}{3} N_c K_1 \left[ \alpha \right] \left. \right\} 
\]

where

\[ f^2 m_\pi^2 = \rho \mu^2 (m_u^2 + m_d^2) \]

which is renormalisation point invariant.

In order to facilitate comparisons with Ref. [4] we have defined

\[ \alpha_1 = N_c (2K_1 - 2K_2 - K_3) \]
\[ \alpha_2 = 2N_c K_3 \]

in Eq. (28).

It is now a simple matter to proceed from Eq. (28) and obtain

\[ A(s,t,u) = \frac{m_\pi^2 + 2s - t - u}{3f_\pi^2} \left[ K_1 \left( -12m_\pi^4 + 6m_\pi^2 (s + t + u) + 2s^2 - t^2 - u^2 - 2(st + su + tu) \right) \\
+ K_2 \left( -2m_\pi^2 + s \right) \left( -2m_\pi^2 + t + u \right) + K_3 \left( -2m_\pi^4 + m_\pi^2 (s + t + u) - tu \right) \right] . \]

[It will be remembered that Eq. (28) is a Euclidean space action. In deriving Eq. (32) we calculated an algebraic expression for \( A(s_E, t_E, u_E) \) and used this to define an analytic continuation to the physical \( (s, t, u) \) domain.] It is clear that the Weinberg result [22] is contained within Eq. (32): one must simply set \( K_1 = 0 = K_2 = K_3 \); i.e., ignore the terms at \( O(E^4) \).

### B. Tree level \( \pi-\pi \) partial wave amplitudes

The expressions in Eq. (29) can now be used to obtain the scattering amplitudes at tree level and here we present our calculated expressions for the first five. All of these expressions have no imaginary part because we are working at tree level:
\[ T_0^0 = \frac{1}{32\pi f_\pi^2} \left( 7m_{\pi}^2 + \frac{8m_\pi^4}{f_\pi^2} \left[ 5(\alpha_1 + \alpha_2) + \frac{32}{3}NcK_1 \right] \right) \]
\[ + 2x \left[ 1 + \frac{4m_\pi^2}{f_\pi^2} \left[ 4\alpha_1 + 3\alpha_2 + 4NcK_1 \right] + \frac{x}{3f_\pi^2} \left[ 11\alpha_1 + 7\alpha_2 \right] \right] \]
\[ + x \left[ 1 + \frac{4m_\pi^2}{f_\pi^2} \left[ 2\alpha_1 + 3\alpha_2 + 4NcK_1 \right] - \frac{x^2}{3f_\pi^2} \left[ \alpha_1 + 3\alpha_2 + 4NcK_1 \right] \right] \] (33)

\[ T_1^1 = \frac{x}{96\pi f_\pi^2} \left( 1 + \frac{4m_\pi^2}{f_\pi^2} \left[ (\alpha_2 - 2\alpha_1 + 4NcK_1) + \frac{x}{f_\pi^2}(\alpha_2 - 2\alpha_1) \right] \right), \] (34)

\[ T_0^2 = \frac{-1}{32\pi f_\pi^2} \left( 2m_{\pi}^2 \left[ 1 - \frac{8m_\pi^2}{f_\pi^2} \left[ \alpha_1 + \alpha_2 - \frac{8}{3}NcK_1 \right] \right] \right) \]
\[ + x \left[ 1 - \frac{4m_\pi^2}{f_\pi^2} \left[ 2\alpha_1 + 3\alpha_2 + 4NcK_1 \right] - \frac{x^2}{3f_\pi^2} \left[ \alpha_1 + 3\alpha_2 \right] \right] \] (35)

\[ T_2^0 = \frac{(\alpha_1 + 2\alpha_2)}{240\pi f_\pi^4} x^2, \] (36)

\[ T_2^2 = \frac{(2\alpha_1 + \alpha_2)}{480\pi f_\pi^4} x^2, \] (37)

where \( x = s - 4m_\pi^2 \) and hence threshold corresponds to \( x = 0 \). The violation of unitarity by these tree level amplitudes will become important as one moves away from threshold.

Near threshold the partial wave amplitudes have the form:

\[ \text{Re} \left[ T_l^I(u) \right] = u^l \left( a_l^I + u b_l^I + O(u^2) \right), \] (38)

where \( u = x/(2m_\pi)^2 \), and a first test of the GCM in this sector is the evaluation of these threshold parameters. The dimensionless scattering lengths

\[ a_0^0 = T_0^0 \bigg|_{u=0}, \quad a_1^1 = \frac{d}{du} T_1^1 \bigg|_{u=0}, \quad a_0^2 = T_0^2 \bigg|_{u=0}, \quad a_2^0 = \frac{d^2}{du^2} T_0^0 \bigg|_{u=0}, \quad a_2^2 = \frac{d^2}{du^2} T_2^2 \bigg|_{u=0} \] (39)

are not influenced by the imaginary part of the partial wave amplitudes and hence they provide a useful, simple first test of tree level calculations in a given model.

The best known of the scattering lengths are \( a_0^0 \) and \( a_0^2 \) and in Ref. [23] it is argued that, experimentally,

\[ a_0^0 = 0.20 \pm 0.01, \quad a_0^2 = -0.037 \pm 0.004. \] (40)

The others are rather less well known. In Ref. [23], however, based on the experiments of Ref. [24], the following values are presented:

\[ a_0^0 = 0.26 \pm 0.005, \quad a_0^2 = -0.028 \pm 0.012, \quad a_1^1 = 0.038 \pm 0.002, \]
\[ a_2^0 = 0.0017 \pm 0.0003, \quad a_2^2 = 0.00013 \pm 0.0003. \] (41)

These were the values fitted in Ref. [3].
C. Fitting the $\pi$-$\pi$ scattering lengths

In Ref. [4], ChPT was used to fit the scattering lengths: a best fit being obtained in that parametrisation with:

$$\alpha_1 = -0.0092 \quad \alpha_2 = 0.0080$$

which yields:

$$a_0^0 = 0.152, \quad a_0^2 = -0.0451, \quad a_1^0 = 0.0363, \quad a_0^0 = 0.00142, \quad a_2^2 = -0.00109.$$  \hspace{1cm} (43)

Before proceeding to a calculation of the scattering length $s$ in the GCM it is of interest to determine whether a fit is possible at all and how good it might be; i.e., if there are values of the coefficients $K_1$, $K_2$ and $K_3$ in the GCM which provide a good fit to the scattering lengths. This is not a well posed problem since in general, as we have seen, the scattering lengths are not well known. For the purpose of illustration we will then simply choose to attempt to fit a “model experimental data set”: $a_0^0$ and $a_0^2$ from Eq. (40); $a_1^0$ and $a_0^2$ from Eq. (41); and $a_2^2$ from Eq. (43) (based on the plots in Ref. [4]):

$$a_0^0 = 0.20, \quad a_0^2 = -0.037, \quad a_1^0 = 0.038, \quad a_0^0 = 0.0017, \quad a_2^2 = -0.0011.$$ \hspace{1cm} (44)

We find a best fit (defined as that set of $K$s which minimise the sum of the squares of the relative difference between the model experimental data and the calculated values) with

$$K_1 = -0.000455, \quad K_2 = 0.000414, \quad K_3 = 0.00150,$$ \hspace{1cm} (45)

which corresponds to $\alpha_1 = -0.00972$ and $\alpha_2 = 0.00901$, and these values yield

$$a_0^0 = 0.15, \quad a_0^2 = -0.042, \quad a_1^0 = 0.035, \quad a_0^0 = 0.0017, \quad a_2^2 = -0.0011.$$ \hspace{1cm} (46)

This fit has a mean deviation in the magnitude of the relative error of 10%.

It should be remarked at this point that, as far as the at-threshold parameters are concerned, the only contribution that pion loops will make is to modify the values of the $K$-parameters calculated from the expressions in the appendix; i.e., pion loops will only modify the expressions defined implicitly via Eq. (39) by requiring that

$$K_{\text{tree}} \rightarrow K_{\text{loops}}$$ \hspace{1cm} (47)

and hence these equations are actually form invariant and may be looked upon as providing a very general parametrisation of these at-threshold parameters.

If we arbitrarily set $K_1 = 0$ then the expressions in Eqs. (40-43) reduce to exactly those of Ref. [4] (with a minor correction of a typographical error in $T_0^0$ therein). In this case the “best fit values” of $\alpha_1$ and $\alpha_2$ are:
\[ \alpha_1 = -0.00955 \quad \alpha_2 = 0.00897 \]  

which differ little from Eq. (42) and correspond to

\[ K_1 = 0.0, \quad K_2 = 0.000844, \quad K_3 = 0.00149. \]  

[Obviously, \( K_2 \) has changed in order to compensate as well as possible for the loss of the \( K_1 \) degree of freedom: In Eq. (45) \( K_2 - K_1 = 0.000869 \) to be compared with \( K_2 \) in Eq. (49).] Using Eq. (49) one obtains:

\[ a_0^0 = 0.15, \quad a_0^2 = -0.045, \quad a_1^1 = 0.037, \quad a_0^3 = 0.0018, \quad a_2^2 = -0.0011. \]  

This fit has a mean deviation in the magnitude of the relative error of 11%. In this illustrative example the extra freedom associated with the coefficient \( K_1 \) allows for a marginally improved fit in the GCM.

We have summarised these results in Table 1.

### IV. MODEL EVALUATION OF THE SCATTERING LENGTHS

#### A. Point Meson Limit

We turn now to the direct evaluation of the scattering lengths in the GCM using the formulae in the appendix that relate the coefficients \( K_1, K_2 \) and \( K_3 \) to the quark-gluon substructure in the GCM. The first and simplest calculation is the point meson limit. Writing the fermion propagator in the form

\[ \tilde{G}(p) = K(p^2) \left( -i\gamma \cdot pA(p^2) + B(p^2) \right) \]  

with

\[ K(p^2) = \frac{1}{p^2 A(p^2)^2 + B(p^2)^2} \]  

the point meson limit is obtained with

\[ A(p^2) = 1 \quad \text{and} \quad B(p^2) = \text{const.} \]  

This effectively reproduces the results one would obtain in the NJL [10]. In this case \( f_\pi \) and \( \langle \bar{q}q \rangle \) are both given by divergent momentum integrals which must be regularised but the \( K \)'s are finite and, in fact,

\[ K_1 = K_2 = K_3 = \frac{1}{96\pi^2}. \]  

(This corrects a typographical error in Ref. [11].) In this limit \( r_\pi \) is also finite and from Eq. (51) one obtains:
\[ r_\pi = \frac{\sqrt{N_c}}{2\pi f_\pi} . \] (55)

which agrees with that in the infinite cutoff limit of the NJL model, \[24\] as it should.

In order to present results in this case we simply take the experimental values of \( f_\pi = 0.093 \) GeV and \( m_\pi = 0.1385 \) GeV (these values can always be arranged in the NJL) and we find that:

\[
\begin{align*}
    r_\pi &= 0.58 \text{ fm}, \\
    a_0 &= 0.17, \quad a_0' = -0.048, \quad a_1 = 0.036, \quad a_2 = 0.0020, \quad a_2' = 0.0 . 
\end{align*}
\] (56)

One sees that even in the (properly regularised) point meson limit the GCM provides a reasonably good description of the scattering lengths.

### B. Finite Size Pion

It is simple matter to go beyond the NJL limit of the GCM. It will be recalled that in the GCM the fermion propagator is obtained as a solution of the fermion SDE: Eq. (12); which, in general, is a pair of coupled, nonlinear, integral equations the solution of which is completely determined once the model gluon propagator is specified. The solution of these equations in phenomenological models of QCD is an important area of research in its own right and has received a good deal of attention \[14,28,29,30,31\]. The characteristic features of the fermion propagator obtained in all of these studies are the same and may be distilled: In the spacelike region, \( A \) is a slowly varying function of \( p^2 \) with \( A(p^2 = 0) > 1 \) and \( A(p^2 = \infty) = 1 \); \( B \) provides most of the structure in the propagator with \( B(p^2 = 0) \approx 1 \) GeV and with the asymptotic form expected from the Operator Product Expansion \[32\] and QCD Renormalisation Group:

\[ b(x)|_{x \to \infty} \to \frac{4\pi^2 \lambda}{3} \frac{\kappa}{x(\ln x)^{1-\lambda}} \] (57)

with \( \lambda = 12/(33 - 2N_f) \) [we use \( N_f = 3 \) throughout], \( \kappa = (\ln \mu^2)^{-\lambda} < \bar{q}q >_\mu \), a dimensionless renormalisation point invariant, and \( x = p^2/\Lambda^2_{\text{QCD}} \).

Taking these considerations into account, and following Ref. \[33\] in which fits to various SDE solutions were made, the following model forms of \( A \) and \( B \), that incorporate all of the physical information and QCD input available from the SDE studies, suggest themselves:

\[ A(p^2) = \frac{2 + p^2}{1 + p^2} \] (58)

\[ B(p^2) = b_1 \left( 1 - \tanh \left[ b_2 \left( b_3 + p^2 \right) \right] \right) + \frac{p^4}{1 + p^4} \frac{4\pi^2 \lambda}{3} \frac{\left[ \ln[1/\Lambda^2_{\text{QCD}}] \right]^{-\lambda}}{p^2 \ln \left( b_4 + \frac{p^2}{\Lambda^2_{\text{QCD}}} \right)} . \] (59)

(We remark that these fitting forms should only be used at spacelike momenta: \( p^2 > 0 \); i.e, they are useful for interpolating the SDE solutions but not for extrapolating them into the complex...
$p^2$ plane. Studies of the solution of model fermion SDEs in the complex plane can be found in Refs. [34].

The experimentally determined QCD mass scale, $\Lambda_{\text{QCD}}$, introduced into QCD by renormalisation, can be inferred from a number of experiments and the average value for $N_f = 4$ in the modified minimal subtraction renormalisation scheme is:

$$\Lambda_{\text{QCD}} = 238 \pm 30 \pm 68 \text{ MeV}.$$  \hspace{1cm} (60)

For $N_f = 3$ this corresponds to

$$\Lambda_{\text{QCD}} \simeq 290 \text{ MeV}$$  \hspace{1cm} (61)

with a charm quark mass of 1.5 GeV and this is the value we use herein.

A reasonable choice for the value of the quark condensate is

$$\rho_{1\text{GeV}^2} = \langle \overline{u}u \rangle_{1\text{GeV}^2} = \langle \overline{d}d \rangle_{1\text{GeV}^2} = \langle \overline{q}q \rangle_{1\text{GeV}^2} = (0.255 \text{GeV})^3$$  \hspace{1cm} (62)

which follows from Refs. [29,36] and $m_{\pi}$ is then determined by Eq. (29) once $f_{\pi}$ is calculated from Eq. (17). [We fix $(m_{u_{1\text{GeV}^2}} + m_{d_{1\text{GeV}^2}}) = 10 \text{ MeV}$].

The undetermined parameters in the SDE based model fermion propagator are $b_1$-$b_4$. Choosing values for these parameters is completely equivalent to choosing a model form for the IR (small $k^2$) behaviour of the model gluon propagator in the GCM. This is because the UV (large $p^2$) behaviour of $A$ and $B$ is guaranteed by the requirement that the model gluon propagator manifest asymptotic freedom at large $k^2$; i.e.,

$$F_T(k^2) \bigg|_{k^2 \text{ large}} = \left. \frac{\lambda_{\pi}}{\ln \left(\frac{k^2}{\Lambda_{\text{QCD}}^2}\right)} \right\}.$$  \hspace{1cm} (63)

This is established by the SDE studies of Refs. [14,28,29,30,31].

In the fit of Ref. [33]

$$b_1 = 0.7336 \text{ GeV} \quad b_2 = 4.779 \text{ GeV}^{-2} \quad b_3 = -0.1435 \text{ GeV}^2.$$  \hspace{1cm} (64)

which were chosen to give the best possible fit to the SDE results of Ref. [37] on the domain $s = p^2 \in [0, 1] \text{ GeV}^2$. It is worth remarking that the parameter $b_4$ in Eq. (59) is important only in that it should be greater that one so that the small $p^2$ behaviour is governed by the “tanh” piece of the model. As a consequence we set

$$b_4 = 2$$  \hspace{1cm} (65)

and consider it fixed hereafter since our results are insensitive to increasing it by two orders of magnitude. This is also the reason for the factor $p^4/(1 + p^4)$ in Eq. (59).
The model quark propagator obtained from Eqs. (58) and (59) can be used to calculate $f_\pi$, $m_\pi$, $r_\pi$ and the partial wave amplitudes. As a first step we calculated these quantities using the parameters of Eq. (64) which tests the model SDE solution of Ref. [37]. This yields

$$f_\pi = 0.074 \text{ GeV}, \quad m_\pi = 0.17 \text{ GeV}, \quad r_\pi = 0.72 \text{ fm}, \quad a_0^0 = 0.37, \quad a_0^2 = -0.15, \quad a_1^1 = 0.072, \quad a_2^0 = -0.0030, \quad a_2^2 = -0.0067.$$ (66)

The value of $f_\pi$ agrees with that in Ref. [37] while the other quantities have not previously been calculated in this model.

These results are not generally satisfactory, a fact that is not surprising since the parametrisation of the gluon propagator in Ref. [37] was simply a “first guess”; i.e., no systematic attempt was made to improve the fit to static properties of mesons. The obvious next step is to determine whether better agreement with experiment can be obtained simply by varying $b_1$, $b_2$ and $b_3$.

We allowed $b_1$, $b_2$ and $b_3$ to vary in order to perform a least squares fit to $f_\pi = 0.093$ GeV, $m_\pi = 0.1385$ GeV, $r_\pi = 0.66$ fm, $a_0^0 - a_2^2$ from our model experimental data set, Eq. (44), and values of $T_0^0(x)$, $T_0^2(x)$ and $T_1^1(x)$ at $x = 0.28$ GeV$^2$ taken from the plots of Ref. [4]. This procedure yields

$$b_1 = 0.5901 \text{ GeV}, \quad b_2 = 3.0508 \text{ GeV}^{-2}, \quad b_3 = -0.2961 \text{GeV}^2.$$ (67)

The mass function obtained with this parameter set is plotted in Fig. 3 and, for comparison, we also plot $B(x = p^2)$ obtained with the parameters of Eq. (64) in this figure. It will be observed that these functions differ only for $|p|/\Lambda_{\text{QCD}} \lesssim 4$; i.e., for infrared momenta. It is important to recall that Eq. (64) yields an exact fit to the solution of the fermion SDE obtained when the model gluon propagator has an integrable IR singularity $\propto \delta^4(k)$. This can be interpreted as a regularisation of the $1/k^4$ behaviour argued for in Refs. [38,32,10,11].

The parameters of Eq. (67) lead to significantly improved agreement with the data:

$$f_\pi = 0.091 \text{ GeV}, \quad m_\pi = 0.141 \text{ GeV}, \quad r_\pi = 0.59 \text{ fm}, \quad a_0^0 = 0.17, \quad a_0^2 = -0.052, \quad a_1^1 = 0.032, \quad a_2^0 = 0.0011, \quad a_2^2 = -0.00085.$$ (68)

(The values of the $K$-coefficients are listed in Table. 1.) It is of interest to note that $r_\pi$ is too small by $\approx 10\%$ which, consistent with the lattice analysis of Ref. [20], allows for a small correction from the $\pi$-loop at the physical value of $m_\pi$.

It is clear now that, apart from $m_\pi$, the characteristic quantities of the pionic sector are a measure of the IR structure of the model gluon propagator; i.e., the effective interaction between quarks at small $k^2$. It is worth emphasising here that this follows because there is a one to one correspondence between the IR behaviour of the quark propagator and that of the model gluon propagator. In fact, in principle it is possible to invert the fermion SDE, Eq. (12), to obtain the gluon propagator that corresponds to Eq. (68). For our purposes herein, however, this is not necessary. (Although it should be remarked that the form of $A$ and $B$ is consistent with the
presence of an integrable singularity in the model gluon propagator at \( k^2 = 0 \). \([12,23,24]\) The one thing it is necessary to recall in this connection is that once one has a form for \( B(p^2) \) then the form of the pion’s Bethe-Salpeter amplitude, \( \Gamma \), follows immediately because of the identity between \( B \) and \( \Gamma \) in vector exchange theories with DCSB: \([12]\)

\[
\Gamma_\pi(p^2, P^2 = 0) \propto B(p^2) .
\] (69)

To further illustrate the importance of the IR part of the fermion propagator we have calculated the characteristic parameters of the \( \pi \)-sector using Eq. (67) but suppressing the UV tail contribution in Eq. (59); i.e., neglecting the \( \ln \) term, in which case:

\[
\begin{align*}
  f_\pi &= 0.090 \text{ GeV}, \quad r_\pi = 0.58 \text{ fm}, \\
  a_0^0 &= 0.18, \quad a_0^2 = -0.055, \quad a_1^1 = 0.034, \quad a_2^0 = 0.0012, \quad a_2^2 = -0.00094 .
\end{align*}
\] (70)

Clearly, the UV tail only contributes a small amount. In this connection we have also studied the sensitivity of our results to variations in \( \Lambda_{\text{QCD}} \) in our parametrisation. Changes of \( \pm 50\% \) produce insignificant (<1%) changes in the characteristic parameters. This is somewhat artificial, however, since in reality the coefficients \( b_1-b_3 \) depend on \( \Lambda_{\text{QCD}} \) through the solution of the fermion SDE.

We present the partial wave amplitudes of Eqs. (??-(??), calculated with the parameter set of Eq. (67), in Fig. 4. The agreement with the experimental data, even away from threshold (out to \( E \approx 0.7 \) GeV), is quite good.

Clearly, the GCM is able to provide a good qualitative and quantitative description of the pionic sector of QCD with only three parameters.

V. SUMMARY AND CONCLUSIONS

We have studied the pionic sector of QCD using a model field theory which incorporates asymptotic freedom, dynamical chiral symmetry breaking (DCSB) and quark confinement: the Global Colour-symmetry Model (GCM). Since the GCM manifests DCSB, the results of chiral perturbation theory (ChPT) are necessarily contained within it. In this connection, the GCM allows for a microscopic interpretation of the parameters in ChPT; i.e., it relates them to properties of the model dressed quark and gluon propagators in the GCM which are the fundamental ingredients of this model.

In deriving the \( \mathcal{O}(E^4) \) action for \( \pi \) in the GCM we demonstrated explicitly that ChPT phenomenology is contained within the GCM since our action contained that of Ref. 1 plus an additional term allowed by chiral symmetry. We showed that, at the simplest level, this extra term, characterised by a single parameter, enables a marginally improved fit to the \( \pi-\pi \) scattering lengths.

Calculations in the GCM are tied to the large body of work that has been completed in the area of phenomenological coupled Schwinger-Dyson→Bethe-Salpeter equation (SDBSE) studies in
QCD and, in fact, the application of the GCM can be interpreted solely within that framework. In the calculation reported herein we used the SDBSE studies to motivate a form for the quark propagator - a form that incorporated the physical input of asymptotic freedom and the operator product expansion; i.e., the large $p^2$ (UV) behaviour of the propagator was completely determined - so that our model propagator had only three parameters. These parameters, which determine the small $p^2$ behaviour of the propagator, reflect the fact that little is known about the form of the quark-quark interaction at small $k^2$ (IR). (The SDBSE studies indicate that there is a one to one correspondence between the IR behaviour of the gluon propagator and the IR behaviour of the quark propagator.)

Using the three parameter form of the GCM we calculated the quantities that characterise the pionic sector of QCD: $f_\pi$, $m_\pi$, $r_\pi$ and the lowest five partial wave amplitudes for $\pi-\pi$ scattering at tree level. We allowed the parameters to vary in order to obtain the best fit possible. In Sec. [IVB] we demonstrated that the GCM provides a very good description of this sector of QCD with a good representation of the partial wave amplitudes well beyond threshold.

It is worth reiterating here that in our study there are no infinite regularisation parameters, as there are in ChPT, even if we proceed to incorporate pion loop diagrams. We have a model, which can also be understood as a relativistic potential model, in which the parameters are tied to the behaviour of the quark-quark interaction in the infrared.

The results we have presented herein provide evidence that a “two-point model” of QCD; i.e., a model in which coloured quark currents interact via dressed gluon exchange (where the dressing models the effect of gluon and ghost polarisation diagrams on the gluon propagator but where interactions that depend on explicit three- and four-gluon vertices are neglected) provides a good qualitative and quantitative understanding of low energy phenomena in QCD. It is worth remarking in this connection that a better experimental determination of the scattering lengths would allow for more stringent constraints to be placed upon the nature of the effective interaction between quarks at small $k^2$.

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**COEFFICIENTS IN $\Omega[U]$**

In this appendix we present the expression for $r_\pi$ and expressions for the coefficients $K_1$, $K_2$ and $K_3$ in Eq. (??). The manner in which they are calculated is described in Ref. [7] and the expressions presented here correct some typographical errors therein. Further, they extend those
expressions by incorporating the dependence on the derivatives of $A(p^2)$.

We have

\[ K_1 = -b + c - 2h + f + g \]

(71)

\[ K_2 = aK_2 - d + 2f + g + i - j - 2l + 2m + 2nK_2 \]

(72)

\[ K_3 = aK_3 + e - 2g - 2i + 4k + 2l - 2m + 2nK_3 \]

(73)

where

\[ aK_2 = \int \frac{d^4q}{(2\pi)^4} K^4B^4 \left( A^4 + 2q^2A^2(AA' + q^2[A']^2) + \frac{2}{3}q^4(AA' + q^2[A']^2) \right) , \]

(74)

\[ aK_3 = \int \frac{d^4q}{(2\pi)^4} K^4B^4 \left( A^4 + 2q^2A^2(AA' + q^2[A']^2) - \frac{2}{3}q^4(AA' + q^2[A']^2) \right) , \]

(75)

\[ b = \frac{1}{2} \int \frac{d^4q}{(2\pi)^4} B^2 \left( AKA' + A^2KK' + \frac{q^2}{2}(K^2[A']^2 + 2AKA'K' - A^2[K']^2 + 2AK^2A'' + A^2KK'') \right) 
+ \frac{1}{3} q^4 \left( 4(KK'' - K'^2)(BB')^2 + K^2(2B^2B'B'' - (BB'')^2 - [B']^4) \right) \]

(76)

\[ c = \frac{1}{16} \int \frac{d^4q}{(2\pi)^4} B^2 \left( K'' + q^2K'' + \frac{1}{6}q^4K'' \right) , \]

(77)

\[ d = \frac{1}{4} \int \frac{d^4q}{(2\pi)^4} \left( K^2(-(BB'')^2 + q^2BB'(BB'' - [B']^2)) 
+ \frac{1}{3} q^4 \left( 4(KK'' - K'^2)(BB')^2 + K^2(2B^2B'B'' - (BB'')^2 - [B']^4) \right) \right) \]

(78)

\[ e = \frac{1}{2} \int \frac{d^4q}{(2\pi)^4} q^2 \left( K(2K'(BB')^2 + KB^2B'B'' + KB[B']^3) 
+ \frac{1}{6} q^2 \left( 4(KK'' - K'^2)(BB')^2 + K^2(2B^2B'B'' - (BB'')^2 - [B']^4) \right) \right) \]

(79)

\[ f = \frac{1}{4} \int \frac{d^4q}{(2\pi)^4} K^2 \left( (BB')^2 + q^2BB'([B']^2 + BB'') + \frac{1}{6} q^4 ([B']^2 + BB'')^2 \right) \]

(80)

\[ g = \frac{1}{12} \int \frac{d^4q}{(2\pi)^4} q^4 K^2 \left( [B']^2 + BB'' \right)^2 \]

(81)

\[ h = \frac{1}{4} \int \frac{d^4q}{(2\pi)^4} q^2 K^2BB' \left( [B']^2 + BB'' + \frac{1}{3} q^2(3B'B'' + BB'') \right) \]

(82)

\[ i = - \int \frac{d^4q}{(2\pi)^4} q^2 K^2 B^2 \left( A'B'(ABK' + 2BA'K + ABB')K + \frac{1}{3} q^2 \left( 4ABA'B'K' + 8B [A']^2 B' K + 5 A A' [B']^2 K + 2 A B B' A'' K + A B A' B'' K 
+ q^2 A'(4AB A'B' K' + 5 A' [B']^2 K + 4BB' A'' K + B A' B'' K) \right) \right) \]

(83)

\[ j = \int \frac{d^4q}{(2\pi)^4} K^2 B^2 \left( A^2BB'K 
+ q^2 \left[ \frac{1}{2} A K (2A'BB' + A([B']^2 + BB'')) + \frac{1}{3} q^2 \left( 4ABA'B'K' \right) \right] \right) \]

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In these expressions

\[ H(\pi) = n \]

representing the second derivative, etc.

The \( \pi \) charge radius obtained from the generalised impulse approximation diagram, Fig. 2, is

\[ \langle r_\pi^2 \rangle = \frac{N_c}{2f_\pi^2} \int \frac{d^3q}{(2\pi)^4} H(q^2) \] (91)

where

\[
H(x = q^2) = B \left( -A^2 B^2 B' K'' K^2 x^2 - A^4 B'' K' K^2 x^3 - 36 A^2 B^3 K' K^2 - A^4 B' K'' K^2 x^3 \\
- 18 B^3 A'^2 K^3 x - 24 A^4 B K^3 x^2 - 42 A B^2 A' B' K^3 x - 24 A B^3 A'' K^3 x \\
- 21 A^2 B^2 B'' K^3 x - 4 A^2 B^3 K'' K' K^2 - 6 A B^3 A' K'^2 K^2 x \\
+ 3 A^2 B^2 K'' K'^2 K^2 x^2 - 20 A^2 B A'^2 K^3 x^2 - 40 A^3 A' B' K^3 x^2 - 14 A B A' K'^2 K^3 x^2 \\
+ 2 A^2 B^3 K'^3 x^2 - 36 A^3 B A'' K^3 x^2 - 5 B^3 A' A'' K^3 x^2 - 4 A B^2 B' A'' K^3 x^2 \\
- 8 A^4 K'' K^3 x^2 + 2 A B^2 A' B'' K^3 x^2 + 4 A^2 B B' B'' K^3 x^2 - 4 A^4 B K'^3 x^3 \\
+ 4 A^4 B K'' K' K x^3 - 6 A^3 B A' K'^2 K x^3 - 2 A^4 B' K'' K x^3 - 3 A^3 B A' K'' K^2 x^3 \\
- 10 A^2 B A'^2 K' K^2 x^3 - 10 A^3 A' B' K' K^2 x^3 - 4 A^3 B A'' K' K^2 x^3 - 6 A B A'^3 K^3 x^3 \\
- 6 A^2 A'^2 B' K^3 x^3 - 2 A^3 B' A'' K^3 x^3 - 2 A^3 A' B'' K^3 x^3 \right) \] (92)

which is obtained with the fermion-photon vertex Ansatz of Ref. [12]:

\[ k = \frac{1}{4} \int \frac{d^4q}{(2\pi)^4} q^2 K^3 B^3 K' \]

\[ l = \frac{1}{3} \int \frac{d^4q}{(2\pi)^4} \left( q^4 K^3 B^3 K' \right) \]

\[ m = \frac{1}{3} \int \frac{d^4q}{(2\pi)^4} q^4 K^3 B^3 K' \]

\[ n_{K_0} = \int \frac{d^4q}{(2\pi)^4} q^2 K^4 B^4 K' \]

\[ n_{K_3} = \int \frac{d^4q}{(2\pi)^4} q^2 K^4 B^4 K' \]

\[ F' = \frac{d}{dq^2} F(q^2) \] (90)

with \( F'' \) representing the second derivative, etc.
\[ \Gamma_\mu(p, q) = \frac{1}{2} [A(p) + A(q)] \gamma_\mu \]

\[ + \frac{(p + q)_\mu}{p^2 - q^2} \left\{ [A(p) - A(q)] \frac{1}{2} (\gamma \cdot p + \gamma \cdot q) - i [B(p) - B(q)] \right\} . \]

which ensures local current conservation in this calculation.

Computer code for all of the quantities listed in this appendix in either MATHEMATICA or FORTRAN format is available upon request.
REFERENCES


FIGURES

FIG. 1. This quark loop diagram contributes to $f_\pi$. The filled circles represent $\langle \pi | \bar{q} q \rangle$ Bethe-Salpeter amplitudes, the thick external lines represent the $\pi$ field and the thin internal lines represent dressed quark propagators.

FIG. 2. This diagram provides the piece of $r_\pi$, Eq. (91), that is regular in the chiral limit. The thick, straight external lines represent the incoming and outgoing $\pi$, the filled circles at the $\pi$ legs represent the $\langle \pi | \bar{q} q \rangle$ Bethe-Salpeter amplitudes, the wiggly line represents the photon, $\gamma$, the shaded circle at the $\gamma$ leg represents the regular part of the dressed quark-photon vertex (which satisfies the Ward Identity), Eq. (93), and the thin internal lines represent the dressed quark propagator in the model.

FIG. 3. The function $B(p^2)$ of Eq. (59) obtained with the parameters in Eq. (67): solid line; compared with that obtained with Eq. (64): dashed line.

FIG. 4. The partial wave amplitudes of Eqs. (37–39) obtained with the best fit described in connection with Eq. (68): solid line. In the plots of $T_0^0(x)$, $T_1^1(x)$ and $T_0^2(x)$ we show, for comparison, the results that would be obtained with the Weinberg result for $A(s, t, u)$ in Ref. [22] (short-dashed line): this form gives $T_2^0(x) \equiv 0 \equiv T_2^2(x)$. In these plots we also show the extrapolation to threshold of Roy equation fits to high-energy $\pi-\pi$ scattering data for these partial waves: long-dashed line. The data are extracted from Fig. 4 in Ref. [4]. We have defined $x = E^2 - 4m_\pi^2$. 

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TABLES

TABLE I. In this table we summarise our results for the scattering lengths calculated in Secs. III and IV. The column labelled “Exp” lists the “model experimental data” set that we constructed; see Eq. (44). Fit 1 is the best fit to “Exp” possible (in the sense of minimising the sum of the squares of the differences) in the GCM when the $K$s are allowed to vary; see Eq. (46). Fit 2 is the best fit possible when one sets $K_1 = 0$ and allows $K_{2,3}$ to vary and corresponds to the best fit to “Exp” possible with the parametrisation of Ref. [4]; see Eq. (50). Cal. 1 is the calculated result obtained in the point meson limit of the GCM and corresponds to the result that would be obtained in the Nambu–Jona-Lasinio model $[1/(96\pi^2) = 0.00106]$; see Eq. (56). Cal. 2 is the calculated result using the quark propagator given by Eqs. (58) and (59) and using the parameters of Eq. (67); see Eq. (68).
| \( K_1 = \) | \(-0.000455\) | 0.0 | \( \frac{1}{96\pi^2} \) | 0.000460 |
| \( K_2 = \) | 0.000414 | 0.000844 | \( \frac{1}{96\pi^2} \) | 0.00104 |
| \( K_3 = \) | 0.00150 | 0.00149 | \( \frac{1}{96\pi^2} \) | 0.000868 |

| \( a_0 \) | 0.200 | 0.15 | 0.15 | 0.17 | 0.17 |
| \( a_0^2 \) | -0.037 | -0.042 | -0.045 | -0.048 | -0.052 |
| \( a_1^2 \) | 0.038 | 0.035 | 0.037 | 0.036 | 0.032 |
| \( a_2^0 \) | 0.0017 | 0.0017 | 0.0018 | 0.0020 | 0.0011 |
| \( a_2^2 \) | -0.0011 | -0.0011 | -0.0011 | 0.0 | -0.00085 |

\[ \frac{1}{5} \sum_{i=1}^{5} \left( \frac{a_{\text{fit}}}{a_{\text{Exp}^\prime}} - 1 \right) \]

| | 0.10 | 0.11 | 0.34 | 0.20 |