Renormalization in the Gauged Nambu-Jona-Lasinio Model

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Abstract

Based on the Cornwall-Jackiw-Tomboulis effective potential, we extensively study nonperturbative renormalization of the gauged Nambu-Jona-Lasinio model in the ladder approximation with standing gauge coupling. Although the pure Nambu-Jona-Lasinio model is not renormalizable, presence of the gauge interaction makes it possible that the theory is renormalized as an interacting continuum theory at the critical line in the ladder approximation. Extra higher dimensional operators ("counter terms") are not needed for the theory to be renormalized. By virtue of the effective potential approach, the renormalization ("symmetric renormalization") is performed in a phase-independent manner both for the symmetric and the spontaneously broken phases of the chiral symmetry. We explicitly obtain $\beta$ function having a nontrivial ultraviolet fixed line for the renormalized coupling as well as the bare one. In both phases the anomalous dimension is very large ($\geq 1$) without discontinuity across the fixed line. Operator product expansion is explicitly constructed, which is consistent with the large anomalous dimension owing to the appearance of the nontrivial extra power behavior in the Wilson coefficient for the unit operator. The symmetric renormalization breaks down at the critical gauge coupling, which is cured by the generalized renormalization scheme ("$\bar{M}$-dependent renormalization"). Also emphasized is the formal resemblance to the four-fermion theory in less than four dimensions which is renormalizable in $1/N$ expansion.
1 Introduction

This is an expanded version of our previous paper[1] on the renormalization of the gauged Nambu-Jona-Lasinio (NJL) models (gauge theories plus NJL-type [2] four-fermion interactions).

The gauged NJL models have recently become very popular in the context of modern versions[3] of the dynamical electroweak symmetry breaking such as the walking technicolor[4], technicolor models with strong coupling ETC[5, 6], top quark condensate model (top mode standard model)[7], etc., and also in the context of a possible existence of nontrivial (interacting) QED[8, 9]. Among others the most important feature of the gauged NJL model is a very large anomalous dimension[10]

\[ 1 \leq \gamma_m < 2, \]

which corresponds to a very slowly damping behavior of the fermion dynamical mass,

\[ \Sigma(-p^2) \sim (-p^2)^{-1+\gamma_m/2}. \]

The simplest version of the gauged NJL model, quenched QED plus chiral invariant four-fermion interaction \((G/2)[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\psi)^2]\), was first studied by Bardeen, Leung and Love[11] in the ladder Schwinger-Dyson (SD) equation. A full set of spontaneous chiral symmetry breaking (S\(\chi\)SB) solutions of the ladder SD equation and the critical line were discovered by Kondo, Mino and Yamawaki[12] and independently by Appelquist, Soldate, Takeuchi and Wijewardhana[13]. The critical line reads(Fig. 1);

\[ g = \frac{1}{4} \left( \frac{1}{1 + \frac{\alpha}{\alpha_c}} \right)^2 \equiv g^* \quad (0 < \alpha < \alpha_c = \frac{\pi}{3}), \]

\[ \alpha = \alpha_c = \frac{\pi}{3} \quad (g < \frac{1}{4}), \]

where \(\alpha \equiv e^2/4\pi\) and \(g \equiv G\Lambda^2/4\pi^2\), with \(\Lambda\) being the ultraviolet cutoff. This is the line separating the S\(\chi\)SB phase \((g > g^*)\) and the unbroken (symmetric) phase \((g < g^*)\).

The critical line Eq.(1.4) is actually the nontrivial ultraviolet (UV) fixed line [12, 14, 15], with the line of \(\alpha = \text{constant}\) being identified as the renormalization-group (RG) flow. This identification is in accord with the usual expectation that the gauge coupling \(\alpha\) may not be renormalized in the absence of the vacuum polarization in the ladder approximation. Actually, this model, having non-running gauge coupling, may
be regarded[16] as the “standing” (non-running) limit of the “walking” (slowly running) gauge theories plus four-fermion interaction in the “improved” ladder approximation (ladder SD equation with the gauge coupling simply replaced by the one-loop running one)[10, 17, 18, 16, 19]. Once the RG flow is so identified, the scaling relation[12, 13]

$$\frac{M_d}{\Lambda} \sim \left(\frac{g - g^*}{g - \tilde{g}^*}\right)^{1 \over 2 \sqrt{1 - \alpha / \alpha_c}} (g > g^*)$$

implies an explicit form of the $\beta$ function for $g$[20, 21]:

$$\beta_g(g, \alpha) \equiv \Lambda \frac{\partial g}{\partial \Lambda}|_{\alpha, M_d} = -2(g - g^*)(g - \tilde{g}^*) (g > g^*),$$

with $M_d \equiv \Sigma(0)$ and $\tilde{g}^* \equiv \frac{1}{4} \left(1 - \sqrt{1 - \alpha / \alpha_c}\right)^2$. Eq.(1.6) indeed has a nontrivial UV fixed line at $g = g^*$.

The very large anomalous dimension was in fact found at the UV fixed line by Miransky and Yamawaki[10]:

$$\gamma_m = 1 + \sqrt{1 - \frac{\alpha}{\alpha_c}} (g = g^*),$$

which corresponds to the slowly damping $S\chi SB$ solutions obtained by Refs.[12, 13]:

$$\Sigma(-p^2) \sim (-p^2)^{-\left(1 - \sqrt{1 - \frac{\alpha}{\alpha_c}}\right)/2}.$$ 

It was further suggested by Miransky and Yamawaki[10] that such a large anomalous dimension $\gamma_m \geq 1$ would imply the (nonperturbative) renormalizability of the four-fermion interaction, since the four-fermion operators would then become relevant/marginal, $d(\bar{\psi}\psi)^2 = 2d\bar{\psi}\psi = 2(3 - \gamma_m) \leq 4$, in the ladder approximation. Moreover, due to the presence of gauge coupling $\alpha \neq 0$ ($\gamma_m < 2$) the theory might have a nontrivial (interacting) continuum limit $\Lambda \to \infty$ in contrast to the pure NJL model with $\alpha = 0$ ($\gamma_m = 2$)[22, 15, 16].

However, the above nonperturbative renormalization procedure, originally proposed by Miransky[8] in the ladder QED (without four-fermion interaction), has so far been made only for the bare couplings and for the $S\chi SB$ phase. We wish to find the RG property in terms of the renormalized couplings and in the symmetric phase as well.

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Footnote 1: For the effects of the vacuum polarization in QED plus four-fermion interaction see Ref.[9]
Indeed, running of the bare couplings based on such a renormalization in the restricted coupling space may not correspond to the conventional \( \beta \) function of the renormalized couplings in the continuum theory, unless the RG flow is correctly identified.

Furthermore, the above renormalization procedure was crucially based on the nontrivial (S\( \chi \)SB) solution of the SD gap equation for the fermion mass function, which is no longer possible in the symmetric phase where the gap equation has only a trivial solution. This would yield an identically vanishing \( \beta \) function for the bare couplings in the symmetric phase (non-running for \( g \) as well as \( \alpha \)). Accordingly, the anomalous dimension in the symmetric phase of the gauged NJL model was considered to be small \(( \gamma_m = 1 - \sqrt{1 - \frac{\alpha}{\alpha_c}} < 1) [11] \) even in the vicinity of the UV fixed line, which is contrasted with that of the S\( \chi \)SB phase \(( \gamma_m = 1 + \sqrt{1 - \frac{\alpha}{\alpha_c}} > 1) [10] \), thus implying a paradoxical discontinuity of the anomalous dimension across the UV fixed line.

However, it was pointed out by Kikukawa and Yamawaki\[23\] that such a discontinuity would be an artifact of non-running treatment of the four-fermion coupling \( g \) in the symmetric phase.\(^2\) A possible resolution was in fact demonstrated\[23\] in the four-fermion theory in \( D \) (2 \( < D \) \( < 4 \)) dimensions (to be generically denoted by NJL\(_{D<4}\) hereafter) where nonperturbative renormalization can be explicitly done in the \( 1/N \) expansion\[25\]. Through the renormalization of the fermion four-point function (auxiliary field propagator) as well as the two-point function (fermion propagator), one obtains running of the coupling in the symmetric phase as well as the S\( \chi \)SB phase. The \( \beta \) function does have a nontrivial UV fixed point not only for the bare coupling but for the renormalized coupling. This in fact gives rise to a large anomalous dimension \(( \gamma_m = D - 2 ) \) near the UV fixed point, thus filling in the would-be discontinuity of the anomalous dimension across the UV fixed point in NJL\(_{D<4}\)[23]. The large anomalous dimension does exist not only for the bare coupling but also for the renormalized coupling in the continuum theory. Such a large anomalous dimension was in fact explicitly shown\[23\] to be consistent with the operator product expansion (OPE). Most remarkably, the Wilson coefficient for the unit operator does have an extra nontrivial power behavior (other than the anomalous dimension) due to nonperturbative effects.

In the previous paper\[1\] we showed that thanks to the presence of gauge interactions \(( \alpha \neq 0 ) \), the gauged NJL model can be renormalized in the ladder approximation in a quite similar fashion to NJL\(_{D<4}\). The crucial point was that the renormalization was done through the effective potential in the symmetric as well as the S\( \chi \)SB phase in a

\(^2\)The nontrivial scaling behavior near the critical line in the symmetric phase was also suggested through the gap equation for the pure NJL model in Ref.\[24\].
phase-independent manner, in contrast to the earlier works based on the SD gap equation. To demonstrate such an advantage of the effective potential approach, we first reformulated the renormalization procedure of Ref.[23] for NJL$_{D<4}$ through the effective potential (Similar reformulation for NJL$_{D<4}$ was also made by Ref.[26]). Then, for the gauged NJL model in four dimensions we considered the Cornwall-Jackiw-Tomboulis (CJT) effective potential[27] and rewrote it only in terms of the local auxiliary fields à la Bardeen and Love[28]. This effective potential, an analogue of the effective potential for NJL$_{D<4}$, was then renormalized in a very similar manner to NJL$_{D<4}$ and was explicitly written in terms of the renormalized parameters of the continuum limit theory ($\Lambda \to \infty$). Remarkably enough, as in NJL$_{D<4}$ the auxiliary field propagator[29] was shown to be simultaneously renormalized through the above renormalization of the effective potential. We explicitly computed the $\beta$ function for $g$ for the renormalized as well as the bare coupling, which in either case has a nontrivial UV fixed point for each $\alpha$ (fixed line in $(\alpha, g)$ plane). In either case we obtained a large anomalous dimension both in the $S\chi$SB and the symmetric phases without paradoxical discontinuity across the UV fixed line, in accord with Ref.[23]. As in NJL$_{D<4}$ the OPE was explicitly given in a consistent manner with such a large anomalous dimension in both phases. As in NJL$_{D<4}$ the Wilson coefficient for the unit operator acquires an extra nontrivial power behavior other than the anomalous dimension.

In this paper we present detailed description of the results of Ref.[1] on the renormalization of the simplest gauged NJL model, gauge theories with standing gauge coupling plus four-fermion interaction, in the ladder approximation. As a basis of our analysis we consider the CJT effective potential written in terms of the auxiliary fields. Here we give a more general form than the simplest one (the BL form[28]) discussed in Ref.[1]. By use of this effective potential, it is shown in the $S\chi$SB phase that all the amputated multi-fermion Green functions at zero momentum are finite at the critical line (including the end point $\alpha = \alpha_c$) in the continuum limit $\Lambda \to \infty$. We then give an explicit procedure to renormalize this effective potential in that limit. This renormalization is possible owing to the presence of gauge interactions ($\alpha \neq 0$). The $\beta$ function and the large anomalous dimension are obtained through this renormalization both in the symmetric and the $S\chi$SB phases. The OPE is explicitly constructed, which is consistent with the large anomalous dimension in both phases. Corresponding to the generalized form of the effective potential, we consider a generalization ("$\bar{M}$-dependent renormalization") of the simplest renormalization scheme ("symmetric renormalization")[1] made on the symmetric vacuum. Wilson coefficients and RG
functions, etc. are calculated in the $\bar{M}$-dependent renormalization as well as in the symmetric renormalization. In particular, whereas the symmetric renormalization breaks down at the end point $\alpha = \alpha_c$ of the critical line, the $\bar{M}$-dependent renormalization still remains valid there.

The paper is organized as follows. In the next section we derive the CJT effective potential and its variants of the gauged NJL model (in the equivalent Yukawa form rewritten in terms of the local auxiliary fields) in the ladder approximation. In section 3 all the amputated multi-fermion Green functions at zero momentum are explicitly calculated from the effective potential in the $S\chi$SB phase and shown to be finite even at the end point $\alpha = \alpha_c$ of the critical line in the continuum limit $\Lambda \to \infty$. Section 4 critically reviews calculation of the auxiliary field propagator made by Appelquist et al.[29]. Then in section 5 we present an explicit procedure of renormalization (symmetric renormalization done on the symmetric vacuum) which is made through the effective potential (BL effective potential[28]) in an analogous manner to the renormalization of NJL$_{D<4}$[23]. In section 6 explicit construction of OPE is given, which is shown to be consistent with the large anomalous dimension in both the symmetric and the $S\chi$SB phases in a quite nontrivial manner: The Wilson coefficient for the unit operator possesses an extra power behavior other than the anomalous dimension. Section 7 is devoted to the $\bar{M}$-dependent renormalization, a generalization of the symmetric renormalization, which remains valid at the end point $\alpha = \alpha_c$ where the symmetric renormalization breaks down. Section 8 is the conclusion and discussion: We comment on the “renormalizability” of NJL$_{D<4}$ and the gauged NJL model in the language of the usual RG equation of the equivalent Yukawa model. In Appendix A the effective potential and auxiliary field propagators in NJL$_{D<4}$ are given. Appendix B presents OPE in NJL$_{D<4}$. In Appendices C and D we present RG study of the bare parameters $\dot{a}$ la Miransky through the SD equation and through the effective potential, respectively.

## 2 Effective Potentials of Gauged NJL model

Let us start with the lagrangian of the $SU(N)$ gauge theory plus NJL-type four-fermion interaction:

$$
\mathcal{L} = \bar{\psi}(i\partial - eA)\psi - m_0\bar{\psi}\psi + \frac{G}{2N}\left[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\psi)^2\right] - \frac{1}{2}\text{tr}(F_{\mu\nu}F^{\mu\nu}),
$$

(2.1)
where $m_0$ is the bare fermion mass, $e$ the gauge coupling constant, and $G$ the four-fermion coupling. By using auxiliary fields $\sigma$, $\pi$, Eq.(2.1) is cast into an equivalent lagrangian

$$L = \bar{\psi}(i\partial - eA)\psi - \bar{\psi}(\sigma + \pi\gamma_5)\psi - V_{(cl)}(\sigma, \pi) - \frac{1}{2}\text{tr}(F_{\mu\nu}F^{\mu\nu}),$$

(2.2)

where the classical part of the potential of auxiliary fields $\sigma$ and $\pi$ is given by

$$V_{(cl)}(\sigma, \pi) = \frac{1}{G}\left[\frac{N}{2} \left(\sigma^2 + \pi^2\right) - m_0\sigma\right].$$

(2.3)

It is more convenient to study Eq.(2.2) than Eq.(2.1) for the discussions of renormalization.

The point is that as is demonstrated in NJL$_D<4[1]$, the renormalization is studied most transparently through the effective action (potential) written only in terms of the auxiliary fields. Such an effective potential in the gauged NJL model is derived from Eq.(2.2) by integrating out the degrees of freedom of fermion ($\psi$) and gauge boson ($A_\mu$). An effective potential of this kind was first derived by Bardeen and Love (BL)[28] through discussion of the vacuum condensate of fermion composite operator. Here we take an alternative approach[1] using the effective action of Cornwall, Jackiw and Tomboulis (CJT)[27, 30] and derive systematically several variants of the CJT effective potential including the BL effective potential as a special case. We thus clarify the relation among various kinds of effective potentials including the BL potential.

In the CJT formalism[27] we introduce a bilocal external source in a similar manner to the usual external source terms:

$$L_{\text{source}} = -\bar{\psi}(x)J(x, y)\psi(y),$$

(2.4)

in the generating functional:

$$W[J; \sigma, \pi] = -i \ln \int [d\psi][d\bar{\psi}][\text{gauge}] \exp[i \int d^4x (L + L_{\text{source}})].$$

(2.5)

Corresponding to the bilocal external source term $J$, the fermion propagator $S$ becomes

\footnote{It was also noted in Ref.[35] that the BL potential can be derived from the CJT effective potential.}
a variational variable in the CJT effective action through the Legendre transformation,

\[ \Gamma[S; \sigma, \pi] \equiv W[J; \sigma, \pi] - \text{Tr}(J \cdot S) \]

\[ = -i \text{Tr} \left( \ln S^{-1} + S_0^{-1} S \right) + \kappa^{2\text{PI}}[S] - \int d^4x V_{\text{cl}}(\sigma, \pi), \]  \tag{2.6} \]

where \( \kappa^{2\text{PI}}[S] \) is the sum of all the two-particle (fermion) irreducible diagrams written in terms of \( S \), and \( S_0 \) is the function of \( \sigma \),

\[ iS_0^{-1} = i\partial - \sigma - \pi i \gamma_5. \]  \tag{2.7} \]

The stationary condition for \( S \) gives the SD equation of the fermion propagator

\[ 0 = J \equiv i \frac{\delta \Gamma}{\delta S} = -S^{-1} + S_0^{-1} + i \frac{\delta \kappa^{2\text{PI}}}{\delta S}. \]  \tag{2.8} \]

Since the SD equation Eq.(2.8) can be regarded as the condition for the bilocal external source to vanish, Eq.(2.6) reads

\[ \Gamma[S_{\text{sol}}; \sigma, \pi] = W[J = 0; \sigma, \pi], \]  \tag{2.9} \]

with \( S_{\text{sol}} \) being the solution of Eq.(2.8). This in fact yields the desired effective action. Note that \( S_{\text{sol}} \) depends on the auxiliary fields \( \sigma \) and \( \pi \).

The Yukawa-type vertex \( \Gamma_S \) is given by:

\[ \Gamma_S(x, y; z) = -i \frac{\delta}{\delta \sigma(z)} S^{-1}_{\text{sol}}(x, y), \]  \tag{2.10} \]

which satisfies the following SD equation:

\[ \Gamma_S(x, y; z) = \delta^{(4)}(x - y) \delta^{(4)}(y - z) \]

\[ + \frac{i \delta^2 \kappa^{2\text{PI}}[S]}{\delta S(y, x) \delta S(x', y')} \bigg|_{S=S_{\text{sol}}} S_{\text{sol}}(x', x'') \Gamma_S(x'', y''; z) S_{\text{sol}}(y'', y'), \]  \tag{2.11} \]

where integration over the repeated indices \( x', x'', y', y'' \) is understood.

Now, noting the translational invariance, \( S(x, y) = S(x - y), \sigma(x) = \sigma \) and \( \pi(x) = \pi \), we define the CJT effective potential \( V \);

\[ V[S; \sigma, \pi] = -\Gamma[S, \sigma = \text{const}, \pi = \text{const}] / \Omega, \]  \tag{2.12} \]
with \( \Omega \) being the space-time volume. Then Eq.(2.12) may be rewritten as

\[
V[S; \sigma, \pi] = V_{(cl)}(\sigma, \pi) + V_{(qu)}[S; \sigma, \pi],
\]

(2.13)

where the quantum part \( V_{(qu)}[S; \sigma, \pi] \) is given by

\[
-\Omega V_{(qu)}[S; \sigma, \pi] = -i \text{Tr}(\ln S^{-1} + S_0^{-1}S) + \kappa^{2\pi I}[S].
\]

(2.14)

### 2.1 Ladder approximation

For actual calculation we need to make an approximation for \( \kappa^{2\pi I}[S] \) which contains infinite number of diagrams. Here we consider the simplest choice, namely, the lowest diagram (two-loop diagram) depicted in Fig. 2. This corresponds to the ladder approximation with the fixed gauge coupling. Although the ladder approximation is not a systematic expansion, it picks up at least the leading log behavior of fermion mass function which is actually important for our present purpose to renormalize the gauged NJL model. Since the running effects of the gauge coupling is left out of account in this approximation, it is certainly not a good approximation for the QCD-like gauged NJL model with a normal running gauge coupling. However, it can be regarded[16] as the standing limit of the walking gauge theories (gauge theories with a slowly running coupling) plus four-fermion interactions. For \( N = 1 \) this approximation also corresponds to the quenched QED plus NJL-type four-fermion interaction.

In the ladder approximation we may parameterize the fermion propagator \( S \) (in Landau gauge) by

\[
iS^{-1}(p) = \not{p} - \Sigma(-p^2) - i\gamma_5 \Sigma(-p^2).
\]

(2.15)

Then we evaluate each term of the quantum part of the CJT effective potential Eq.(2.14):

\[
\frac{\text{Tr} \ln S^{-1}}{i\Omega N} = \int_{\Lambda} d^4p \frac{\text{tr} \ln \frac{1}{2} \ln(1 + \Sigma(p_E^2) + \Sigma_5(p_E^2))}{p_E^2},
\]

(2.16)

\footnote{Our effective potential is defined so as to keep the auxiliary fields not path-integrated out and hence does not include the “two-loop” graph with the \( \sigma, \pi \) line sitting on the diameter of the fermion line circle. Or, even if we included such a diagram, it would vanish identically anyway, because \( \sigma, \pi \) are not propagating at this stage.}
\[
\frac{\text{Tr}(S_0^{-1} S)}{i \Omega N} = -i \int_0^\Lambda \frac{d^4p}{(2\pi)^4 i} \text{tr} \left[ (\phi - \sigma - i\gamma_5 \pi) S(p) \right]
\]
\[
= \frac{1}{4\pi^2} \int_0^\Lambda 2p_E^2 p_{E_0}^2 \left[ \frac{\sigma \Sigma(p_E^2) + \pi \Sigma_5(p_E^2)}{p_E^2 + \Sigma(p_E^2) + \Sigma_5(p_E^2)} - \frac{\Sigma^2(p_E^2) + \Sigma_5^2(p_E^2)}{p_E^2 + \Sigma^2(p_E^2) + \Sigma_5^2(p_E^2)} \right],
\] (2.17)

\[
\frac{\kappa^{2\Pi} |S|}{\Omega N} = 2\pi C_F \alpha i \int_0^\Lambda \frac{d^4p}{(2\pi)^4 i} \frac{d^4k}{(2\pi)^4 i} \text{tr} \left[ \gamma^\mu S(p) \gamma^\nu S(k) \right] D_{\mu\nu}(p - k),
\]
\[
= \frac{1}{8\pi^2} \int_0^\Lambda 2p_E^2 p_{E_0}^2 \int_0^\Lambda \frac{dk_E^2 k_{E_0}^2 \Sigma(p_E^2) \Sigma(k_E^2) + \Sigma_5(p_E^2) \Sigma_5(k_E^2)}{(p_E^2 + \Sigma + \Sigma_5)(k_E^2 + \Sigma + \Sigma_5) - K(p_E^2, k_E^2),}
\] (2.18)

where the Euclidean momentum integral \((p_E^2 \equiv -p^2)\) is regularized by the ultraviolet cutoff \(\Lambda\), \(C_F\) is the quadratic Casimir of the fermion representation, and the gauge boson propagator \(D_{\mu\nu}\) in Landau gauge takes the form \(D_{\mu\nu}(q) = (-i/q^2)(g_{\mu\nu} - q_\mu q_\nu/q^2)\), and

\[
K(p_E^2, k_E^2) \equiv \frac{3C_F \alpha / 4\pi}{\max(p_E^2, k_E^2)}.
\] (2.19)

We have defined our effective potential by subtracting a variable-independent divergence from Eq.(2.14) at the origin \(\Sigma = \Sigma_5 = 0, \sigma = \pi = 0\): \(V_{(\text{qu})}[\Sigma = 0, \Sigma_5 = 0; \sigma = 0, \pi = 0] = 0\).

Plugging Eqs.(2.16–2.18) into Eq.(2.14), we obtain

\[
- \frac{4\pi^2}{N} V[\Sigma, \Sigma_5; \sigma, \pi]
\]
\[
= -\frac{\Lambda^2}{g} \left[ \frac{1}{2} (\sigma^2 + \pi^2) - m_0 \sigma \right]
\]
\[
+ \int_0^\Lambda 2 p_E^2 p_{E_0}^2 \left\{ \frac{1}{2} \ln \left( 1 + \frac{\Sigma + \Sigma_5}{p_E^2} \right) - \frac{\Sigma + \Sigma_5}{p_E^2 + \Sigma + \Sigma_5} + \frac{\sigma \Sigma + \pi \Sigma_5}{p_E^2 + \Sigma + \Sigma_5} \right\}
\]
\[
+ \frac{1}{2} \int_0^\Lambda 2 p_E^2 p_{E_0}^2 \int_0^\Lambda dk_E^2 k_{E_0}^2 \frac{\Sigma(p_E^2) \Sigma(k_E^2) + \Sigma_5(p_E^2) \Sigma_5(k_E^2)}{(p_E^2 + \Sigma + \Sigma_5)(k_E^2 + \Sigma + \Sigma_5)} K(p_E^2, k_E^2),
\] (2.20)

with

\[
g \equiv \frac{G \Lambda^2}{4\pi^2},
\] (2.21)
where we included the classical part Eq.(2.3). It should be noted that our cutoff regularization does not violate the chiral symmetry:

$$
\begin{pmatrix}
\sigma \\
\pi
\end{pmatrix} \rightarrow \begin{pmatrix}
\cos \theta, & \sin \theta \\
-\sin \theta, & \cos \theta
\end{pmatrix}\begin{pmatrix}
\sigma \\
\pi
\end{pmatrix},
\begin{pmatrix}
\Sigma \\
\Sigma_5
\end{pmatrix} \rightarrow \begin{pmatrix}
\cos \theta, & \sin \theta \\
-\sin \theta, & \cos \theta
\end{pmatrix}\begin{pmatrix}
\Sigma \\
\Sigma_5
\end{pmatrix}.
$$

(2.22)

2.2 $V[\Sigma, \Sigma_5]$

Starting with the CJT effective potential Eq.(2.20), we now investigate its variants in what follows. We first derive a form of the CJT potential written only in terms of dynamical mass $\Sigma$ and $\Sigma_5$, by eliminating $\sigma$ and $\pi$ through their stationary conditions. The stationary condition of the effective potential for auxiliary fields gives

$$
0 = \frac{4\pi^2}{N} \frac{\partial}{\partial \sigma} V[\Sigma, \Sigma_5; \sigma, \pi] = -\int_0^{\Lambda^2} dp_E^2 \frac{p_E^2 \Sigma}{p_E^2 + \Sigma^2 + \Sigma_5^2} + \frac{\Lambda^2}{g} (\sigma - m_0),
$$

(2.23a)

$$
0 = \frac{4\pi^2}{N} \frac{\partial}{\partial \pi} V[\Sigma, \Sigma_5; \sigma, \pi] = -\int_0^{\Lambda^2} dp_E^2 \frac{p_E^2 \Sigma_5}{p_E^2 + \Sigma^2 + \Sigma_5^2} + \frac{\Lambda^2}{g} \pi.
$$

(2.23b)

Plugging the solution of Eq.(2.23) back into $V[\Sigma, \Sigma_5; \sigma, \pi]$, we obtain the effective potential written only in terms of the mass function of the fermion:

$$
-\frac{4\pi^2}{N} V[\Sigma, \Sigma_5] = \int_0^{\Lambda^2} dp_E^2 \left\{ \frac{1}{2} \ln \left( 1 + \frac{\Sigma^2 + \Sigma_5^2}{p_E^2} \right) - \frac{\Sigma^2 + \Sigma_5^2}{p_E^2 + \Sigma^2 + \Sigma_5^2} + \frac{m_0 \Sigma}{p_E^2 + \Sigma^2 + \Sigma_5^2} \right\}
+ \frac{1}{2} \int_0^{\Lambda^2} dp_E^2 \int_0^{\Lambda^2} dk_E \frac{\Sigma(p_E^2) \Sigma(k_E^2) + \Sigma_5(p_E^2) \Sigma_5(k_E^2)}{(p_E^2 + \Sigma^2 + \Sigma_5^2)(k_E^2 + \Sigma^2 + \Sigma_5^2)} \left[ K(p_E^2, k_E^2) + \frac{g}{\Lambda^2} \right].
$$

(2.24)

This type of CJT potential was obtained by Nonoyama, Suzuki and Yamawaki[14] directly from the original lagrangian Eq.(2.1) with the lowest $\kappa^{2P1}$ being given by Fig. 3.
Actually, the stationary condition of this effective potential Eq.(2.24) leads to the usual ladder SD gap equation for the fermion mass function Eq.(C.6) in Appendix C.

2.3 \[ V[\Sigma^{\text{sol}}, \Sigma_5^{\text{sol}}; \sigma, \pi] \]

We now derive explicit form of the effective potential written solely in terms of the auxiliary fields through several intermediate steps described in this and the next two subsections.

As such an intermediate step we first obtain a variant of the CJT effective potential \[ V[\Sigma^{\text{sol}}, \Sigma_5^{\text{sol}}; \sigma, \pi] \] by plugging the solution \( \Sigma^{\text{sol}} \) and \( \Sigma_5^{\text{sol}} \) of the stationary conditions, \( \delta V/\delta \Sigma = 0, \delta V/\delta \Sigma_5 = 0 \), back into \( V[\Sigma, \Sigma_5, \sigma, \pi] \). The stationary conditions (still not “gap equations”) are equivalent to the ladder SD equations

\[
\Sigma(p_E^2) = \sigma + \int_0^{\Lambda^2} dK^2 K(p_E^2, K^2) \frac{k_E^2 \Sigma(k_E^2)}{k_E^2 + S^2 + \Sigma_5^2}, \tag{2.25a}
\]

\[
\Sigma_5(p_E^2) = \pi + \int_0^{\Lambda^2} dK^2 K(p_E^2, K^2) \frac{k_E^2 \Sigma_5(k_E^2)}{k_E^2 + S^2 + \Sigma_5^2}, \tag{2.25b}
\]

which are reduced to Eq.(C.5), the stationary condition of Eq.(2.24), when \( \sigma \) and \( \pi \) are eliminated by use of Eq.(2.23).

By using the chiral symmetry, we can always rotate \( \Sigma, \Sigma_5, \sigma \) and \( \pi \) so as to set \( \Sigma_5 = \pi = 0 \), where \( \Sigma \) and \( \Sigma_5 \) are understood to be the solution of Eqs.(2.25a–2.25b). Thus, it is sufficient to study the case of \( \Sigma_5 = 0 \) and \( \pi = 0 \). The SD equation Eq.(2.25a) now reads

\[
\Sigma(p_E^2) = \sigma + \int_0^{\Lambda^2} dK^2 K(p_E^2, K^2) \frac{k_E^2 \Sigma(k_E^2)}{k_E^2 + S^2 + \Sigma_5^2}. \tag{2.26}
\]

For actual evaluation of \( V[\Sigma^{\text{sol}}, \Sigma_5 = 0; \sigma, \pi = 0] \) it is useful to note[36, 14] that \( V[\Sigma, 0, \sigma, 0] \) obeys a simple scaling relation:

\[
V_{(qu)}[\Sigma_\kappa, 0; \sigma, 0; \Lambda / \kappa] = \kappa^4 V_{(qu)}[\Sigma, 0; \sigma / \kappa, 0; \Lambda / \kappa], \tag{2.27}
\]

with \( \Sigma_\kappa(p_E^2) \equiv \kappa \Sigma(p_E^2 / \kappa^2) \), where we made explicit the \( \Lambda \)-dependence of \( V_{(qu)} \). Taking \( \kappa \) derivative of Eq.(2.27) at \( \kappa = 1 \), we find

\[
V_{(qu)}[\Sigma^{\text{sol}}, 0; \sigma, 0; \Lambda] = \frac{1}{4} \left[ 2\Lambda^2 \frac{\partial}{\partial \Lambda^2} + \sigma \frac{\partial}{\partial \sigma} \right] V_{(qu)}[\Sigma^{\text{sol}}, 0; \sigma, 0; \Lambda], \tag{2.28}
\]
where we have used $\delta V_{\text{qu}}/\delta \Sigma|_{\Sigma=\Sigma_{\text{sol}}}=0$. By using quantum part of Eq.(2.20) and Eq.(2.28), we obtain

$$
-\frac{4\pi^2}{N} V_{\text{qu}}[\Sigma_{\text{sol}}, 0; \sigma, 0; \Lambda] = \frac{\Lambda^4}{2} \left\{ \frac{1}{2} \ln \left( 1 + \frac{\Sigma^2_{\Lambda}}{\Lambda^2} \right) - \frac{\Sigma^2_{\Lambda}}{\Lambda^2 + \Sigma^2_{\Lambda}} + \frac{\sigma \Sigma_{\Lambda}}{\Lambda^2 + \Sigma^2_{\Lambda}} \right\} \\
+ \frac{3C_F}{8\pi} \frac{\Lambda^2 \Sigma_{\Lambda}}{\Lambda^2 + \Sigma^2_{\Lambda}} \int_0^\Lambda dk_E \frac{k^2_E \Sigma^2_{\text{sol}}(k^2_E)}{k^2_E + (\Sigma^2_{\text{sol}})^2} + \frac{\sigma}{4} \int_0^\Lambda dp_E \frac{p^2_E \Sigma^2_{\text{sol}}(p^2_E)}{p^2_E + (\Sigma^2_{\text{sol}})^2},
$$

(2.29)

where

$$
\Sigma_{\Lambda} \equiv \Sigma_{\text{sol}}(p^2_E = \Lambda^2).
$$

(2.30)

Putting $p^2_E = \Lambda^2$ in the SD equation Eq.(2.26), we obtain

$$
\Sigma_{\Lambda} - \sigma = \frac{3C_F}{4\pi} \frac{\alpha}{\Lambda^2} \int_0^\Lambda dk^2_E \frac{k^2_E \Sigma^2_{\text{sol}}(k^2_E)}{k^2_E + (\Sigma^2_{\text{sol}})^2}.
$$

(2.31)

Plugging Eq.(2.31) into Eq.(2.29), we obtain another expression for the CJT effective potential

$$
-\frac{4\pi^2}{N} V[\Sigma_{\text{sol}}, 0; \sigma, 0; \Lambda] = -\frac{\Lambda^2}{g} \left[ \frac{1}{2} \sigma^2 - m_0 \sigma \right] + \frac{\Lambda^4}{4} \ln \left( 1 + \frac{\Sigma^2_{\Lambda}}{\Lambda^2} \right) + \frac{4\pi}{3C_F} \frac{\Lambda^2 (\Sigma_{\Lambda} - \sigma) \sigma}{\alpha}.
$$

(2.32)

where the classical part Eq.(2.3) was included. Note that $\Sigma_{\Lambda}$ is a function of $\sigma$ as determined by the SD equation Eq.(2.26) and hence Eq.(2.32) can in principle be written only in terms of $\sigma$.

### 2.4 Fermion mass function

In order to solve $\Sigma_{\Lambda}$ as an explicit function of $\sigma$, we now discuss the solution $\Sigma_{\text{sol}}$ of the SD equation Eq.(2.26)[12, 13]. Eq.(2.26) is equivalent to the differential equation

$$
\left[ \frac{p^2_E}{\left( \frac{d}{dp^2_E} \right)^2} + 2 \frac{d}{dp^2_E} + 3C_F \frac{\alpha}{4\pi} \frac{\Lambda^2 (\Sigma_{\Lambda} - \sigma) \sigma}{p^2_E + \Sigma^2(p^2_E)} \right] \Sigma(p^2_E) = 0,
$$

(2.33)
with infrared (IR) boundary condition (BC):

$$\lim_{p^2_{E} \to 0} p^4_{E} \frac{d}{dp^2_{E}} \Sigma(p^2_{E}) = 0, \quad (2.34)$$

and ultraviolet (UV) BC:

$$\left[ 1 + p^4_{E} \frac{d}{dp^2_{E}} \right] \Sigma(p^2_{E}) \Bigg|_{p^2_{E} = \Lambda^2} = \sigma. \quad (2.35)$$

In high energy region $p^2_{E} \gg \Sigma^2(p^2_{E})$, the differential equation Eq.(2.33) can be safely approximated by the linearized equation

$$0 = \left[ \frac{p^4_{E}}{M^4} \left( \frac{d}{dp^2_{E}} \right)^2 + 2 \frac{d}{dp^2_{E}} + \frac{3C_F}{4\pi} \frac{\alpha}{p^2_{E}} \right] \Sigma(p^2_{E}) + \mathcal{O}\left(\frac{\Sigma^3(p^2_{E})}{(p^2_{E})^2}\right). \quad (2.36)$$

Then the solution of Eq.(2.36) is written by the linear combination of two independent solutions:

$$\frac{\Sigma(p^2_{E})}{M} = c_1 \left( \frac{p^2_{E}}{M^2} \right)^{-\frac{(1-\omega)}{2}} + d_1 \left( \frac{p^2_{E}}{M^2} \right)^{-\frac{(1+\omega)}{2}} + \mathcal{O}\left(\left( \frac{p^2_{E}}{M^2} \right)^{-\frac{3(1-\omega)}{2}-1}\right), \quad (2.37)$$

where $M$ is an infrared scale of fermion mass function and

$$\omega \equiv \sqrt{1 - \frac{\alpha}{\alpha_c}}, \quad (2.38)$$

with

$$\alpha_c \equiv \frac{\pi}{3C_F}. \quad (2.39)$$

The UVBC Eq.(2.35) determines the scale $M$ as a function of $\sigma$:

$$\frac{\sigma}{\Lambda} = C_1 \left( \frac{M}{\Lambda} \right)^{2-\omega} + D_1 \left( \frac{M}{\Lambda} \right)^{2+\omega} + \mathcal{O}\left(\left( \frac{M}{\Lambda} \right)^{3(2-\omega)}\right), \quad (2.40)$$
where $C_1$ and $D_1$ are defined by
\[
C_1 \equiv \left( \frac{1}{2} + \frac{\omega}{2} \right) c_1, \quad D_1 \equiv \left( \frac{1}{2} - \frac{\omega}{2} \right) d_1.
\] (2.41)

For strong gauge coupling region $\alpha > \alpha_c$, $\omega$ becomes pure imaginary and the fermion mass function becomes oscillating:
\[
\frac{\Sigma(p^2_E)}{M} = \frac{A'}{\omega'} \sqrt{\frac{M^2}{p^2_E}} \sin \left[ \frac{\omega'}{2} \ln \frac{p^2_E}{M^2} + \omega' \delta' \right] + \mathcal{O}\left( \left( \frac{p^2_E}{M^2} \right)^{-5/2} \right),
\] (2.42)
where
\[
\omega' \equiv \sqrt{\frac{\alpha}{\alpha_c} - 1}
\] (2.43)
and
\[
A' = 2\omega' \sqrt{-c_1 d_1}, \quad \delta' = \frac{1}{2i\omega'} \ln \left( -\frac{c_1}{d_1} \right).
\] (2.44)
The coefficients $c_1$ and $d_1$ are complex conjugate to each other so as to guarantee that the fermion mass is real. The mass scale $M$ defined in Eq.(2.40) becomes multivalued function of $\sigma$ for $\alpha > \alpha_c$ due to the oscillating behavior of the fermion mass function. We then take $M$ with the largest absolute value (no-node solution), since it minimizes the effective potential. Note that the SD equation has a nontrivial solution even for $\sigma = 0$ in this region [31].

In the weak gauge coupling region $0 < \alpha < \alpha_c$, on the other hand, the fermion mass function may be written as
\[
\frac{\Sigma(p^2_E)}{M} = A \sqrt{\frac{M^2}{p^2_E}} \sinh \left[ \frac{\omega}{2} \ln \frac{p^2_E}{M^2} + \omega \delta \right] + \mathcal{O}\left( \left( \frac{p^2_E}{M^2} \right)^{-3(1-\omega)/2-1} \right),
\] (2.45)
where
\[
A = 2\omega \sqrt{-c_1 d_1}, \quad \delta = \frac{1}{2\omega} \ln \left( -\frac{c_1}{d_1} \right).
\] (2.46)
The coefficients $c_1$ and $d_1$ in Eq.(2.37) are determined by the IRBC Eq.(2.34). However, the non-linearity in the infrared region makes it difficult to calculate them in an analytical method. In the following, we evaluate $c_1$ and $d_1$ by using linearizing techniques of the ladder SD equation. Although the result varies slightly according to the choice of such a linearizing technique, we will find in sections 5, 6 and 7 that the
structure of the renormalization does not depend on such a detail of $c_1$ and $d_1$.

There exist two familiar linearizing techniques of the ladder SD equation. One is to replace the $\Sigma(p_E^2)$ in the denominator of Eq.(2.26) by $M$ [32, 14, 21]:

$$\Sigma(p_E^2) = \sigma + \int^\Lambda_2 d k_E^2 K(p_E^2, k_E^2) \frac{k_E^2 \Sigma(k_E^2)}{k_E^2 + M^2}. \quad (2.47)$$

Such a linearization leads to the solution:

$$\frac{\Sigma(p_E^2)}{M} = F\left(\frac{1}{2} + \frac{\omega}{2}, \frac{1}{2} - \frac{\omega}{2}, \frac{-p_E^2}{M^2}\right) = \sum_{n=1}^\infty c_n \left(\frac{p_E^2}{M^2}\right)^{-\left(\frac{1-\omega}{2}\right)+1-n} + \sum_{n=1}^\infty d_n \left(\frac{p_E^2}{M^2}\right)^{-\left(\frac{1+\omega}{2}\right)+1-n}, \quad (2.48)$$

with

$$c_n = \frac{(-1)^{n-1} \Gamma(\omega)}{\Gamma\left(\frac{1}{2} + \omega\right) \Gamma\left(\frac{1}{2} - \omega\right)} \frac{\Gamma\left(\frac{1}{2} + \omega + n - 1\right) \Gamma\left(-\frac{1}{2} + \omega + n - 1\right) \Gamma(-\omega + 1)}{\Gamma\left(\frac{1}{2} + \omega\right) \Gamma\left(-\frac{1}{2} + \omega\right) \Gamma(-\omega + n)},$$

$$d_n = c_n (\omega \rightarrow -\omega). \quad (2.49)$$

In particular, we find

$$c_1 = \frac{1}{\omega} \frac{\Gamma(1 + \omega)}{\Gamma\left(\frac{1}{2} + \omega\right) \Gamma\left(\frac{3}{2} + \frac{\omega}{2}\right)}, \quad d_1 = -\frac{1}{\omega} \frac{\Gamma(1 - \omega)}{\Gamma\left(\frac{1}{2} - \omega\right) \Gamma\left(\frac{3}{2} - \frac{\omega}{2}\right)}, \quad (2.50)$$

which lead to

$$A = \sqrt{\frac{8 \omega \cot\left(\frac{\pi}{2} \omega\right)}{\pi (1 - \omega^2)}}, \quad \delta = \frac{1}{2 \omega} \ln \left[\frac{\Gamma(1 + \omega) \Gamma\left(\frac{1}{2} - \omega\right) \Gamma\left(\frac{3}{2} - \frac{\omega}{2}\right)}{\Gamma(1 - \omega) \Gamma\left(\frac{1}{2} + \omega\right) \Gamma\left(\frac{3}{2} + \frac{\omega}{2}\right)}\right], \quad (2.51)$$

for $0 < \alpha < \alpha_c$ and

$$A' = \sqrt{\frac{8 \omega' \coth\left(\frac{\pi}{2} \omega'\right)}{\pi (1 + \omega'^2)}}, \quad \delta' = \frac{1}{2 i \omega'} \ln \left[\frac{\Gamma(1 + i \omega') \Gamma\left(\frac{1}{2} - \frac{i \omega'}{2}\right) \Gamma\left(\frac{3}{2} - \frac{i \omega'}{2}\right)}{\Gamma(1 - i \omega') \Gamma\left(\frac{1}{2} + \frac{i \omega'}{2}\right) \Gamma\left(\frac{3}{2} + \frac{i \omega'}{2}\right)}\right], \quad (2.52)$$

for $\alpha > \alpha_c$. Here we took a normalization $\Sigma(p_E^2 = 0) = M$. It should be noted that
this linearizing method overestimates the order of error in Eq.(2.37).

Another method is the bifurcation technique[33, 12]. As a result of the bifurcation theory of the non-linear integral equation, the bifurcation solution from the trivial one satisfies the integral equation which is obtained by ignoring the $\Sigma$ in the denominator and placing the infrared cutoff $M$ in Eq.(2.26):

$$\Sigma(p_E^2) = \sigma + \int_{M^2}^{\Lambda^2} dk_E^2 K(p_E^2, k_E^2) \Sigma(k_E^2).$$

(2.53)

The normalization of the solution is given by $\Sigma(p_E^2 = M^2) \equiv M$. By using the bifurcation technique, we obtain

$$c_1 = \frac{1 + \omega}{2\omega}, \quad d_1 = -\frac{1 - \omega}{2\omega},$$

(2.54)

which leads to

$$A = \sqrt{1 - \omega^2}, \quad \delta = \frac{1}{\omega} \tanh^{-1} \omega, \quad (0 < \alpha < \alpha_c),$$

(2.55a)

and

$$A' = \sqrt{1 + \omega'^2}, \quad \delta' = \frac{1}{\omega'} \tan^{-1} \omega', \quad (\alpha > \alpha_c).$$

(2.55b)

Note that $A, \delta$ evaluated by these linearizing methods are finite in $\omega \to 0 (\omega' \to 0)$:

$$A_0 \equiv \lim_{\omega \to 0} A = \frac{4}{\pi}, \quad \delta_0 \equiv \lim_{\omega \to 0} \delta = \ln 4 - 1,$$

(2.56a)

for the linearization Eq.(2.47), and

$$A_0 \equiv \lim_{\omega \to 0} A = 1, \quad \delta_0 \equiv \lim_{\omega \to 0} \delta = 1,$$

(2.56b)

for the bifurcation method.

At $\alpha = \alpha_c$ the fermion mass function can be obtained directly from Eq.(2.36) plus IRBC Eq.(2.34). It can also be obtained by taking $\omega \to 0 (\omega' \to 0)$ limit of Eq.(2.45) (Eq.(2.42)):

$$\frac{\Sigma(p_E^2)}{M} = A_0 \sqrt{\frac{M^2}{p_E^2}} \left[ \frac{1}{2} \ln \frac{p_E^2}{M^2} + \delta_0 \right] + \mathcal{O}\left(\left(\frac{p_E^2}{M^2}\right)^{-5/2}\right).$$

(2.57)

In this limit Eq.(2.40) reads

$$\frac{\sigma}{\Lambda} = \frac{A_0}{2} \left(\frac{M}{\Lambda}\right)^2 \left[ 1 + \delta_0 - \ln \frac{M}{\Lambda} \right] + \mathcal{O}\left(\frac{M}{\Lambda}\right)^6,$$

(2.58)
where we have used the following relation derived from Eq.(2.46) and Eq.(2.41):

\[
\lim_{\omega \to 0} \omega C_1 = - \lim_{\omega \to 0} \omega D_1 = \frac{A_0}{4}, \quad \lim_{\omega \to 0} (C_1 + D_1) = \frac{A_0}{2} (\delta_0 + 1). \tag{2.59}
\]

### 2.5 \( V(M) \)

Let us now return to the CJT effective potential Eq.(2.32). By using the explicit solution \( \Sigma^\text{sol}(p^2_E) \) of the SD equation Eq.(2.37), we find:

\[
\frac{\Sigma}{\Lambda} = c_1 \left( \frac{M}{\Lambda} \right)^{2-\omega} + d_1 \left( \frac{M}{\Lambda} \right)^{2+\omega} + \mathcal{O}\left( \left( \frac{M}{\Lambda} \right)^{3(2-\omega)} \right). \tag{2.60}
\]

Plugging Eq.(2.60) and Eq.(2.40) into Eq.(2.32), we obtain an effective potential solely expressed in terms of \( M \):

\[
-8\pi^2 \frac{V(M)}{N\Lambda^4} = \left( \frac{1}{g^*} - \frac{1}{g} \right) C_1 \left[ C_1 \left( \frac{M}{\Lambda} \right)^{4-2\omega} + \frac{2 + \omega}{2} D_1 \left( \frac{M}{\Lambda} \right)^4 \right]
+ \left( \frac{1}{\tilde{g}^*} - \frac{1}{\tilde{g}} \right) D_1 \left[ D_1 \left( \frac{M}{\Lambda} \right)^{4+2\omega} + \frac{2 - \omega}{2} C_1 \left( \frac{M}{\Lambda} \right)^4 \right] + \mathcal{O}\left( \left( \frac{M}{\Lambda} \right)^{4(2-\omega)} \right)
+ \frac{2m_0}{\Lambda g} \left[ C_1 \left( \frac{M}{\Lambda} \right)^{2-\omega} + D_1 \left( \frac{M}{\Lambda} \right)^{2+\omega} + \mathcal{O}\left( \left( \frac{M}{\Lambda} \right)^{3(2-\omega)} \right) \right], \tag{2.61}
\]

where \( C_1 \) and \( D_1 \) are defined in Eq.(2.41) and

\[
g^* \equiv \frac{1}{4} (1 + \omega)^2, \quad \tilde{g}^* \equiv \frac{1}{4} (1 - \omega)^2. \tag{2.62}
\]

It should be stressed again that \( M \) is a function of \( \sigma \) determined by Eq.(2.40).

### 2.6 \( V(\sigma, \pi) \)

In this subsection we make more explicit the \( \sigma \)-dependence of the effective potential Eq.(2.61). To this end it is convenient to rewrite Eq.(2.61) so as to leave \( M \) partly

\[\text{19}\]
unsolved:

\[-\frac{4\pi^2 V(\sigma, \pi = 0)}{N} \frac{1}{\Lambda^4} = \frac{1}{g} m_0 \sigma \frac{1}{\Lambda^2} \frac{1}{g^*} - \frac{1}{g} \right) \frac{\sigma^2}{2\Lambda^2} + \left( \frac{1}{g^*} - \frac{1}{g} \right) \frac{\sigma^2}{2\Lambda^2} \left[ D_1 \left( \frac{M}{\Lambda} \right)^{\omega} + \frac{1}{2 - \omega} - C_1 \left( \frac{M}{\Lambda} \right)^{\omega} \right] \]

\[+ \mathcal{O} \left( \frac{\sigma^4}{\Lambda^4} \right), \quad (2.63)\]

where we have used Eq.(2.40). This effective potential is actually the basis for studying the renormalization in this paper, particularly in section 7.

As a special case of our effective potential Eq.(2.63), we obtain the BL potential[28] derived through a different method, which is valid only for $1 > \omega > 0$ ($0 < \alpha < \alpha_c$). In this region Eq.(2.40) can be solved in a recursive way,

\[
\left( \frac{M}{\Lambda} \right)^{2-\omega} = \frac{1}{C_1 \Lambda} - \frac{D_1}{C_1} \left( \frac{1}{\sigma} \right)^{2(2+\omega)/(2-\omega)} \frac{\sigma^2}{\alpha/\alpha_c} \frac{2 - \omega}{4} \left( \frac{\sigma}{\Lambda} \right)^{4/(2-\omega)} \mathcal{O} \left( \frac{\sigma^4}{\Lambda^4} \right), \quad (2.64)
\]

with \( \eta \) being

\[
\eta \equiv \min \left( 4, \frac{2(2 + \omega)}{2 - \omega} \right). \quad (2.65)
\]

Plugging Eq.(2.64) into Eq.(2.63) (or Eq.(2.61)), we obtain

\[-\frac{4\pi^2 V(\sigma, \pi = 0)}{N} \frac{1}{\Lambda^4} = \frac{1}{g} m_0 \sigma \frac{1}{\Lambda^2} \frac{1}{g^*} - \frac{1}{g} \right) \frac{\sigma^2}{2\Lambda^2} \left[ D_1 \left( \frac{M}{\Lambda} \right)^{\omega} + \frac{1}{2 - \omega} - C_1 \left( \frac{M}{\Lambda} \right)^{\omega} \right] \]

\[+ \mathcal{O} \left( \frac{\sigma^4}{\Lambda^4} \right), \quad (2.66)\]

where \( \zeta_\omega \) is defined by:

\[
\zeta_\omega \equiv -\omega \left( \frac{2}{1 + \omega} \right)^{4/(2-\omega)} \frac{c_1 (2+\omega)/(2-\omega)}{d_1} > 0. \quad (2.67)
\]

It is easy to recover the pseudoscalar auxiliary field \( \pi \) in the effective potential Eq.(2.66):

\[-\frac{4\pi^2 V(\sigma, \pi)}{N} \frac{1}{\Lambda^4} = \frac{1}{g} m_0 \sigma \frac{1}{\Lambda^2} \frac{1}{g^*} - \frac{1}{g} \right) \frac{\sigma^2 + \pi^2}{2\Lambda^2} \left[ \frac{1}{\alpha/\alpha_c} \frac{2 - \omega}{4} \left( \frac{\sigma^2 + \pi^2}{\Lambda^2} \right)^{2/(2-\omega)} \mathcal{O} \left( \frac{\sigma^4}{\Lambda^4} \right), \quad (2.68)\]
\[ +\mathcal{O}\left(\frac{\sigma^2 + \pi^2}{\Lambda^2}\right)^{\eta/2}. \] (2.68)

The expansion Eq.(2.64) obviously breaks down at \( \alpha = \alpha_c \) where \( \mathcal{O}\left((\sigma/\Lambda)^{\eta-1}\right) \) term in Eq.(2.64) gives the same order contribution as others. A remarkable feature of our expression Eq.(2.63) for the effective potential is that it has a definite value in the limit \( \omega \downarrow 0 \) (\( \alpha \uparrow \alpha_c \)):

\[ - \frac{4\pi^2 V(\sigma, \pi = 0)}{N} \left(1 \frac{m_0}{g} \frac{\sigma}{\Lambda^2} + \left(1 \frac{1}{g^*} - \frac{1}{g}\right) \frac{\sigma^2}{2\Lambda^2} - 8 \frac{\sigma^2}{2\Lambda^2} \left[\frac{3}{4} + \delta_0 - \ln \frac{M}{\Lambda}\right] + \mathcal{O}\left(\frac{\Lambda^4}{\Lambda^2}\right)\right), \] (2.69)

where we have used Eq.(2.59). Since \( M \) is written in terms of \( \sigma \) through the UVBC at \( \alpha = \alpha_c \) Eq.(2.58), the above effective potential can be further expressed in terms of the auxiliary field:\(^5\)

\[ - \frac{4\pi^2 V(\sigma, \pi = 0)}{N} \frac{1}{g} \frac{\sigma}{\Lambda^2} + \left(1 \frac{1}{g^*} - \frac{1}{g}\right) \frac{\sigma^2}{2\Lambda^2} - 16 \frac{\sigma^2}{\Lambda^2} \ln \left(\frac{\Lambda^2}{\sigma^2}\right) + \mathcal{O}\left(\ln \frac{\Lambda^2}{\sigma^2}\right)^2, \] (2.70)

where we have used the relation

\[ - \ln \frac{M}{\Lambda} = \frac{1}{4} \ln \left(\frac{\Lambda^2}{\sigma^2}\right) + \mathcal{O}(\ln \frac{\Lambda^2}{\sigma^2}), \] (2.71)

derived from Eq.(2.58). Actually, in the limit \( \Lambda \gg \sigma \) this potential agrees with the effective potential derived by Bardeen and Love\(^28\) for \( \alpha = \alpha_c \).

Eq.(2.63) is also applicable to the strong gauge coupling region \( \alpha > \alpha_c \) by performing the analytic continuation \( \omega = i\omega', \omega' = \sqrt{\alpha/\alpha_c - 1} \).

Finally, we comment on the relation of our derivation of the effective potential Eq.(2.66) to that by Bardeen and Love\(^28\) which was derived based on the observation:

\[ \langle \bar{\psi}\psi \rangle = \frac{d}{d\sigma} V_{(qu)}(\sigma, \pi = 0). \] (2.72)

Actually, Eq.(2.72) is manifest in the CJT formalism, since \( V_{(qu)}(\sigma, \pi) \) is identified as

---

\(^5\)This expression actually coincides with the one obtained directly from the solution at \( \alpha = \alpha_c \).
\[ V_{\text{qu}}[\Sigma_{\text{sol}}, \Sigma_{s}^0, \sigma, \pi] \] which satisfies

\[
\frac{d}{d\sigma} V_{\text{qu}}[\Sigma_{\text{sol}}, \sigma, 0, 0] = \frac{\partial}{\partial \sigma} V_{\text{qu}}[\Sigma_{\text{sol}}, \sigma, 0, 0] + \int_0^{\Lambda^2} dp_E^2 \frac{\partial \Sigma_{\text{sol}}(p_E^2)}{\partial \sigma} \frac{\delta}{\delta \Sigma(p_E^2)} V_{\text{qu}}[\Sigma, \sigma, 0, 0] \bigg|_{\Sigma = \Sigma_{\text{sol}}}
\]

\[ \equiv \langle \bar{\psi} \psi \rangle = -\frac{N}{4\pi^2} \int_0^{\Lambda^2} dp_E^2 p_E^2 \frac{\Sigma_{\text{sol}}(p_E^2)}{p_E^2 + (\Sigma_{\text{sol}})^2}, \]  

(2.73)

where in the last line we have used the stationary condition

\[ \frac{\delta}{\delta \Sigma(p_E^2)} V_{\text{qu}}[\Sigma, \sigma, 0, 0] \bigg|_{\Sigma = \Sigma_{\text{sol}}} = \frac{\delta}{\delta \Sigma(p_E^2)} V[\Sigma, \sigma, 0, 0] \bigg|_{\Sigma = \Sigma_{\text{sol}}} = 0. \]

### 2.7 Gap equation

Now, we are interested in the solution \( \sigma = \sigma_{\text{sol}} \) of the stationary condition \( 0 = \partial V(\sigma, \pi = 0) / \partial \sigma \) of the effective potential Eq.(2.68):

\[
0 = \frac{1}{g} m_0 + \left( \frac{1}{g^*} - \frac{1}{g} \right) \frac{\sigma}{\Lambda} - \frac{4\zeta_{\omega}}{\alpha / \alpha_c} \left( \frac{\sigma}{\Lambda} \right)^{(2+\omega)/(2-\omega)} + \cdots.
\]

(2.74)

The instability of the symmetric vacuum \( \langle \sigma \rangle = \langle \pi \rangle = 0 \) for \( g > g^* \) in the chiral symmetric limit \( m_0 = 0 \) is manifest in the effective potential. Thus it is readily seen that

\[
g = g^* \equiv \frac{1}{4} (1 + \omega)^2 \quad (0 < \alpha < \alpha_c)
\]

(2.75)

is the critical line[12, 13]. Actually, we can read off the S\( \chi \)SB solution \( \sigma = \sigma_{\text{spont}} \) from Eq.(2.74) at \( m_0 = 0 \). The scaling at \( g \approx g^* \) of the S\( \chi \)SB solution to Eq.(2.74) is given by

\[
\frac{\sigma_{\text{spont}}}{\Lambda} = \left[ \frac{\alpha / \alpha_c}{4\zeta_{\omega}} \left( \frac{1}{g^*} - \frac{1}{g} \right) \right]^{(2-\omega)/2\omega} + \cdots,
\]

(2.76)

where \( \cdots \) stands for terms with higher power of \( (1/g^* - 1/g) \). Eq.(2.76) combined with Eq.(2.64) leads to the scaling of dynamical mass of fermion \( M_d = \langle M \rangle \) [12, 13]

\[
\frac{M_d}{\Lambda} \sim \left( \frac{1}{g^*} - \frac{1}{g} \right)^{1/2\omega}.
\]

(2.77)
However, Eq.(2.68) is not valid in the $\omega \to 0$ ($\alpha \to \alpha_c$) limit, in which case we need to return to Eq.(2.63) or Eq.(2.61), the form before the expansion Eq.(2.64) is applied. The stationary condition of Eq.(2.61) with $m_0 = 0$ leads to

$$
\frac{M_d}{\Lambda} = \left( -\frac{C_1}{D_1} \frac{g^* - \frac{1}{g}}{\tilde{g}^* - \frac{1}{g}} \right)^{1/2\omega} + O\left( \left( \frac{1}{g^* - \frac{1}{g}} \right)^{(5-4\omega)/2\omega} \right).
$$

(2.78)

In the $\omega \to 0$ ($g^* \to 1/4$) limit we find the essential singularity-type scaling:[12, 13]

$$
\frac{M_d}{\Lambda} = \exp \left[ 1 + \delta_0 - \frac{8}{4 - 1/g} \right].
$$

(2.79)

## 2.8 Yukawa-type vertex

As shown in Eq.(2.10), the Yukawa-type vertex is calculated by the $\sigma$ derivative of $S^{-1}$. Since we have already evaluated the fermion mass function $\Sigma$ under the ansatz of constant $\sigma$, it is easy to determine the Yukawa-type vertex at $q = 0$:

$$
\Gamma_S(-p^2) \equiv \Gamma_S(p, q = 0) = \frac{\partial}{\partial \sigma} \Sigma(-p^2),
$$

(2.80)

where the momentum assignment of Yukawa-type vertex $\Gamma_S(p, q)$ is depicted in Fig 4. Plugging Eq.(2.64) into Eq.(2.37), we find

$$
\frac{\Sigma(-p^2)}{\Lambda} = \frac{2}{1 + \omega} \left( \frac{-p^2}{\Lambda^2} \right)^{-(1-\omega)/2} \frac{\sigma}{\Lambda} + O\left( \frac{\sigma}{\Lambda} \right)^{(2+\omega)/(2-\omega)},
$$

(2.81)

which leads to the Yukawa-type vertex on the chiral symmetric vacuum $\langle \sigma \rangle = 0$:

$$
\Gamma_S(-p^2) = \left. \frac{\partial}{\partial \sigma} \Sigma(-p^2) \right|_{\sigma = 0} = \frac{2}{1 + \omega} \left( \frac{-p^2}{\Lambda^2} \right)^{-(1-\omega)/2}.
$$

(2.82)

In the $\omega \to 0$ ($\alpha \to \alpha_c$), however, the expansion of the fermion mass function around the symmetric vacuum Eq.(2.81) loses its validity. Thus, we need to evaluate the Yukawa-type vertex without employing such an expansion around the symmetric vacuum at $\omega \to 0$. Here we calculate the Yukawa-type vertex corresponding to the
expansion around non-zero $\sigma$:

$$\Gamma_S(-p^2; M) \equiv \frac{\partial}{\partial \sigma} \Sigma(-p^2) = \frac{\partial}{\partial M} \frac{\Sigma(-p^2)}{\partial \sigma}. \quad (2.83)$$

Combining the $M$ derivative of $\sigma$ from Eq.(2.40),

$$\frac{\partial \sigma}{\partial M} = (2 - \omega) C_1 \left( \frac{M}{\Lambda} \right)^{1-\omega} + (2 + \omega) D_1 \left( \frac{M}{\Lambda} \right)^{1+\omega} + O\left( \frac{M}{\Lambda} \right)^{5-3\omega}, \quad (2.84)$$

and the $M$ derivative of $\Sigma(-p^2)$ from Eq.(2.37),

$$\frac{\partial \Sigma(-p^2)}{\partial M} = 2 \frac{2 - \omega}{1 + \omega} C_1 \left( \frac{-p^2}{M^2} \right)^{-(1-\omega)/2} + 2 \frac{2 + \omega}{1 - \omega} D_1 \left( \frac{-p^2}{M^2} \right)^{-(1+\omega)/2} + O\left( \frac{-p^2}{M^2} \right)^{-(5-3\omega)/2}, \quad (2.85)$$

we obtain

$$\Gamma_S(-p^2; M) = \frac{2}{1 + \omega} \left( \frac{-p^2}{\Lambda^2} \right)^{-(1-\omega)/2} \left[ 1 + \frac{1 + \omega}{1 - \omega} K \left( \frac{-p^2}{M^2} \right)^{-\omega} + O\left( \frac{-p^2}{M^2} \right)^{-2+\omega} \right] \frac{1 + K \left( \frac{M}{\Lambda} \right)^{2\omega} + O\left( \frac{M}{\Lambda} \right)^{2(2-\omega)}}{1 + 2 \delta_0 + \ln \left( \frac{M^2}{\Lambda^2} \right)}, \quad (2.86)$$

where $K$ is defined by

$$K \equiv \frac{(2 + \omega) D_1}{(2 - \omega) C_1}. \quad (2.87)$$

As expected, Eq.(2.86) is indeed applicable to $\omega \to 0$ limit:

$$\Gamma_S(-p^2; M) = 2 \left( \frac{-p^2}{\Lambda^2} \right)^{-1/2} - 1 + 2 \delta_0 + \ln \left( \frac{-p^2}{M^2} \right) + O\left( \frac{-p^2}{M^2} \right)^{-2} \frac{1 + 2 \delta_0 + \ln \left( \frac{M^2}{\Lambda^2} \right)}{1 + 2 \delta_0 - \ln \left( \frac{M^2}{\Lambda^2} \right)} \quad (2.88)$$

3 Finiteness of Amputated Green Functions

In this section we calculate from the effective potential all the (amputated) multi-fermion Green functions at zero momentum of $\sigma$ in the $S\chi$SB phase and show that they are finite in the continuum limit $\Lambda \to \infty$, once the four-fermion coupling $g$ is fine
tuned so as to fix the fermion dynamical mass $M_d$. This strongly suggests the existence of an explicit renormalization scheme to make the effective potential finite at the full critical line including the end point $\alpha = \alpha_c$.

### 3.1 Green functions for $0 < \alpha < \alpha_c$

Let us first calculate the second derivative of the effective potential Eq.(2.63),

$$V^{(2)}(\sigma) \equiv \left( \frac{\partial}{\partial \sigma} \right)^2 V(\sigma, \pi = 0),$$

so as to evaluate the auxiliary field propagator at zero-momentum. By using Eq.(2.84), we obtain

$$-\frac{4\pi^2 V^{(2)}(\sigma)}{N \Lambda^2} = \frac{1}{g^*} - \frac{1}{g} + \left( \frac{1}{g^*} - \frac{1}{g^*} \right) \frac{K \left( \frac{M}{\Lambda} \right)^{2\omega}}{1 + K \left( \frac{M}{\Lambda} \right)^{2\omega}},$$

where $K$ is defined in Eq.(2.87) and we have neglected higher order in $M/\Lambda$. The solution of the gap equation in chiral limit, Eq.(2.78), reads

$$\frac{1}{g^*} - \frac{1}{g} = - \left( \frac{1}{g^*} - \frac{1}{g^*} \right) \frac{(2 - \omega)K \left( \frac{M_d}{\Lambda} \right)^{2\omega}}{(2 + \omega) + (2 - \omega)K \left( \frac{M_d}{\Lambda} \right)^{2\omega}}.$$  

Thus, Eq.(3.2) reads

$$-\frac{4\pi^2 V^{(2)}(\sigma_{\text{spont}})}{N \Lambda^2} = \frac{2\omega K \left( \frac{M_d}{\Lambda} \right)^{2\omega} \left( \frac{1}{g^*} - \frac{1}{g^*} \right)}{1 + K \left( \frac{M_d}{\Lambda} \right)^{2\omega} \left[ 2 + \omega + (2 - \omega)K \left( \frac{M_d}{\Lambda} \right)^{2\omega} \right]}.$$  

The cutoff dependence of $V^{(2)} \sim \Lambda^{2(1-\omega)}$ cancels exactly that of the Yukawa-type vertex Eq.(2.86) and we obtain *finite* amputated four-point Green function in $\Lambda/M_d \to \infty$ limit:

$$\Gamma_4(p_1), -p_1, p_2), -p_2)$$
\[ V^{(2)}(\sigma_{\text{spont}}) \prod_{j=1}^{2} \frac{1}{V^{(2)}(\sigma_{\text{spont}})} \Gamma_S(-p_{(j)}^2, M_d) \]

\[ = \frac{4\pi^2}{N} \prod_{j=1}^{2} \left( \frac{2}{1 + \omega} \left( \frac{-p_{(j)}^2}{M_d^2} \right)^{-(1+\omega)/2} + \frac{2K}{1 - \omega} \left( \frac{-p_{(j)}^2}{M_d^2} \right)^{-(1+\omega)/2} + O\left( \frac{-p_{(j)}^2}{M_d^2} \right)^{-(5+\omega)/2} \right) \]

\[ \Gamma_{2n}(p_{(1)}, -p_{(1)}, p_{(2)}, -p_{(2)}, \ldots, p_{(n)}, -p_{(n)}) \]

To discuss higher-point Green functions (see Fig. 5), we need to evaluate multi-\(\sigma\) vertices. At zero-momentum of \(\sigma\), they are calculated from \(n = m + 2\) \((m \geq 1)\)-th derivative of effective potential:

\[ V^{(n)}(\sigma) \equiv \left( \frac{\partial}{\partial \sigma} \right)^n V(\sigma, \pi = 0). \] (3.6)

After a straightforward calculation using Eq.(2.84), we find

\[ -\frac{4\pi^2}{N} V^{(m+2)}(\sigma) = 2\omega KM^2 - m \left( \frac{M}{\Lambda} \right)^{-(2+m)(1-\omega)} \left( \frac{1}{g^*} - \frac{1}{g^*} \right) P_m(\omega, K(M/\Lambda)^{2\omega}) \]

\[ \left[ (2 - \omega)C_1 \right]^m \left[ 1 + K \left( \frac{M}{\Lambda} \right)^{2\omega} \right]^{2m+1}, \] (3.7)

where \(P_m(\omega, z)\) is given by

\[ P_m(\omega, z) = \sum_{\ell=0}^{m-1} Q_{m,\ell}(\omega)(2\omega z)^\ell(1 + z)^{m-\ell-1}, \] (3.8)

with \(Q_{m,\ell}(\omega)\) being certain polynomial in \(\omega\) with degree \(m - \ell - 1\), e.g.,

\[ Q_{m,0}(\omega) = -1 \prod_{\ell=2}^{m} ((\ell + 1) \omega - 2(\ell - 1)) \] (m \geq 2) .

(3.9)

Now, we are ready to evaluate the amputated \(2n\)-point Green function \((n \geq 3)\) at zero momentum. It is remarkable to see the cancellation of \(\Lambda\) dependence of Eq.(2.86), Eq.(3.4) and Eq.(3.7) in the fermion amputated Green functions. Actually, we find the Green functions remain \textit{finite} in continuum limit \((\Lambda/M_d \to \infty)\):
\[
V^{(n)}(\sigma_{\text{s spont}}) = \left( \frac{4\pi^2}{N} \right)^{n-1} Q_{n-2,0}(\omega) \prod_{j=1}^{n} \left[ \frac{2}{1 + \omega \left( \frac{-p^2_{(j)}}{M_d^2} \right)} + \frac{2K}{1 - \omega \left( \frac{-p^2_{(j)}}{M_d^2} \right)} \right]^{-(1+\omega)/2} + \mathcal{O}\left( \frac{-p^2_{(j)}}{M_d^2} \right)^{-(5-3\omega)/2}.
\]

\[
3.2 \text{ Green functions at } \alpha = \alpha_c
\]

We next consider Green functions at \( \alpha = \alpha_c \), i.e., \( \omega = 0 \). This limit should be taken carefully, since the Yukawa-type vertex Eq.(2.88) and the effective potential depend on \( \Lambda \) not only in power of the cutoff but also in logarithm of the cutoff. As shown in the following, however, finiteness of the amputated Green functions persists also in this limit, thanks to the cancellation of logarithmic dependence on \( \Lambda \).

Taking \( \omega \to 0 \) limit of Eq.(3.4) and Eq.(3.7), we find

\[
\frac{4\pi^2}{N} V^{(2)}(\sigma_{\text{s spont}}) = \frac{16\Lambda^2}{\left[ 1 + 2\delta_0 - \ln \left( \frac{M_d^2}{\Lambda^2} \right) \right] \left[ 2 + 2\delta_0 - \ln \left( \frac{M_d^2}{\Lambda^2} \right) \right]} (3.11)
\]

and

\[
\frac{4\pi^2}{N} V^{(m+2)}(\sigma) = \frac{32M^{2-m}}{\left( \frac{A_0}{2} \right)^m} \left[ 1 + 2\delta_0 - \ln \left( \frac{M^2}{\Lambda^2} \right) \right]^{m+2} \left[ Q_{m,0}(0) + \frac{2Q_{m,1}(0)}{1 + 2\delta_0 - \ln \left( \frac{M^2}{\Lambda^2} \right)} + \cdots \right], (3.12)
\]

respectively. Thus, it is clear that the logarithmic divergence of Eq.(2.88) cancels out that of Eq.(3.11) and Eq.(3.12) in the amputated Green functions. Actually, by noting \( Q_{m,0}(0) = (-2)^{m-1} (m-1)! \), we obtain finite Green functions in the continuum limit
\[ \frac{\Lambda}{M_d} \to \infty: \]

\[ \Gamma_4(p_1, -p_1, p_2, -p_2) = \pi^2 \frac{1}{N M_d^2} \prod_{j=1}^{2} \left[ \frac{-1 + 2 \delta_0 + \ln \left( \frac{-p^2_{(j)}}{M_d^2} \right) + \mathcal{O} \left( \frac{-p^2_{(j)}}{M_d^2} \right)^{-2} }{\sqrt{-p^2_{(j)}/M_d^2}} \right] \]

for the four-point Green function and

\[ \Gamma_{2n}(p_1, -p_1, p_2, -p_2, \cdots, p_n, -p_n) = \]

\[ -\frac{4\pi^2}{N} \left( \frac{2^n A_0^2}{2^n M_d^{3n-4}} \right) \prod_{j=1}^{n} \left[ \frac{-1 + 2 \delta_0 + \ln \left( \frac{-p^2_{(j)}}{M_d^2} \right) + \mathcal{O} \left( \frac{-p^2_{(j)}}{M_d^2} \right)^{-2} }{\sqrt{-p^2_{(j)}/M_d^2}} \right] \]

for the \(2n\) \((n \geq 3)\)-point Green functions.

### 4 Auxiliary Field Propagators

We have shown in the previous section that all the multi-fermion Green functions are finite at zero momentum of \(\sigma\) in the S\(\chi\)SB phase. This strongly suggests that the effective potential can be renormalized. We further wish to show finiteness of the Green functions also at non-zero momentum. This requires knowledge of the effective action not restricted to the effective potential, which is, however, a far-reaching problem even in the ladder approximation. Directly relevant to such a problem are the propagators of the auxiliary fields. If the renormalization of the effective potential can simultaneously renormalize these propagators, it would be very promising for the validity of such a renormalization. This is actually the case in the NJL\(_{D<4}\) model[23, 1, 26] which is renormalizable in \(1/N\) expansion.

Thus we are interested in the calculation of the propagators of the auxiliary fields. Difficulty in such a calculation resides in the lack of our knowledge on the Yukawa-type vertex function \(\Gamma_S(p, q)\) at non-vanishing momentum of the auxiliary fields \(q \neq 0\). Recently, Appelquist, Terning and Wijewardhana[29] made an interesting resummation technique based on a further approximation (besides the ladder approximation) to evaluate the auxiliary field propagator without knowing \(\Gamma_S(p, q \neq 0)\). Most amazingly, as we will show in the next section, their \(\sigma\) propagator is in fact renormalized through
our renormalization conditions of the effective potential.\cite{1}

Here we briefly review the calculation of Ref.\cite{29} and discuss the validity and its possible modifications. Let us consider the propagator of auxiliary field $\sigma$ in the symmetric vacuum $\sigma_{\text{sol}} = 0$ (see Fig.6):

$$iD_{\sigma\sigma}^{-1}(-q^2) = -\int \frac{d^4k}{(2\pi)^4 i} \text{tr} \left[ \Gamma_S(k, q) \frac{1}{k} \frac{1}{k - \not{q}} \right] - V''(\text{cl})(\sigma = 0). \quad (4.1)$$

For $q^2 = 0$, it is given by the second derivative of the effective potential:

$$iD_{\sigma\sigma}^{-1}(-q^2 = 0) = -V''(\sigma_{\text{sol}} = 0) = \frac{N \Lambda^2}{4\pi^2} \left( \frac{1}{g^*} - \frac{1}{g} \right). \quad (4.2)$$

Now we consider the second derivative of Eq.(4.1) on $q_\mu$ (see Fig.7)\cite{34, 35, 29}. By using the self-consistent equation for $\Gamma_S$ (see Fig.8), we can easily show that Fig.7 can be expressed as

$$\frac{1}{N} \frac{\partial^2}{\partial q_\mu \partial q_\nu} iD_{\sigma\sigma}^{-1}(-q^2)$$

$$= -\int \frac{d^4k}{(2\pi)^4 i} \text{tr} \left[ \Gamma_S(k, q) \frac{1}{k} \Gamma_S(k - q, -q) \left( \frac{\partial^2}{\partial q_\mu \partial q_\nu} \frac{1}{k - \not{q}} \right) \right]$$

$$- \int \frac{d^4k}{(2\pi)^4 i} \text{tr} \left[ \left( \frac{\partial}{\partial q_\nu} \Gamma_S(k, q) \right) \frac{1}{k} \Gamma_S(k - q, -q) \left( \frac{\partial}{\partial q_\mu} \frac{1}{k - \not{q}} \right) \right] + (\mu \leftrightarrow \nu) \right], \quad (4.3)$$

which is diagrammatically depicted in Fig.9. In the same fashion, we find a symbolic expression of $N$-th derivative:

$$\frac{1}{N} \left( \frac{\partial}{\partial q} \right)^\ell iD_{\sigma\sigma}^{-1}(-q^2)$$

$$= -\int \frac{d^4k}{(2\pi)^4 i} \sum_{n=0}^{\ell-1} \epsilon C_n \text{tr} \left[ \left( \frac{\partial^n}{\partial q^n} \Gamma_S(k, q) \right) \frac{1}{k} \Gamma_S(k - q, -q) \frac{\partial^{\ell-n}}{\partial q^{\ell-n}} \frac{1}{k - \not{q}} \right]. \quad (4.4)$$

It was then observed\cite{29} that the contribution from $\partial^n \Gamma_S(k, q) / \partial q^n |_{q=0}$ is negligible for $n \neq 0$, which implies

$$\frac{1}{N} \left( \frac{\partial}{\partial q} \right)^\ell iD_{\sigma\sigma}^{-1}(-q^2) |_{q=0} = -\int \frac{d^4k}{(2\pi)^4 i} \text{tr} \left[ \Gamma_S(-k^2) \frac{1}{k} \Gamma_S(-k^2)(-1)^\ell \frac{\partial^\ell}{\partial k^{\ell}} \frac{1}{k} \right]. \quad (4.5)$$
Thus, the resummation of the Taylor expansion around $q = 0$ leads to a compact formula:

$$\frac{1}{N} i D^{-1}_{\sigma\sigma}(-q^2) = - \int \frac{d^4k}{(2\pi)^4i} \text{tr} \left[ \Gamma_S(-k^2) \frac{1}{k} \Gamma_S(-k^2) \frac{1}{k-q} \right] + \text{constant.} \quad (4.6)$$

After subtraction at $q^2 = 0$, the integral of Eq.(4.6) yields finally:[29]

$$i D^{-1}_{\sigma\sigma}(q_E^2) - i D^{-1}_{\sigma\sigma}(q_E^2 = 0)$$

$$= \frac{N}{8\pi^2} \int_0^{\Lambda^2} dk_E^2 k_E^2 \Gamma^2_S(k_E^2) \left[ \left( \frac{k_E^2}{q_E^2} - 2 \right) \theta(q_E^2 - k_E^2) - \frac{q_E^2}{k_E^2 \theta(k_E^2 - q_E^2)} \right] \quad (4.7)$$

$$= - \frac{N}{8\pi^2 q_E^2} \left[ \frac{2\alpha c}{\omega \alpha} \Gamma^2_S(q_E^2) - \frac{1}{1 - \omega} \Gamma^2_S(\Lambda^2) \right]$$

$$= - \frac{N}{8\pi^2 q_E^2} \frac{4}{(1 + \omega)^2} \left[ \frac{2\alpha c}{\omega \alpha} \left( \frac{q_E^2}{\Lambda^2} \right)^{1+\omega} - \frac{1}{1 - \omega} \right]. \quad (4.8)$$

Now, let us discuss the validity of Eq.(4.8). It was argued[29] that the contributions coming from

$$\frac{\partial^n \Gamma_S(k, q)}{\partial q^n} \bigg|_{q=0} \quad (4.9)$$

are to be in higher order of $\alpha$:

$$\frac{\partial^n \Gamma_S(k, q)}{\partial q^n} \bigg|_{q=0} \sim \frac{1}{k^n} \frac{\alpha}{4\alpha c} \Gamma_S(-k^2). \quad (4.10)$$

It was also shown numerically[29] that these terms give smaller contributions in $D^{-1}_{\sigma\sigma}$ than that from the zero-derivative of $\Gamma_S$, when the integral is regularized by an IR cutoff.

However, we note that the dominant integral in Eq.(4.6) comes from IR region where $\Gamma_S(k, q = 0)$ does have a singularity which is actually absent in the original $\Gamma_S(k, q \neq 0)$. Thus we must be careful about the effect of higher derivatives which compensates such a singularity around $k = 0$. Here we evaluate the order of magnitude of such an effect, by considering possible modifications of $\Gamma_S$ in Eq.(4.6) which could change the IR behavior of the integral of Eq.(4.6) at $q \neq 0$.

The simplest such modification of $\Gamma_S$ would be the replacement in Eq.(4.7):

$$\Gamma_S^2(k_E^2) \rightarrow \Gamma_S^2(k_E^2) \theta(k_E^2 - q_E^2) + \Gamma_S^2(q_E^2) \theta(q_E^2 - k_E^2). \quad (4.11)$$
Then we obtain

\[ iD^{-1}_{\sigma\sigma}(q_E^2) - iD^{-1}_{\sigma\sigma}(0) = -\frac{Nq_E^2}{8\pi^2} \left( \frac{3}{2} + \frac{1}{1 - \omega} \right) \Gamma_S^2(q_E^2) + \mathcal{O}(q_E^2). \] (4.12)

Another possible modification is the symmetric calculation under the exchange of \( k \) and \( k - q \) in Eq.(4.6):

\[ iD^{-1}_{\sigma\sigma}(q_E^2) = -N \int \frac{d^4k}{(2\pi)^4} \text{tr} \left[ \Gamma_S(-k^2) \frac{1}{k} \Gamma_S(-(k - q)^2) \frac{1}{k - q} \right] + \text{constant}. \] (4.13)

Such a modification leads to

\[ iD^{-1}_{\sigma\sigma}(q_E^2) - iD^{-1}_{\sigma\sigma}(0) = \frac{N}{4\pi^2} \frac{\Gamma(-\omega)}{\Gamma(2 + \omega)} \left[ \frac{\Gamma\left(\frac{3}{2} + \frac{\omega}{2}\right)}{\Gamma\left(\frac{3}{2} - \frac{\omega}{2}\right)} \right]^2 q_E^2 \Gamma_S^2(q_E^2) + \mathcal{O}(q_E^2). \] (4.14)

Although these modifications are somewhat arbitrary, the results seem to suggest validity of the functional form of the \( \sigma \) propagator:

\[ iD^{-1}_{\sigma\sigma}(-q^2) = \frac{N\omega}{\alpha/\alpha_c} q^2 \Gamma_S^2(-q^2) - V''(\sigma = 0), \] (4.15)

with the precise form of \( \xi_\omega \) being subject to the details of the IR treatment. Whereas various modifications agree with each other near the pure NJL model \( \alpha = 0 \), the deviation becomes significant at \( \alpha = \alpha_c \). Actually, \( \xi_\omega \) diverges at \( \alpha = \alpha_c \) for Eq.(4.8), while Eq.(4.12) does not. In the next section we shall renormalize the generic form of Eq.(4.15) not restricted to Eq.(4.8), the original form of Ref.[29].

5 Symmetric Renormalization

In this section we formulate the renormalization of the gauged NJL model, based on the effective potential Eq.(2.63) expanded around the symmetric vacuum (“symmetric renormalization”). This is done in a similar manner to the pure NJL model in \( D(2 < D < 4) \) dimensions (NJLD\( _{D<4} \)) [1]. We first reformulate [1, 26] the renormalization of NJLD\( _{D<4} \) [23] at the leading order of \( 1/N \) through the effective potential approach. Then in the gauged NJL model we apply the same method through the effective potential written in terms of the auxiliary fields.
5.1 \(\text{NJL}_{D<4}\) model

The lagrangian of \(\text{NJL}_{D<4}\) model is given by:

\[
\mathcal{L} = \bar{\psi}i\partial\psi - m_0\bar{\psi}\psi + \frac{G}{2N} \left[ (\bar{\psi}\psi)^2 + (\bar{\psi}\gamma_5\psi)^2 \right],
\]

where the fermion field \(\psi\) belongs to the fundamental representation of \(SU(N)\), with the summation of \(SU(N)\) indices being understood. In Eq.(5.1), the fermion spinor is given by that in four dimensional space-time, so that the model possesses \(U(1)_L \times U(1)_R\) symmetry for \(m_0 = 0\) besides the \(SU(N)\) symmetry.

We first demonstrate the symmetric renormalization through effective potential. Introducing the auxiliary fields \(\sigma, \pi\) in the same way as the gauged NJL model, we obtain the effective potential \(V(\sigma, \pi)\) [1, 26]:

\[
-\frac{(4\pi)^{D/2}\Gamma(D/2)}{4N\Lambda^D}V(\sigma, \pi) = \frac{1}{g} \frac{m_0\sigma}{\Lambda^2} + \left( \frac{1}{g^*} - \frac{1}{g} \right) \frac{\sigma^2 + \pi^2}{2\Lambda^2} - \frac{1}{2-D} \frac{\zeta_D}{D} \left( \frac{\sigma^2 + \pi^2}{\Lambda^2} \right)^{D/2} + \mathcal{O}\left( \left( \frac{\sigma^2 + \pi^2}{\Lambda^2} \right)^2 \right),
\]

with \(\Lambda\) being the ultraviolet cutoff in the loop integral, and \(g^*\) and \(\zeta_D\) are defined by \(g^* \equiv D/2 - 1\) and \(\zeta_D \equiv B(D/2 - 1, 3 - D/2)\), respectively. Note that \(g^* \to 0\) and \(\zeta_D^{-1} \to 0\) as \(D \to 2\), so that the divergence in the second and the third terms cancel each other to give a well-known logarithmic factor in the Gross-Neveu model [37]. For a detailed derivation of Eq.(5.2), see Appendix A.

Propagators of the auxiliary fields are calculated as Eq.(A.26). In the symmetric vacuum \(\langle \sigma \rangle = 0\) they read

\[
-iD^{-1}_{\sigma\sigma}(-q^2) = -iD^{-1}_{\pi\pi}(-q^2)
\]

\[
= V''(\sigma = 0, \pi = 0) + \frac{4N}{(4\pi)^{D/2}\Gamma(D/2)} \frac{\xi_D}{2-D} (-q^2)^{D/2-1},
\]

where \(\xi_D \equiv B(3 - D/2, D/2 - 1)/\Gamma(D - 1)\) and

\[
V''(\sigma = 0, \pi = 0) \equiv \frac{\partial^2 V(\sigma, \pi = 0)}{\partial \sigma^2} \bigg|_{\sigma=0} = \frac{4N\Lambda^{D-2}}{(4\pi)^{D/2}\Gamma(D/2)} \left( \frac{1}{g} - \frac{1}{g^*} \right),
\]
Eq. (5.4) is negative for \( g > g^* \), implying appearance of a tachyon pole in the auxiliary field propagators, another signal of instability of the symmetric vacuum.

Let us next consider the renormalization of NJL\(_{D<4}\) model at the 1/\( N \) leading order. Due to absence of the divergence in the fermion propagator and the vertex function \( \Gamma_S \), wave function renormalizations of the fermion and the auxiliary fields are not required at this stage:

\[
\psi = \psi_R, \quad \sigma_R = \sigma.
\]  

(5.5)

Thus we concentrate our attention to the renormalization of the effective potential Eq.(5.2). The divergence \( \Lambda^{D-2} \) in Eq.(5.2) can be absorbed into the redefinition of renormalized parameters \( g_R, m_R \):

\[
\Lambda^{D-2} \left( \frac{1}{g} - \frac{1}{g^*} \right) = \mu^{D-2} \left( \frac{1}{g_R} - \frac{1}{g_R^*} \right)
\]  

(5.6)

and

\[
\frac{\Lambda^{D-2}}{g} m_0 = \frac{\mu^{D-2}}{g_R} m_R,
\]  

(5.7)

where \( \mu \) is the renormalization scale. These are precisely the same renormalization conditions as those imposed by Kikukawa and Yamawaki [23] through the renormalization of the propagator of \( \sigma \).

With the above definition of renormalized parameters Eqs.(5.6–5.7) we obtain a renormalized effective potential:

\[
-\frac{(4\pi)^{D/2} \Gamma(D/2)}{4N \mu^D} V_R(\sigma_R, \pi_R)
\]

\[
= \frac{1}{g_R} \frac{m_R \sigma_R}{\mu^2} + \left( \frac{1}{g^*_R} - \frac{1}{g_R} \right) \frac{\sigma_R^2 + \pi_R^2}{2\mu^2} - \frac{1}{2 - \frac{D}{2} \zeta_D} \left( \frac{\sigma_R^2 + \pi_R^2}{\mu^2} \right)^{D/2},
\]  

(5.8)

where higher power terms \( O(\sigma^4) \) disappear in \( \Lambda \to \infty \) (even at \( D = 2 \)). This renormalization breaks down at \( D = 4 \), which is signaled by the singularity \( (2 - \frac{D}{2})^{-1} \) in the last term of Eq.(5.8), corresponding to the logarithmic divergence of \( (\sigma^2 + \pi^2)^2 \) coupling in \( D = 4 \) NJL model. (We would need extra “counter term” such as the eight-fermion operators \( (\bar{\psi} \psi)^4 \) to “renormalize” the model in \( D = 4 \).)

In our renormalization Eqs.(5.6–5.7), the critical value of renormalized four-fermion coupling is left undetermined. Actually, it is a free parameter which corresponds to
varieties of renormalization schemes. In the $D \to 2$ limit, however, we need to define the singular part of $g^*_R$ as a function of $D$:

$$\frac{1}{g^*_R} = \frac{1}{D/2 - 1} + \text{regular function of } D,$$

so that we can reproduce the usual renormalization of the Gross-Neveu model [37].

Again, the choice of the regular part corresponds to the choice of renormalization scheme. In the following, we take the simplest choice $g^*_R = D/2 - 1 = g^*$ (different from that in Ref.[23]).

It is straightforward to get RG functions $\beta_g$ and $\gamma_m$ from Eqs.(5.6–5.7): 6

$$\beta_g(g_R) = (D - 2)g_R \left(1 - \frac{g_R}{g^*_R}\right),$$

$$\gamma_m(g_R) = (D - 2)\frac{g_R}{g^*_R},$$

where $\beta_g \equiv \mu \partial g_R/\partial \mu$ and $\gamma_m m_R \equiv -\mu \partial m_R/\partial \mu$. This agrees with Ref.[23].

It is also obvious that the propagators of $\sigma$ and $\pi$ Eq.(5.3) can also be renormalized by the same condition as Eq.(5.6), as was originally done in Ref.[23]:

$$-iD_{\sigma\sigma}^{(R)}(-q^2) = -iD_{\pi\pi}^{(R)}(-q^2)$$

$$= V_R''(\sigma_R = 0, \pi_R = 0) + \frac{4N}{(4\pi)^{D/2}\Gamma(D/2)} \frac{\xi_D}{2 - D/2} (-q^2)^{D/2-1}. (5.12)$$

It should be stressed again that this renormalization breaks down at $D = 4$, signaled by the appearance of $(2 - \frac{D}{2})^{-1}$ singularity in the term of $(q_{L}^2)^{\frac{D}{2}-1}$ as $D \to 4$), which corresponds to the logarithmic divergence of the kinetic term of $\sigma$ and $\pi$ in $D = 4$ NJL model (We would need higher dimensional operator $\partial_\mu (\bar{\psi}\psi) \partial^\mu (\bar{\psi}\psi)$ as a “counter

---

6The next-to-leading order corrections to this result have been calculated [26, 38, 39].
term” to “renormalize” the model in $D = 4$).

The fact that we can renormalize the theory without higher dimensional operators $(\bar{\psi}\psi)^4$ and $\partial^\mu(\bar{\psi}\psi)\partial^\mu(\bar{\psi}\psi)$ at $1/N$ leading order simply reflects the following fact: $(\bar{\psi}\psi)^2$ is a relevant operator due to a large anomalous dimension $\gamma_m = D - 2$ at $g_R = g_R^*$, i.e., $\text{dim}(\bar{\psi}\psi)^2 = 2(D - 1 - \gamma_m) = 2 < D$, while the would-be “counter terms” $(\bar{\psi}\psi)^4$ and $\partial^\mu(\bar{\psi}\psi)\partial^\mu(\bar{\psi}\psi)$ are irrelevant operators, $\text{dim}(\bar{\psi}\psi)^4 = 4(D - 1 - \gamma_m) = 4 > D$, $\text{dim}[\partial^\mu(\bar{\psi}\psi)\partial^\mu(\bar{\psi}\psi)] = 2(D - \gamma_m) = 4 > D$. At $D = 4$, however, all these operators equally have dimension $4(= D)$ and become marginal operators. Hence they should be included in order to make the theory renormalizable, in which case the NJL model in its renormalizable version becomes identical to the Higgs-Yukawa system (“standard model”) [40, 41].

5.2 Symmetric renormalization of the gauged NJL model

Now, we are ready to study the renormalization properties of the gauged NJL model, based on the simplest effective potential Eq.(2.68) expanded around the symmetric vacuum (symmetric renormalization)[1]. The gauge coupling $\alpha$ does not get renormalized, in accord with the absence of vacuum polarization in the gauge boson propagator in this approximation. Such an approximation becomes realistic in the standing gauge theory as a limit of walking gauge theory [4]. Thus the renormalization is operative only on the four-fermion coupling $g$. Actually, this renormalization ($0 < \omega < 1$) is done in a very similar manner to NJL$_{D<4}(2 < D < 4)$. Both cases break down in the pure NJL limit ($D = 4, \omega = 1$). However it should be noted that while the renormalization of NJL$_{D<4}$ is valid even at $D = 2$ where Eq.(5.2) still remains valid, the renormalization of the gauged NJL model in this scheme breaks down at $\omega = 0$ where Eq.(2.68) becomes no longer valid. Renormalization scheme valid also at $\omega = 0$ should be based on the general form of the effective potential Eq.(2.63), which will be discussed in section 7.

In our definition of (bare) auxiliary field, the (bare) Yukawa-type vertex $\Gamma_S$ does not depend on the four-fermion coupling and vanishes in the $\Lambda \to \infty$ limit. Thus, it should be renormalized via redefinition of auxiliary fields to obtain a finite interacting theory in the $\Lambda \to \infty$ limit:

$$\sigma_R \equiv \left(\frac{\Lambda}{\mu}\right)^{1-\omega} \sigma, \quad \pi_R \equiv \left(\frac{\Lambda}{\mu}\right)^{1-\omega} \pi,$$  \hspace{1cm} (5.13)
with \( \mu \) being “renormalization point”.\(^7\) According to this definition of renormalized fields, the Yukawa-type vertex Eq.(2.82) is renormalized:

\[
\Gamma_R^{(R)}(-p^2) = \left( \frac{\mu}{\Lambda} \right)^{1-\omega} \Gamma_R(-p^2) = \frac{2}{1+\omega} \left( \frac{-p^2}{\mu^2} \right)^{-1-\omega/2}
\]  

(5.14)

for the symmetric vacuum. It should be noted that this definition of the renormalized fields \( \sigma_R, \pi_R \) simultaneously renormalizes the Yukawa-type vertex in \( S\chi SB \) vacuum Eq.(2.86):

\[
\Gamma_S^{(R)}(-p^2; M_d) = \left( \frac{\mu}{\Lambda} \right)^{1-\omega} \Gamma_S(-p^2; M_d) = \frac{2}{1+\omega} \left( \frac{-p^2}{\mu^2} \right)^{-(1-\omega)/2} \left[ 1 + \frac{(1+\omega)(2+\omega) D_1}{(1-\omega)(2-\omega) C_1} \left( \frac{-p^2}{M_d^2} \right)^{-\omega} \right] + \mathcal{O}\left( \left( \frac{-p^2}{M_d^2} \right)^{-2+\omega} \right)
\]  

(5.15)

Then the effective potential Eq.(2.68) is expressed in terms of the renormalized auxiliary fields:

\[
\frac{4\pi^2}{N} \frac{V_R(\sigma_R, \pi_R)}{\mu^4} = \left( \frac{\Lambda}{\mu} \right)^{1+\omega} \left( \frac{m_0 \sigma_R}{g \mu^2} \right) + \left( \frac{\Lambda}{\mu} \right)^{2\omega} \left( \frac{1}{g^*} - \frac{1}{g} \right) \frac{\sigma_R^2 + \pi_R^2}{2 \mu^2} - \frac{4\zeta_\omega}{\alpha/\alpha_c} \frac{2-\omega}{4} \left( \frac{\sigma_R^2 + \pi_R^2}{\mu^2} \right)^{2/(2-\omega)},
\]  

(5.16)

where we have dropped out the contributions which vanish in the \( \Lambda \to \infty \) limit.

Now, the parallelism between NJL\(_{D<4}\) and gauged NJL model is manifest. The effective potential Eq.(2.68) can be renormalized by the definition of renormalized four-fermion coupling

\[
\Lambda^{2\omega} \left( \frac{1}{g^*} - \frac{1}{g} \right) = \mu^{2\omega} \left( \frac{1}{g^*_R} - \frac{1}{g_R} \right),
\]  

(5.17)

\(^7\)Similar redefinition of the auxiliary fields was also made [28], with \( \mu \) being taken as the dynamical fermion mass \( \mu = M_d \).
and renormalized current mass of the fermion:

\[
\Lambda^{1+\omega} \frac{m_0}{g} = \mu^{1+\omega} \frac{m_R}{g_R}.
\] (5.18)

Again, the ambiguity of \( g^*_R \) corresponds to the choice of renormalization scheme. According to this renormalization, we find the renormalized effective potential

\[
-\frac{4\pi^2}{N} \frac{V_R(\sigma_R, \pi_R)}{\mu^4} = \frac{1}{g_R} \frac{m_R \sigma_R}{\mu^2} + \left( \frac{1}{g^*_R} - \frac{1}{g_R} \right) \frac{\sigma^2_R + \pi^2_R}{2\mu^2} - \frac{4\zeta}{\omega} \frac{2 - \omega}{\alpha/\alpha_c} \frac{\sigma^2_R + \pi^2_R}{\mu^2} \left( \frac{\sigma_R}{\mu} \right)^2 \left( \frac{\pi_R}{\mu} \right)^2.
\] (5.19)

Thus, all the multi-fermion Green functions have been renormalized at zero momentum of the auxiliary field.

How about the renormalization at non-zero momentum, then? Remarkably enough, the auxiliary field propagators Eq.(4.15) are also renormalized via the above definition of the renormalized parameters:

\[
-iD^{(R)}_{\sigma\sigma}(-q^2) = -iD^{(R)}_{\pi\pi}(-q^2)
\]

\[
= V''_R(\sigma_R = 0, \pi_R = 0) + \frac{N\zeta}{\alpha/\alpha_c}(-q^2)\Gamma^{(R)2}_S(-q^2).
\] (5.20)

Thus the four-fermion Green function can also be renormalized at non-zero momentum in a similar manner to NJL\(_{D<4}\). In spite of the formal resemblance, however, one should note that the gauged NJL model has only been shown to be renormalized at the level of the ladder approximation, in sharp contrast to NJL\(_{D<4}\) which is shown to be renormalizable in the systematic 1/\(N\) expansion[25].

We should also emphasize that the renormalization Eqs.(5.17–5.18) is based on the effective potential and thus it holds both in the chiral symmetric \((g_R < g^*_R)\) and the S\(_\chi\)SB \((g_R > g^*_R)\) phases.

The definition of the renormalized parameters \( g_R \) and \( m_R \) Eqs.(5.17–5.18) leads to the RG functions \( \beta_g \) and \( \gamma_m \) (Fig. 12):

\[
\beta_g(g_R, \alpha) = 2\omega g_R \left( 1 - \frac{g_R}{g^*_R} \right),
\] (5.21)
\[ \gamma_m(g_R, \alpha) = 1 - \omega + 2\omega \frac{g_R}{g_R^*}. \]  

(5.22)

Thus the theory does have a nontrivial ultraviolet fixed line \( g_R = g_R^* \), on which the mass operator of fermion acquires a large anomalous dimension \( \gamma_m = 1 + \omega \). It should also be noted that the anomalous dimension \( \gamma_m \) is continuous across the critical line \( g_R = g_R^* \), in contrast to the earlier phase-dependent calculation through the fermion propagator based on the solution of the SD gap equation. This is actually in accord with the suggestion\[23\] that the renormalization through the four-fermion Green function (auxiliary field propagator) in the gauged NJL model may lead to the large anomalous dimension in the symmetric phase as well as in the S\( \chi \)SB phase.

At \( \alpha = \alpha_c \) (\( \omega = 0 \)) Eqs.(5.21–5.22) would yield \( \beta_g \equiv 0 \) and \( \gamma_m \equiv 1 \). This may be an artifact of the symmetric renormalization based on the effective potential Eq.(2.68) which is no longer valid at \( \alpha = \alpha_c \). A possible modification at \( \alpha = \alpha_c \) will be given in Section 7.

We thus have found that the gauged NJL model is renormalized within the ladder approximation for non-vanishing gauge coupling \( \alpha > 0 \). It is well known, however, that the pure (non-gauged) NJL model cannot be renormalized in \( D = 4 \) due to uncontrollable logarithmic divergence even in the leading approximation of \( 1/N \) expansion (or chain approximation). We here discuss how the existence of gauge interaction improves the structure of divergence. The anomalous dimension of composite operator \( \bar{\psi}\psi \) at critical point of the pure NJL model is given by \( \gamma_m = 2 \). Such a large anomalous dimension makes the scaling dimension of \( \bar{\psi}\psi \) very small: \( \dim \bar{\psi}\psi = 3 - \gamma_m = 1 \), and hence higher dimensional operators such as \( (\bar{\psi}\psi)^4 \) and \( \partial_\mu (\bar{\psi}\psi) \partial^\mu (\bar{\psi}\psi) \) become marginal operators. As a result, logarithmic divergence associated with these operators appear. On the other hand, the gauge interaction makes the scaling behavior of \( \bar{\psi}\psi \) softer: \( \dim \bar{\psi}\psi = 3 - \gamma_m = 2 - \omega > 1 \). Thus, the above higher dimensional operators becomes irrelevant operators and there are no uncontrollable divergence.

In the diagrammatic picture, the softness of the scaling dimension of \( \bar{\psi}\psi \) corresponds to the softness of high energy behavior of Yukawa-type vertex function \( \Gamma^{(R)}_S \) Eq.(5.14). Thanks to such a soft behavior of \( \Gamma^{(R)}_S \), the structure of divergence is improved and the logarithmic divergence disappears in the presence of gauge interaction.
6 Operator Product Expansion

Now that we have obtained a renormalized theory having a nontrivial UV fixed line with a very large anomalous dimension, $\gamma_m = 1 + \omega \left( g_R = g_R^\ast \right)$, we can explicitly construct OPE both in the symmetric and the SSB phases and see how such a large anomalous dimension fits in the general framework of OPE [1].

The OPE relevant to the fermion mass function takes the form

$$-i\mathcal{FT}\left[ \bar{\psi}(x)\psi(0) \right] = c_1(p; g_R, \alpha, m_R, \mu) \mathbb{1} + c_{\bar{\psi}\psi}(p; g_R, \alpha, m_R, \mu) \left[ (\bar{\psi}\psi)_R + \gamma_5(\bar{\psi}\gamma_5\psi)_R \right] + \cdots. \quad (6.1)$$

We explicitly calculate the Wilson coefficients $c_1$ and $c_{\bar{\psi}\psi}$ on the nonperturbative solution which we know already. Taking the vacuum expectation value of Eq.(6.1), we write

$$-iS(p) = c_1 + c_{\bar{\psi}\psi}\langle (\bar{\psi}\psi)_R \rangle + \cdots. \quad (6.2)$$

The fermion propagator in LHS takes the form

$$-iS(p) = \frac{p^\dagger}{p^2} + \frac{\Sigma(-p^2)}{p^2} + \cdots. \quad (6.3)$$

Comparing Eq.(6.3) with Eq.(6.2), we write

$$\Sigma(-p^2) = p^2m_R(\mu)c_1'(p; g_R, \alpha, \mu) + p^2\langle (\bar{\psi}\psi)_R \rangle c_{\bar{\psi}\psi}(p; g_R, \alpha, 0, \mu) + \cdots, \quad (6.4)$$

where we have expanded the Wilson coefficients around the chiral symmetric limit $m_R = 0$:

$$c_1(p; g_R, \alpha, m_R, \mu) = \frac{p^\dagger}{p^2} + m_R(\mu)c_1'(p; g_R, \alpha, \mu) + \cdots, \quad (6.5a)$$

$$c_{\bar{\psi}\psi}(p; g_R, \alpha, m_R, \mu) = c_{\bar{\psi}\psi}(p; g_R, \alpha, 0, \mu) + \cdots. \quad (6.5b)$$

We denote by $\Sigma_{\text{explicit}}(-p^2)$ the part of the fermion mass function owing to the explicit chiral symmetry breaking $m_R \neq 0$:

$$\Sigma_{\text{explicit}}(-p^2) = p^2m_R(\mu)c_1'(p; g_R, \alpha, \mu) + \cdots, \quad (6.6)$$
which actually takes the form

\[
\Sigma_{\text{explicit}}(-p^2) = \begin{cases} 
\Gamma_S(-p^2; M_d) \sigma_{\text{explicit}}, & (\text{S}\chi\text{SB vacuum}) \\
\Gamma_S(-p^2) \sigma_{\text{explicit}}, & (\text{symmetric vacuum}) 
\end{cases}
\]  

(6.7)

where \( \sigma_{\text{explicit}} \) is defined by

\[
\sigma_{\text{explicit}} \equiv \sigma_{\text{sol}} - \sigma_{\text{spont}} = \left. \frac{\partial \sigma_{\text{sol}}}{\partial m_0} \right|_{m_0=0} m_0 + \cdots ,
\]  

(6.8)

with \( \sigma_{\text{sol}} \) being the solution of the stationary condition of the effective potential Eq.(2.63), \( \partial V(\sigma, \pi = 0)/\partial \sigma = 0 \). For \( 0 < \alpha < \alpha_c \) it is straightforward to calculate \( \sigma_{\text{explicit}} \) from Eq.(2.74) as a Taylor series in the fermion bare mass \( m_0 \):

\[
\sigma_{\text{explicit}} = \begin{cases} 
\frac{2 - \omega}{2 \omega} \frac{m_0}{g^2 - 1} & (\text{S}\chi\text{SB vacuum}) \\
\frac{m_0}{1 - \frac{g}{g^*}} & (\text{symmetric vacuum})
\end{cases}
\]  

(6.9)

By comparing Eq.(6.6) with Eq.(6.7) and Eq.(6.9), we find

\[
c'_4(p; g_R, \mu) = \begin{cases} 
\frac{1}{p^2} \frac{1}{2 \omega} \frac{1}{g_R} \frac{\Gamma^{(R)}_S(-p^2; M_d)}{g^* - 1} & (\text{S}\chi\text{SB vacuum}) \\
\frac{1}{p^2} \frac{1}{1 - \frac{g_R}{g^*_R}} \Gamma^{(R)}_S(-p^2) & (\text{symmetric vacuum})
\end{cases}
\]  

(6.10)

with the renormalized vertex \( \Gamma^{(R)}_S \) being given by Eq.(5.15), where we have used Eq.(5.18);

\[
\frac{m_0}{m_R} = \left( \frac{\mu}{\Lambda} \right)^{1+\omega} \frac{g}{g_R} (\equiv Z_m).
\]  

(6.11)
Now, it is easy to show that the Wilson coefficients satisfy the RG equation:

\[ 0 = \left[ D + 2 + \gamma_m(g_R) \right] c'_1(kp; g_R, \alpha, \mu), \quad (6.12a) \]
\[ 0 = \left[ D + 4 - \gamma_m(g_R) \right] c'_{\bar{\psi}\psi}(kp; g_R, \alpha, 0, \mu), \quad (6.12b) \]

where

\[ D \equiv \kappa \frac{\partial}{\partial \kappa} - \beta_g(g_R) \frac{\partial}{\partial g_R}. \quad (6.13) \]

Eq.(6.12a) is readily solved to yield

\[ c'_1(kp; g_R, \alpha; \mu) \simeq \kappa^{-(2+\gamma^*_m)} c'_1(p; \bar{g}(\kappa), \alpha; \mu) \quad (6.14) \]

where \( \gamma^*_m \equiv \gamma_m(g^*_R) = 1 + \omega \) and \( \bar{g}(\kappa) \) is the solution of

\[ \kappa \frac{d}{d\kappa} \bar{g}(\kappa) = \beta_g(\bar{g}), \quad \text{with} \quad \bar{g}(\kappa = 1) = g_R(\mu). \quad (6.15) \]

Eq.(6.15) can be solved using the \( \beta \) function Eq.(5.21):

\[ \frac{1}{1 - \bar{g}(\kappa) g_R^* \kappa} = 1 + \kappa^{2\omega} \frac{g_R}{g_R^*} \quad (6.16) \]

Then Eq.(6.10) implies a quite unusual situation, i.e., the Wilson coefficient \( c'_1 \) does have a strong momentum dependence

\[ c'_1(p; \bar{g}(\kappa), \mu) \sim \kappa^{2\omega}. \quad (6.17) \]

This is combined with Eq.(6.9), yielding finally the high energy behavior of \( \Sigma_{\text{explicit}} \):

\[ \Sigma_{\text{explicit}}(\kappa^2 p^2) \simeq m_R(\mu) p^2 \kappa^{-(1+\omega)} c'_1(p; \bar{g}(\kappa), \alpha; \mu) \sim \kappa^{-(1-\omega)}. \quad (6.18) \]

Thus the Wilson coefficient for the explicit chiral symmetry breaking term has a nontrivial factor Eq.(6.10), yielding additional momentum dependence \( \kappa^{2\omega} \), which actually compensates the momentum dependence \( \kappa^{-(1+\omega)} \) arising from the anomalous dimension. As a result we obtain \( \kappa^{-(1-\omega)} \) behavior of the fermion mass function.
\[ \Sigma_{\text{explicit}}(-\kappa^2 p^2), \text{ in accord with the solutions of the SD equation (see Appendix C)}: \]

\[ \Sigma_{\text{explicit}}(-\kappa^2 p^2) \simeq \kappa^{-(1-\omega)} \Sigma_{\text{explicit}}(-p^2). \] (6.19)

This peculiar phenomenon with the Wilson coefficient is precisely the same as that in NJL_{D<4} which was discovered by Kikukawa and Yamawaki[23]. Explicit calculation in NJL_{D<4} is given in Appendix B where the Wilson coefficient \( c_1' \) takes the form

\[
c_1'(p; g_R, \mu) = \begin{cases} 
\frac{1}{D-2} \frac{1}{g_R} \frac{1}{p^2} + \cdots & (S\chi \text{SB vacuum}) \\
\frac{1}{g_R} \frac{1}{p^2} + \cdots & (\text{symmetric vacuum}) 
\end{cases}
\] (6.20)

which yields \( c_1'(p; \bar{g}(\kappa), \mu) \simeq \kappa^{D-2} = \kappa^{\gamma_m^*} \). Solving RG equation for \( c_1' \), we obtain

\[
c_1'(\kappa p; g_R, \mu) \simeq c_1'(p; \bar{g}(\kappa), \mu) \kappa^{-(2+\gamma_m^*)}. \] (6.21)

Then in NJL_{D<4} we obtain

\[ \Sigma_{\text{explicit}}(-\kappa^2 p^2) \simeq \kappa^2 p^2 m_R(\mu) c_1'(\kappa p; g_R, \mu) \]

\[ \simeq p^2 m_R(\mu) c_1'(p; \bar{g}(\kappa), \mu) \kappa^{-\gamma_m^*} \]

\[ = \text{constant}, \] (6.22)

which is actually in accord with the explicit solution of the SD gap equation given in Appendix A.

Let us next turn to the Wilson coefficient of \( (\bar{\psi}\psi)_R, c_{\bar{\psi}\psi} \). Such a coefficient function can be determined from the fermion four-point function by taking \( x \to 0 \) limit:

\[ -i \langle [\bar{\psi}(x)\psi(0)\psi(y)\bar{\psi}(z)] \rangle_{\text{connected}} = c_{\bar{\psi}\psi}(x) \langle [\bar{\psi}\psi)_R(0)\psi(y)\bar{\psi}(z)] \rangle_{\text{connected}} + \cdots, \] (6.23)

where \( c_1 \) does not appear in the RHS of Eq.(6.23), since it contributes only to the disconnected diagrams.

In the following calculation \( x,y \) and \( z \) are Fourier transformed to \( p,q \) and \( k \), respectively. It is sufficient to evaluate the case of \( q = k \) to obtain \( c_{\bar{\psi}\psi} \). Hereafter, the external legs for \( q \) and \( k \) are understood to be amputated.
We first calculate the OPE coefficient function on the symmetric vacuum. The $\sigma$-exchange diagram (Fig. 10a) is given by:

$$S(p)i\Gamma_S(-p^2)S(p)D_{\sigma\sigma}(0)\Gamma_S(-q^2) = \frac{1}{p^2}\Gamma_S(-p^2)\frac{1}{-V''(\sigma_{sol})}\Gamma_S(-q^2) + \cdots, \quad (6.24)$$

where we have used a relation $D_{\sigma\sigma}(0) = -i/V''(\sigma_{sol})$. In addition to the above there exist “pure ladder” diagrams (Fig. 10b) contributing to the LHS of Eq.(6.23), which are hard to be calculated. Here we assume that such diagrams have softer high energy behavior than that of Eq.(6.24) and ignore them in the following calculations.

The RHS of Eq.(6.23) can be calculated by (Fig. 11). Since a bubble diagram is given by the second derivative of $V_{(qu)} \equiv V - V_{(cl)}$, we obtain (Fig. 11)

$$Z_m c_{\bar{\psi}\psi}(p; g_R, m_R = 0, \mu) \left(1 - iV''_{(qu)}(\sigma_{sol})D_{\sigma\sigma}(0)\right)\Gamma_S(-q^2)$$

$$= -c_{\bar{\psi}\psi}(p; g_R, m_R = 0, \mu)V''_{(cl)}(\sigma_{sol})\frac{Z_m}{-V''(\sigma_{sol})}\Gamma_S(-q^2), \quad (6.25)$$

where we have used the renormalization of the composite operator

$$(\tilde{\psi}\psi)_R = Z_m(\tilde{\psi}\psi). \quad (6.26)$$

Equating Eq.(6.24) and Eq.(6.25), we finally obtain the OPE coefficient function $c_{\bar{\psi}\psi}$:

$$c_{\bar{\psi}\psi}(p; g_R, \mu) = -\frac{\Gamma_S(-p^2)}{p^2} \frac{Z_m^{-1}}{V''_{(cl)}(\sigma_{sol})} \Gamma^{(R)}_S(-p^2) \frac{G_R}{N}, \quad (6.27)$$

where $G_R$ is defined as

$$G_R \equiv 4\pi^2 \frac{g_R}{\mu^2}. \quad (6.28)$$

We can easily see that Eq.(6.27) does satisfy the RG equation Eq.(6.12b). Thus

$$c_{\bar{\psi}\psi}(k\kappa p; g_R, \mu) \simeq c_{\bar{\psi}\psi}(p; \tilde{g}(t), \mu)\kappa^{-(4-g_m^*)}. \quad (6.29)$$
As for the S\(\chi\)SB vacuum, the OPE coefficient \(c_{\bar{\psi}\psi}\) can be evaluated by replacing the Yukawa-type vertex in the symmetric vacuum in Eq.(6.27) with that in the broken vacuum:

\[
c_{\bar{\psi}\psi}(p; g_R, \mu) = -\frac{\Gamma^{(R)}_{S}(-p^2; M_d)}{p^2} \frac{G_R}{N}.
\]  

(6.30)

It is easy to see from Eq.(6.29) that the coefficient \(c_{\bar{\psi}\psi}\), having no extra momentum dependence, yields a correct high energy behavior of \(\Sigma_{\text{dyn}}(-p^2) = p^2 \langle \bar{\psi}\psi \rangle_R c_{\bar{\psi}\psi}(p; g_R, \alpha, \mu)\):

\[
\Sigma_{\text{dyn}}(-\kappa^2 p^2) = \kappa^2 p^2 \langle \bar{\psi}\psi \rangle_R c_{\bar{\psi}\psi}(p; \bar{g}(\kappa), \mu) \sim \kappa^{-2-\gamma_m} \sim \kappa^{-(1-\omega)},
\]

(6.31)

which indeed agrees with the solution of the SD equation given in Appendix C.

This result is also similar to that of NJL\(_{D>4}\) [23]. See Appendix B for details.

7 \(\bar{M}\)-Dependent Renormalization

In the previous section, we have carried out the symmetric renormalization of the gauged NJL model with \(\alpha < \alpha_c \ (\omega \neq 0)\), based on the effective potential around the symmetric vacuum Eq.(2.68). Such a renormalization cannot directly apply to \(\alpha = \alpha_c \ (\omega = 0)\), since the expansion Eq.(2.68) loses its validity at this point. However, breakdown of the expansion Eq.(2.68) does not necessarily imply the non-renormalizability of the gauged NJL model at \(\alpha = \alpha_c\). Actually, as we have studied in section 3, the fermion scattering amplitudes remain finite even at \(\alpha = \alpha_c\) as well as at \(\alpha < \alpha_c\), once the bare four-fermion coupling is fine-tuned so as to make the fermion mass finite. This fact suggests that the renormalization may be possible by a suitable definition of renormalized parameters even at \(\alpha = \alpha_c\). Actually, the general form of the effective potential Eq.(2.63) remains valid even at \(\alpha = \alpha_c\).

In this section we present yet another renormalization scheme which renormalizes the effective potential Eq.(2.63) instead of Eq.(2.68). The renormalization described in this section introduces a redundant mass parameter \(\bar{M}\) (\(\bar{M}\)-dependent renormalization) and \(\bar{M} = 0\) corresponds to the symmetric renormalization. The \(\bar{M}\)-dependent renormalization is related to the symmetric renormalization via a finite renormalization at \(0 < \alpha < \alpha_c\) and remains valid at \(\alpha = \alpha_c\).

The Yukawa-type vertex Eq.(2.86) at the non-trivial \(M \neq 0\) is renormalized by the
wave function renormalization:

$$\frac{\sigma}{C_1 \left( \frac{M}{\lambda} \right)^{1-\omega} + D_1 \left( \frac{M}{\lambda} \right)^{1+\omega}} = \frac{\sigma_{\tilde{R}}}{C_1 \left( \frac{M}{\mu} \right)^{1-\omega} + D_1 \left( \frac{M}{\mu} \right)^{1+\omega}}, \quad (7.1)$$

which leads to the renormalized Yukawa-type vertex in the $S\chi$SB vacuum $M = M_d$:

$$\Gamma_{S}^{(\tilde{R})}(-p^2; M_d) = \frac{\partial \Sigma(-p^2)/\partial M|_{M=M_d}}{(2-\omega) \left[ C_1 \left( \frac{M}{\mu} \right)^{1-\omega} + D_1 \left( \frac{M}{\mu} \right)^{1+\omega} \right] \left( \frac{M_d}{M} \right)^{1-\omega}}. \quad (7.2)$$

Note that the same renormalization condition simultaneously makes finite the Yukawa-type vertex in the symmetric vacuum:

$$\Gamma_{S}^{(R)}(-p^2) = \Gamma_{S}^{(\tilde{R})}(-p^2; M = 0) = \frac{2}{1+\omega} \left( \frac{-p^2}{\mu^2} \right)^{-\omega/2} \left[ 1 + \frac{D_1}{C_1} \left( \frac{M}{\mu} \right)^{2\omega} \right]^{-1}. \quad (7.3)$$

We next define the renormalized four-fermion coupling $g_{\tilde{R}}$ and renormalized fermion mass $m_{\tilde{R}}$ by:

$$\left[ C_1 \left( \frac{\tilde{M}}{\lambda} \right)^{-\omega} + D_1 \left( \frac{\tilde{M}}{\lambda} \right)^{\omega} \right]^{2} \left( \frac{1}{g^*} - \frac{1}{g} \right) + D_1 \left( \frac{\tilde{M}}{\lambda} \right)^{\omega} \left[ D_1 \left( \frac{\tilde{M}}{\lambda} \right)^{\omega} + \frac{2-\omega}{2} C_1 \left( \frac{\tilde{M}}{\lambda} \right)^{-\omega} \right] \left( \frac{1}{g^*} - \frac{1}{g} \right)$$

$$= \left[ C_1 \left( \frac{\tilde{M}}{\mu} \right)^{-\omega} + D_1 \left( \frac{\tilde{M}}{\mu} \right)^{\omega} \right]^{2} \left( \frac{1}{g_{\tilde{R}}^*} - \frac{1}{g_{\tilde{R}}} \right) + D_1 \left( \frac{\tilde{M}}{\mu} \right)^{\omega} \left[ D_1 \left( \frac{\tilde{M}}{\mu} \right)^{\omega} + \frac{2-\omega}{2} C_1 \left( \frac{\tilde{M}}{\mu} \right)^{-\omega} \right] \left( \frac{1}{g_{\tilde{R}}^*} - \frac{1}{g_{\tilde{R}}} \right), \quad (7.4)$$

and

$$\frac{m_0}{g} \left[ C_1 \left( \frac{\tilde{M}}{\lambda} \right)^{-(1+\omega)} + D_1 \left( \frac{\tilde{M}}{\lambda} \right)^{-(1-\omega)} \right] = \frac{m_{\tilde{R}}}{g_{\tilde{R}}} \left[ C_1 \left( \frac{\tilde{M}}{\mu} \right)^{-(1+\omega)} + D_1 \left( \frac{\tilde{M}}{\mu} \right)^{-(1-\omega)} \right]. \quad (7.5)$$

Here $g^*_R$ and $g^*_\tilde{R}$ are arbitrary parameters corresponding to the choice of $\tilde{M}$ in this renormalization scheme. It is easy to see that the Eq.(7.4) reduces to the symmetric
renormalization Eq.(5.17) in the $\bar{M} \to 0$ limit. Also note that unlike the symmetric renormalization, $g^*_R$ in Eq.(7.4) does not corresponds to a “critical” coupling, unless $\bar{M}/\mu = 0$.

Starting from the bare effective potential Eq.(2.63), we obtain a renormalized effective potential in the $\Lambda \to \infty$ limit:

$$-rac{4\pi^2 V_R(\sigma_R, \pi_R = 0)}{N} = \frac{1}{g_R} \frac{m_R \sigma_R}{\mu^2} + \frac{1}{2} \left( \frac{1}{g^*_R} - \frac{1}{g_R} \right) \left( \frac{\sigma_R}{\mu} \right)^2 + \frac{1}{2} \left( \frac{1}{g^*_R} - \frac{1}{g^*_R} \right) \left( \frac{\sigma_R}{\mu} \right)^2 D_1 \left[ D_1 \left( \frac{\bar{M}}{\mu} \right)^{2\omega} + \frac{2 - \omega}{2} C_1 \right]$$

$$+ \frac{1}{2} \left( \frac{1}{g^*_R} - \frac{1}{g^*_R} \right) \left( \frac{\sigma_R}{\mu} \right)^2 \left[ C_1 \left( \frac{\bar{M}}{\mu} \right)^{-\omega} + D_1 \left( \frac{\bar{M}}{\mu} \right)^{\omega} \right]^2$$

$$+ \frac{1}{2} \left( \frac{1}{g^*_R} - \frac{1}{g^*_R} \right) \left( \frac{\sigma_R}{\mu} \right)^2 \frac{2 - \omega}{2} C_1 D_1 \left[ \left( \frac{\bar{M}}{M} \right)^{2\omega} - 1 \right]$$

$$\left[ C_1 \left( \frac{\bar{M}}{\mu} \right)^{-\omega} + D_1 \left( \frac{\bar{M}}{\mu} \right)^{\omega} \right]^2.$$  \hspace{1cm} (7.6)

Eq.(2.40) and Eq.(7.1) lead to the relation between $M$ and the field $\sigma_R$:

$$\frac{\sigma_R}{\mu} = \left[ C_1 + D_1 \left( \frac{\bar{M}}{\mu} \right)^{2\omega} \right] \left( \frac{M}{\mu} \right)^{2-\omega}. \hspace{1cm} (7.7)$$

Note that $\bar{M}$ is a certain constant but not a field like $M$.

We choose the parameters $g^*_R$ and $\bar{g}^*_R$:

$$\frac{1}{g^*_R - \bar{g}^*_R} = \frac{1}{\omega} + \text{regular function of } \omega, \hspace{1cm} (7.8)$$

so as to take a sensible $\omega \to 0 (\alpha \to \alpha_c)$ limit. Hereafter we take the simplest choice:

$$g^*_R = g^* = \frac{(1 + \omega)^2}{4}, \hspace{1cm} \bar{g}^*_R = \bar{g}^* = \frac{(1 - \omega)^2}{4}. \hspace{1cm} (7.9)$$

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Such a choice of $g_R^e$ and $\tilde{g}_R^e$ leads to the following effective potential at $\omega = 0$ ($\alpha = \alpha_c$):

\[
-\frac{4\pi^2}{N} \frac{V_R(\sigma_R, \pi_R = 0)}{\mu^4} = \frac{1}{g_R^e} \frac{m_R \sigma_R}{\mu^2} + \frac{1}{2} \left( \frac{1}{g_R^e} - \frac{1}{g_R^e} \right) \left( \sigma_R \right)^2 - 4 \left( \frac{\sigma_R}{\mu} \right)^2 \frac{3}{4} + \delta_0 + \ln \frac{\mu M}{M^2} \left[ 1 + \delta_0 - \ln \frac{M}{\mu} \right].
\] (7.10)

The stationary condition of the effective potential Eq.(7.6) yields the (renormalized) gap equation. Solving the gap equation in the chiral symmetric limit $m_R = 0$, we obtain the scaling relation of the dynamical mass of the fermion $M_d \equiv \langle M \rangle$:

\[
\left( \frac{M_d}{M} \right)^{2\omega} = -\frac{D_1}{C_1} \left( \frac{M}{\mu} \right)^{2\omega} - \frac{1}{g_R^e} \left[ \frac{1}{g_R^e} - \frac{1}{g_R^e} \right] \frac{1}{g_R^e} \left[ C_1 \left( \frac{M}{\mu} \right)^{-\omega} + D_1 \left( \frac{M}{\mu} \right)^{\omega}\right]^2.
\] (7.11)

The critical NJL coupling $g_R^{\text{crit}}$ is determined from Eq.(7.11) at $M_d = 0$:

\[
\frac{1}{g_R^{\text{crit}}} = \frac{1}{g_R^e} \left[ \frac{1}{g_R^e} - \frac{1}{g_R^e} \right] \frac{D_1^2}{C_1} \left( \frac{M}{\mu} \right)^{2\omega}.
\] (7.12)

Note here that the critical coupling can be calculated as

\[
g_R^{\text{crit}} = \begin{cases} 
    g_R^e, & \text{for } M/\mu = 0 \\
    \tilde{g}_R^e, & \text{for } M/\mu \to \infty.
\end{cases}
\] (7.13)

If we take $\bar{M} = M_d$ (a natural choice in the S$\chi$SB phase) in Eq.(7.11), we find:

\[
\left( \frac{M}{\mu} \right)^{2\omega} = D_1 \left( \frac{M_d}{\mu} \right)^{2\omega} = -\frac{1}{g_R^e} \left[ \frac{1}{g_R^e} - \frac{1}{g_R^e} \right] \frac{D_1}{C_1} \left( \frac{M_d}{\mu} \right)^{2\omega}.
\] (7.14)

Now, we are ready to discuss the RG properties. It is straightforward to derive the
\[ \beta_g(g_R, \alpha; \bar{M}/\mu) = -2\omega g_R^2 \left( \frac{1 - g_R}{g_R^*} \right) C_1^2 \left( \frac{\bar{M}}{\mu} \right)^{-2\omega} - \left( \frac{1 - g_R}{g_R^*} \right) D_1^2 \left( \frac{\bar{M}}{\mu} \right)^{2\omega}, \quad (7.15) \]

and the anomalous dimension \( \gamma_m \) from Eq. (7.5):
\[ \gamma_m(g_R, \alpha; \bar{M}/\mu) = 1 + \omega \left( \frac{2 g_R^* - 1}{g_R^*} \right) C_1^2 \left( \frac{\bar{M}}{\mu} \right)^{-2\omega} - \left( \frac{2 g_R^* - 1}{g_R^*} \right) D_1^2 \left( \frac{\bar{M}}{\mu} \right)^{2\omega} \left[ C_1 \left( \frac{\bar{M}}{\mu} \right)^{-\omega} + D_1 \left( \frac{\bar{M}}{\mu} \right)^{\omega} \right]^2. \quad (7.16) \]

Putting \( \bar{M} = 0 \) in Eq. (7.15) and Eq. (7.16), we find (Fig. 12)
\[ \beta_g(g_R, \alpha; M = 0) = 2\omega g_R \left( 1 - \frac{g_R}{g_R^*} \right), \quad (7.17) \]
\[ \gamma_m(g_R, \alpha; M = 0) = 1 - \omega + 2\omega \frac{g_R}{g_R^*}, \quad (7.18) \]
which coincide with Eqs. (5.21–5.22) as they should.

Plugging Eq. (7.14) into Eq. (7.15) and Eq. (7.16), on the other hand, we obtain (Fig. 12)
\[ \beta_g(g_R, \alpha; \bar{M} = M_d) = -2(g_R - \bar{g}_R^*)(g_R - g_R^*), \quad (7.19) \]
\[ \gamma_m(g_R, \alpha; \bar{M} = M_d) = 2g_R + \frac{\alpha}{2\alpha_c}, \quad (7.20) \]
which remain valid even at \( \alpha = \alpha_c \) and coincide with the form of the RG functions of bare parameters, Eq. (C.18) and Eq. (C.23), given in Appendix C.

Finally, we make a brief comment on the relation of the symmetric renormalization and the \( \bar{M} \)-dependent renormalization. From Eq. (5.13) and Eq. (7.1), we obtain
\[ \sigma_R = \tilde{Z}_\sigma \sigma_R, \quad \tilde{Z}_\sigma^{-1} \equiv 1 + \frac{D_1}{C_1} \left( \frac{\bar{M}}{\mu} \right)^{2\omega}. \quad (7.21) \]
in the \( \Lambda \to \infty \) limit. Using Eq. (5.17) and Eq. (7.4), we obtain a relation between the renormalized four-fermion coupling of the symmetric renormalization and that of the
\( \tilde{M} \)-dependent one:

\[
\frac{1}{g^*} - \frac{1}{g_R} = \tilde{Z}_\sigma^{-2} \left( \frac{1}{g^*} - \frac{1}{g_R} \right) + \frac{D^2}{C^2_1} \left( \frac{\tilde{M}}{\mu} \right)^{4\omega} \left( \frac{1}{\tilde{g}^*} - \frac{1}{g^*} \right),
\]

(7.22)

where we have used \( g^* = g_R^* = \tilde{g}^* = \tilde{g}_R^* \). It is easy to show that the current mass in the \( \tilde{M} \)-dependent renormalization scheme is expressed by

\[
m_R = \tilde{Z}_m m_R, \quad \tilde{Z}_m = \tilde{Z}_\sigma^{-1} \frac{g_R}{g_R},
\]

(7.23)

in the \( \Lambda \to \infty \) limit. Thus, the symmetric renormalization and the \( \tilde{M} \)-dependent renormalization are connected by a finite renormalization unless \( \omega = 0 \).

The OPE coefficient functions in \( \tilde{M} \)-dependent renormalization \( \tilde{c}_1' \) and \( \tilde{c}_{\bar{\psi}\psi} \) are calculated from

\[
\tilde{c}_1'(p; g_R, \alpha, \tilde{M}/\mu; \mu) = \tilde{Z}_m c_1'(p; g_R, \alpha; \mu), \quad (7.24a)
\]

\[
\tilde{c}_{\bar{\psi}\psi}(p; g_R, \alpha, \tilde{M}/\mu; \mu) = \tilde{Z}_m^{-1} c_{\bar{\psi}\psi}(p; g_R, \alpha; \mu). \quad (7.24b)
\]

By using Eq.(6.10), Eq.(6.27) and Eq.(6.30), we find

\[
\tilde{c}_1'(g_R, \alpha, \tilde{M}/\mu; \mu) = \frac{1}{p^2} \frac{2}{2\omega} \Gamma^{(R)}_S (-p^2; M_d) \frac{g_R}{g^* - 1 + \tilde{Z}_\sigma^2 D^2}{C^2\tilde{M}/\mu}^{4\omega} \left( \frac{1}{\tilde{g}^*} - \frac{1}{g^*} \right),
\]

(7.25a)

\[
\tilde{c}_{\bar{\psi}\psi}(g_R, \alpha, \tilde{M}/\mu; \mu) = -\frac{1}{p^2} \frac{4\pi^2 g_R}{N\mu^2} \Gamma^{(R)}_S (-p^2; M_d)
\]

(7.25b)

for the S\( \chi \)SB vacuum, and

\[
\tilde{c}_1'(g_R, \alpha, \tilde{M}/\mu; \mu) = \frac{1}{p^2} \Gamma^{(R)}_S (-p^2; M = 0) \frac{1}{1 - \frac{g_R}{g^*} - \tilde{Z}_\sigma^2 D^2}{C^2\tilde{M}/\mu}^{4\omega} \left( \frac{1}{\tilde{g}^*} - \frac{1}{g^*} \right),
\]

(7.26a)

\[
\tilde{c}_{\bar{\psi}\psi}(g_R, \alpha, \tilde{M}/\mu; \mu) = -\frac{1}{p^2} \frac{4\pi^2 g_R}{N\mu^2} \Gamma^{(R)}_S (-p^2; M = 0)
\]

(7.26b)
for the symmetric vacuum, where we have used

$$\Gamma^{(R)}_{S}(-p^2; M) = Z_{S}^{-1}\Gamma^{(R)}_{S}(-p^2; M). \quad (7.27)$$

We notice that

$$\bar{Z}_{\sigma} \rightarrow \frac{1}{-2\omega \ln \frac{M}{\mu} + \mathcal{O}(\omega^2)} \quad (7.28)$$

as $\omega \rightarrow 0$. Therefore $\bar{c}_{\bar{\psi}\bar{\psi}}$ on $S\chi_{SB}$ vacuum and $\bar{c}'_{1l}$ remain finite in $\omega \rightarrow 0$ limit, while $\bar{c}_{\bar{\psi}\bar{\psi}}$ in the symmetric vacuum diverges.

Special attention should be paid to the $\omega \rightarrow 0$ limit where two renormalization schemes are not connected by a finite renormalization. The above calculation implies that the OPE coefficient function $\bar{c}_{\bar{\psi}\bar{\psi}}$ cannot be made finite in a vacuum-independent manner. Such a peculiar phenomenon may be an artifact coming from the infrared singularity in the symmetric vacuum at $\alpha = \alpha_c$.

8 Conclusion and Discussion

We have presented renormalization of the simplest gauged NJL model, gauge theories with standing gauge coupling plus four-fermion interaction, in the ladder approximation. Through the CJT effective potential written in terms of the auxiliary fields, we have established, in an analogous manner to NJL$_{D<4}$, a phase-independent renormalization (symmetric renormalization) valid both for the symmetric and the $S\chi_{SB}$ phases as far as $\alpha \neq \alpha_c$. Accordingly, the $\beta$ function and the anomalous dimension were obtained phase-independently in both phases: The theory has a nontrivial UV fixed line and a large anomalous dimension there. The OPE was explicitly constructed, which is consistent with the large anomalous dimension in both phases. The Wilson coefficient for the unit operator has an extra power behavior other than the anomalous dimension. The symmetric renormalization done on the symmetric vacuum breaks down at the end point $\alpha = \alpha_c$ of the critical line, while the $M$-dependent renormalization still remains valid.

The reason why the renormalization is possible in NJL$_{D<4}$ and gauged NJL model is very simple. In NJL$_{D<4}$ the four-fermion operators become relevant/marginal: $\dim\left((\bar{\psi}\psi)^2\right) = 2\dim(\bar{\psi}\psi) = 2(D - 1 - \gamma_m) = 2 \leq D$, while other fermion operators become irrelevant: $\dim\left((\bar{\psi}\psi)^4\right) = \dim\left(\partial_\mu(\bar{\psi}\psi)\partial^\mu(\bar{\psi}\psi)\right) = 4 > D$, etc. for $2 \leq D < 4$. In the same way, the four-fermion operators in the gauged NJL model
become relevant/marginal: \( \dim \left( (\bar{\psi}\psi)^2 \right) = 2(3 - \gamma_m) = 2(2 - \sqrt{1 - \alpha/\alpha_c}) \leq 4, \)
while other operators become irrelevant: \( \dim \left( (\bar{\psi}\psi)^4 \right) = 4(2 - \sqrt{1 - \alpha/\alpha_c}) > 4, \)
\( \dim \left( \partial_\mu(\bar{\psi}\psi)\partial_\nu(\bar{\psi}\psi) \right) = 2(3 - \sqrt{1 - \alpha/\alpha_c}) > 4, \) etc. for \( \alpha_c \geq \alpha > 0.8. \)

At this point it should be emphasized that our renormalization breaks down at the pure NJL limit \( \alpha \to 0. \) In that limit our renormalized effective potential Eq.(5.19) has a singularity of \( 1/\alpha, \) a signal of the appearance of the logarithmic divergence for the four-point vertex of the local auxiliary fields. Such a divergence can only be removed by introduction of eight-fermion operator ("\( \lambda \phi^4 \) counter term") which now becomes a marginal operator; \( \dim \left( (\bar{\psi}\psi)^4 \right) = 4(2 - \sqrt{1 - \alpha/\alpha_c}) \to 4 \) at \( \alpha \to 0. \) The \( 1/\alpha \) singularity also appears in the auxiliary field propagator Eq.(5.20), another signal of the logarithmic divergence in the induced kinetic term of the auxiliary field. This again can only be removed by introduction of derivative-type four-fermion operator ("counter term for kinetic term \( \partial_\mu\phi\partial_\nu\phi \)") which also becomes marginal; \( \dim \left( \partial_\mu(\bar{\psi}\psi)\partial_\nu(\bar{\psi}\psi) \right) = 2(3 - \sqrt{1 - \alpha/\alpha_c}) \to 4 \) at \( \alpha \to 0. \) Without such extra “higher dimensional” operators, the pure NJL model in four dimensions cannot be renormalized even in the nonperturbative sense of \( 1/N \) expansion. The presence of gauge interaction turns these “higher dimensional” operators into irrelevant ones and hence make the renormalization possible without such additional “counter terms”.\[22, 15, 16\] Similar comments also apply to the NJL\( _{D<4} \): The above \( 1/\alpha \) singularity corresponds to the \( 1/(D - 4) \) singularity in Eq.(5.8) and Eq.(5.12) of NJL\( _{D<4}. \)

Such a formal resemblance between the renormalization of NJL\( _{D<4} \) and that of the gauged NJL model may not be a mere accident. As is well known, the renormalizable NJL\( _{D<4} \) model is equivalent to the Yukawa model with the Yukawa coupling lying on the nontrivial IR fixed point in the \( 1/N \) leading order. Actually, the \( \beta \) function of dimensionless Yukawa coupling \( y \) is given by

\[
\beta(y) = \left( \frac{D}{2} - 2 \right) y + 4N \frac{\Gamma(3 - D/2) (\Gamma(D/2))^2}{(4\pi)^D/2\Gamma(D - 1)} y^3, \tag{8.1}
\]

\[8\]These higher dimensional operators can be explicitly shown to be suppressed in part by the power of the cutoff in the low energy physics through renormalization.\[42\]

\[9\]The \( 1/\alpha \) singularity instead of the logarithmic divergence is an artifact of the inadequate approximation taking only the first two dominant terms in the solution of SD equation for \( \alpha > 0. \) Taking account of the third dominant term, we can recover the correct NJL limit\[14, 28\].
which indeed has a nontrivial infrared (IR) fixed point

\[ y^2 = \frac{(4\pi)^{D/2}\Gamma(D-1)}{4N\Gamma(2-D/2)\left(\Gamma(D/2)\right)^2}, \tag{8.2} \]

corresponding to the NJL\(_{D<4}\) model. Although two parameters (Yukawa coupling and scalar boson mass) are needed in order for the Yukawa model to be renormalizable, the number of parameters is now reduced by this constraint to that of NJL\(_{D<4}\). This is the essence of the renormalizability of NJL\(_{D<4}\). This argument obviously breaks down at \( D = 4 \) where the nontrivial IR fixed point disappears. This situation reflects the fact that the NJL model is not renormalizable in four dimensions as an interacting field theory even in \( 1/N \) expansion.

However, in the presence of gauge interaction (\( \alpha \neq 0 \)) the Yukawa model shows a similar structure to Eq.(8.1) even in four dimensions:

\[ \beta_y(\alpha, y) = -\frac{6C_F\alpha}{4\pi} y + \frac{N}{8\pi^2} y^3, \tag{8.3} \]

where \( C_F \) is the quadratic Casimir of the fermion representation and we have ignored the graphs containing scalar particle loop. Now, the \( \beta \) function of gauge coupling may be parameterized as

\[ \beta_\alpha(\alpha, y) = -\frac{3C_F/\pi}{A} \alpha^2, \tag{8.4} \]

where \( A(= 18C_F/(11N - 2N_f)) \) for \( N_f \)-flavored \( SU(N) \) gauge theory) is the measure of the running speed of the gauge coupling, i.e., large \( A(\gg 1) \) means “walking” (slowly running) gauge coupling and \( A \rightarrow \infty \) corresponds to the non-running (“standing”) case studied in this paper. For \( A > 1 \) the above \( \beta \) functions lead to the RG invariant IR stable subspace of the couplings:

\[ y^2 = \frac{12\pi C_F}{N} \frac{A - 1}{A} \alpha, \tag{8.5} \]

in analogue to Eq.(8.2). This is another expression of a suggestion[16, 44] that the presence of gauge interactions with \( A > 1 \) may make the theory renormalizable. The explicit renormalization procedure in the present paper is actually a concrete example of this suggestion in the special case \( A \rightarrow \infty \). Such a possibility in the case with running/walking gauge coupling \( A < \infty \) will be further studied in the subsequent
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\section{NJL_{D<4} model}

In this appendix, we derive the effective potential and the auxiliary field propagators in NJL_{D<4} model.

The auxiliary field technique enables us to rewrite the original lagrangian Eq.(5.1) into an equivalent one:

\[ \mathcal{L} = \bar{\psi} i \partial \psi - \bar{\psi} (\sigma + \pi i \gamma_5) \psi - V_{(cl)}(\sigma, \pi), \]  

(A.1)

where classical potential of auxiliary fields $\sigma, \pi$ is given by

\[ V_{(cl)} = \frac{N}{G} \left[ \frac{1}{2} (\sigma^2 + \pi^2) - m_0 \sigma \right]. \]  

(A.2)

The effective action is evaluated solely from the fermion integral within the $1/N$ leading approximation:

\[ \Gamma[\sigma, \pi] = -i \ln \det \left( i \partial - \sigma - i \gamma_5 \pi \right) - \int d^D x V_{(cl)}(\sigma, \pi). \]  

(A.3)

The effective potential $V$ is calculated from the effective action

\[ V(\sigma, \pi) \equiv -\frac{\Gamma[\sigma = \text{const}, \pi = \text{const}]}{\Omega}, \]  

(A.4)

where $\Omega$ is the space-time volume. Thus, we obtain

\[ V(\sigma, \pi) = V_{(cl)}(\sigma, \pi) + V_{(qu)}(\sigma, \pi), \]  

(A.5)

where

\[ V_{(qu)}(\sigma, \pi) \equiv -2N \int_{\Lambda}^{\Lambda} \frac{d^D p}{(2\pi)^{D/2}} \ln \left( 1 + \frac{\sigma^2 + \pi^2}{-p^2} \right), \]  

(A.6)

\[ = -\frac{2N}{(4\pi)^{D/2} \Gamma(D/2)} \int_0^\Lambda \frac{dp_E^2 (p_E^2)^{D/2-1}}{p_E^2} \ln \left( 1 + \frac{\sigma^2 + \pi^2}{p_E^2} \right), \]  

(A.7)

with the momentum integral being regularized by the ultraviolet cutoff $\Lambda$. Using the
following formula
\[ \int_0^\infty dtt^{-1} \left[ \ln \left( 1 + \frac{1}{t} \right) - \frac{1}{t} \right] = -\frac{1}{z} B(z - 1, 2 - z), \]
(A.8)

it is rather straightforward to integrate Eq.(A.7):
\[
\frac{(4\pi)^{D/2} \Gamma(D/2)}{4N} V_{(qu)}(\sigma, \pi)
= -\frac{\Lambda^{D-2} \sigma^2 + \pi^2}{g^*} \frac{\zeta_D}{2} \frac{(\sigma^2 + \pi^2)^{D/2}}{2 - D/2} \frac{\Lambda^D}{D} - \sum_{n=0}^{\infty} \frac{\Lambda^D(-1)^n n!}{(D - 4 - 2n)(n + 2)} \left( \frac{\sigma^2 + \pi^2}{\Lambda^2} \right)^{n+2}, \quad (A.9)
\]
with \( g^* \equiv D/2 - 1 \) and \( \zeta_D \equiv B(D/2 - 1, 3 - D/2) \). Finally, we find the effective potential
\[
-\frac{(4\pi)^{D/2} \Gamma(D/2)}{4N\Lambda^D} V(\sigma, \pi)
= \frac{m_0\sigma}{g} - \frac{1}{g^*} \frac{(\sigma^2 + \pi^2)}{2\Lambda^2} - \frac{\zeta_D}{2 - D/2} \left( \frac{\sigma^2 + \pi^2}{\Lambda^2} \right)^{D/2} + O \left( \left( \frac{\sigma^2 + \pi^2}{\Lambda^2} \right)^2 \right), \quad (A.10)
\]
with the dimensionless four-fermion coupling \( g \) being defined by
\[
g \equiv \frac{4\Lambda^{D-2}}{(4\pi)^{D/2} \Gamma(D/2)} G. \quad (A.11)
\]
It is easy to see that the last term \( \sim O(\Lambda^{D-4}) \) in Eq.(A.10) becomes negligible for sufficiently large \( \Lambda \) and will be neglected in the following calculation of the propagators of auxiliary fields. Note that \( g^* \to 0 \) and \( \zeta_D \to \infty \) as \( D \to 2 \), so that the divergences in the second and third term cancel each other to give a well-known logarithmic potential in the Gross-Neveu model [37]:
\[
-\frac{\pi}{N} V(\sigma, \pi) = \frac{1}{g} m_0\sigma - \frac{1}{g} \frac{\sigma^2 + \pi^2}{2} + \frac{\sigma^2 + \pi^2}{2} \left( 1 - \ln \left( \frac{\sigma^2 + \pi^2}{\Lambda^2} \right) \right) + \Lambda^2 O \left( \left( \frac{\sigma^2 + \pi^2}{\Lambda^2} \right)^2 \right). \quad (A.12)
\]
Let us now consider solution of the gap equation $\partial V/\partial \sigma = 0$ for Eq.(A.10):

$$0 = \frac{1}{g} \frac{m_0}{\Lambda} + \left( \frac{1}{g^*} - \frac{1}{g} \right) \frac{\sigma}{\Lambda} - \frac{\zeta_D}{2} \frac{D}{2} \left( \frac{\sigma}{\Lambda} \right)^{D-1} + \cdots.$$  \hspace{1cm} (A.13)

It is convenient to parameterize the solution of the gap equation $\sigma_{\text{sol}}$ as:

$$\sigma_{\text{sol}} = \sigma_{\text{spont}} + \sigma_{\text{explicit}},$$  \hspace{1cm} (A.14)

with $\sigma_{\text{spont}}$ being the solution in the chiral limit ($m_0 = 0$):

$$\frac{\zeta_D}{2 - D/2} \sigma_{\text{spont}}^{D-2} = \begin{cases} 
\left( \frac{1}{g^*} - \frac{1}{g} \right) \Lambda^{D-2} + \cdots & \text{for S\chiSB vacuum } (g > g^*) \\
0 & \text{for symmetric vacuum}
\end{cases}.$$  \hspace{1cm} (A.15)

Actually it is easy to see from Eq.(A.10) that the symmetric vacuum $\langle \sigma \rangle = \langle \pi \rangle = 0$ becomes unstable in the strong coupling region $g > g^*$. In this vacuum the fermion acquires the dynamical mass

$$\Sigma_{\text{dyn}}(-p^2) = \Gamma_S(p, q = 0)\sigma_{\text{spont}} = \sigma_{\text{spont}},$$  \hspace{1cm} (A.16)

where $\Gamma_S(p, q)$ is the Yukawa-type vertex defined in an analogous way to Eq.(2.10) and calculated as

$$\Gamma_S(p, q) = 1.$$  \hspace{1cm} (A.17)

The nonvanishing bare fermion mass ($m_0 \neq 0$) causes non-zero $\sigma_{\text{explicit}}$:

$$\sigma_{\text{explicit}} = \sigma_{\text{sol}} - \sigma_{\text{spont}} = \frac{\partial \sigma_{\text{sol}}}{\partial m_0} \bigg|_{m_0=0} m_0 + \cdots.$$  \hspace{1cm} (A.18)
We can evaluate $\sigma_{\text{explicit}}$ as a Taylor series from Eq.(A.13) in $m_0$:

$$
\sigma_{\text{explicit}} = \begin{cases} 
\frac{1}{D-2} \frac{m_0}{g* - 1} + \cdots & \text{for S}\chi\text{SB vacuum} \\
\frac{m_0}{1 - g/g*} + \cdots & \text{for symmetric vacuum} 
\end{cases}.
$$

(A.19)

Hence the current mass of the fermion may be written as

$$
\Sigma_{\text{explicit}}(-p^2) = \Gamma_S(p,q = 0)\sigma_{\text{explicit}} = \sigma_{\text{explicit}}.
$$

(A.20)

Thus, the current mass $\Sigma_{\text{explicit}}$ has the same (constant) high energy behavior as that of the dynamical mass in the NJL$_{D<4}$ model.

The auxiliary field propagators $D_{\sigma\sigma}^{-1}$ and $D_{\pi\pi}^{-1}$ are given by the second derivative of the effective action:

$$
iD_{\sigma\sigma}^{-1}(-q^2) = \mathcal{F} \mathcal{T} \frac{\delta^2 \Gamma[\sigma, \pi]}{\delta \sigma(x) \delta \sigma(0)} \bigg|_{\sigma = \sigma_{\text{sol}}, \pi = 0},
$$

(A.21a)

$$
iD_{\pi\pi}^{-1}(-q^2) = \mathcal{F} \mathcal{T} \frac{\delta^2 \Gamma[\sigma, \pi]}{\delta \pi(x) \delta \pi(0)} \bigg|_{\sigma = \sigma_{\text{sol}}, \pi = 0}.
$$

(A.21b)

They are evaluated as

$$
iD_{\sigma\sigma}^{-1}(-q^2) = -\int \frac{d^Dk}{(2\pi)^D} \text{tr} \left[ \frac{i}{k - \sigma_{\text{sol}}} \frac{i}{\bar{k} - \bar{\sigma}_{\text{sol}}} \right] - \frac{\partial^2 V_{(\text{cl})}(\sigma, \pi)}{\partial \sigma^2} \bigg|_{\sigma = \sigma_{\text{sol}}, \pi = 0},
$$

(A.22a)

$$
iD_{\pi\pi}^{-1}(-q^2) = -\int \frac{d^Dk}{(2\pi)^D} \text{tr} \left[ \gamma_5 \frac{i}{k - \sigma_{\text{sol}}} \gamma_5 \frac{i}{\bar{k} - \bar{\sigma}_{\text{sol}}} \right] - \frac{\partial^2 V_{(\text{cl})}(\sigma, \pi)}{\partial \pi^2} \bigg|_{\sigma = \sigma_{\text{sol}}, \pi = 0}.
$$

(A.22b)

The subtraction at zero momentum makes these expressions finite in the $\Lambda \to \infty$ limit.\textsuperscript{10}

$$
iD_{\sigma\sigma}^{-1}(-q^2) - iD_{\sigma\sigma}^{-1}(0)
$$

\textsuperscript{10}Thus, we do not need to worry about keeping the regularization chiral invariant for the auxiliary field propagator with non-vanishing momentum $q^2 \neq 0$. 

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The second derivative of the effective potential gives the auxiliary field propagator at zero momentum:

\[
\begin{align*}
  i D_{\sigma\sigma}^{-1}(-q^2) &= -\frac{\partial^2}{\partial\sigma^2} V(\sigma, \pi) \bigg|_{\sigma=\sigma_{sol}, \pi=0}, \\
  i D_{\pi\pi}^{-1}(-q^2) &= -\frac{\partial^2}{\partial\pi^2} V(\sigma, \pi) \bigg|_{\sigma=\sigma_{sol}, \pi=0}.
\end{align*}
\]

For the symmetric vacuum \(\sigma_{sol} = 0\) we find

\[
i D_{\sigma\sigma}^{-1}(-q^2) = i D_{\pi\pi}^{-1}(-q^2) = -\frac{4N}{(4\pi)^{D/2}} \frac{\Lambda^{D-2}}{\Gamma(D/2)} \left( \frac{1}{g} - \frac{1}{g^*} \right).
\]

Combining Eq.(A.23) with Eq.(A.25), we find

\[
\begin{align*}
  -\frac{(4\pi)^{D/2}}{4N} \Gamma(D/2) & i D_{\sigma\sigma}^{-1}(-q^2) = \left( \frac{1}{g} - \frac{1}{g^*} \right) \Lambda^{D-2} + \frac{\xi_D}{2 - D/2} (-q^2)^{D-2-1}, \\
  -\frac{(4\pi)^{D/2}}{4N} \Gamma(D/2) & i D_{\pi\pi}^{-1}(-q^2) = \left( \frac{1}{g} - \frac{1}{g^*} \right) \Lambda^{D-2} + \frac{\xi_D}{2 - D/2} (-q^2)^{D-2-1},
\end{align*}
\]

with \(\xi_D\) defined by

\[
\xi_D \equiv \frac{B(3 - D/2, D/2 - 1)}{\Gamma(D - 1)}.
\]

For the S\(\chi\)SB vacuum \(\sigma_{sol} \neq 0\) in the strong coupling region \(g > g^*\), the auxiliary field propagators at zero-momentum are evaluated as

\[
\begin{align*}
  i D_{\sigma\sigma}^{-1}(-q^2) &= -\frac{8N}{(4\pi)^{D/2}} \Gamma(2 - D/2) \sigma_{sol}^{D-2}, \\
  i D_{\pi\pi}^{-1}(-q^2) &= 0.
\end{align*}
\]
which lead to

\[-\frac{(4\pi)^{D/2}}{4N}\Gamma(D/2)iD_{\sigma}^{-1}(-q^2) = 2 \left( \frac{\sigma^2_{\text{sol}} - q^2}{4} \right) \int_0^1 dx \left[ \sigma^2_{\text{sol}} - x(1 - x)q^2 \right]^{D/2-2}, (A.29a)\]

\[-\frac{(4\pi)^{D/2}}{4N}\Gamma(D/2)iD_{\pi\pi}^{-1}(-q^2) = 2 \left( -\frac{q^2}{4} \right) \int_0^1 dx \left[ \sigma^2_{\text{sol}} - x(1 - x)q^2 \right]^{D/2-2}. (A.29b)\]

Thus, the auxiliary field $\sigma$ acquires a pole at $q^2 = 4\sigma^2_{\text{sol}} = 4m^2$, while $\pi$ becomes massless NG boson. Note that $m_\sigma = 2m$ is independent of $D$.

**B OPE in NJL$_{D<4}$ model**

Let us consider the time ordered bilocal operator $T[\psi(x)\bar{\psi}(0)]$. The OPE for it reads

\[-i\mathcal{F}\mathcal{T}\left[\psi(x)\bar{\psi}(0)\right] = c_1(p; g_R, m_R, \mu) + c_{\bar{\psi}\psi}(p; g_R, m_R, \mu) \left[ (\bar{\psi}\psi)_R + \gamma_5(\bar{\psi}\gamma_5\psi)_R \right] + \cdots, (B.1)\]

It is useful to expand $c_1$ by $m_R$:

\[c_1(p; g_R, m_R, \mu) = \frac{\not{p}}{p^2} + m_R c_1'(p; g_R; \mu) + \cdots. (B.2)\]

Corresponding to Eq.(6.6), we have

\[\Sigma_{\text{explicit}}(-p^2) = p^2 m_R(\mu)c_1'(p; g_R; \mu) + \cdots. (B.3)\]

From Eq.(A.20) and Eq.(A.19) we obtain

\[\Sigma_{\text{explicit}}(-p^2) = \begin{cases} 
\frac{1}{D - 2} \frac{Z_m m_R}{g^* - 1} + \cdots & \text{for S}\chi\text{SB vacuum} \\
\frac{Z_m m_R}{1 - \frac{g}{g^*}} + \cdots & \text{for symmetric vacuum}
\end{cases}, (B.4)\]

where $Z_m \equiv m_0/m_R$ is given by Eq.(5.7)

\[Z_m \equiv m_0/m_R = \frac{g}{g_R} \left( \frac{\mu}{\Lambda} \right)^{D-2}. (B.5)\]
Eq. (B.5) and Eq. (5.6) lead to
\[ \frac{Z_m}{1 - \frac{g}{g^*}} = \frac{1}{1 - \frac{g_R}{g_R^*}}. \] (B.6)

Comparing Eq. (B.3) with Eq. (B.4) and Eq. (B.6), we obtain
\[
1 - \frac{g}{g^*} = 1 - \frac{g_R}{g_R^*}
\]
(B.7)

Note that the Wilson coefficient \( c_1' \) in the \( S\chi SB \) vacuum remains finite in two dimensions, while \( c_1' \) calculated in the symmetric vacuum vanishes in \( D \to 2 \). This is consistent with the one phase structure in two dimensions.

Let us next consider the coefficient function of \( \bar{\psi}\psi \), \( c_{\bar{\psi}\psi} \). This can be calculated from the fermion four-point function by taking \( x \to 0 \) limit:
\[
-i \langle T[\bar{\psi}(x)\bar{\psi}(0)\psi(y)\bar{\psi}(z)] \rangle_{\text{connected}} = c_{\bar{\psi}\psi}(x) \langle T(\bar{\psi}\psi)_R(0)\psi(y)\bar{\psi}(z) \rangle_{\text{connected}} + \cdots. \] (B.8)

The Wilson coefficients of the unit operator do not appear in the RHS of Eq. (B.8), since they only contribute to disconnected diagrams.

In the following calculation \( x, y, z \) are Fourier transformed to \( p, q, k \), respectively. The calculation at \( q = k \) is sufficient to determine \( c_{\bar{\psi}\psi} \). The LHS of Eq. (B.8) is evaluated:
\[
S(p)i\Gamma_S(-p^2)S(p)\frac{1}{-V''(\sigma_{\text{sol}})}i\Gamma_S(-q^2) = \frac{1}{p^2 - V''(\sigma_{\text{sol}})}\Gamma_S(-q^2) + \cdots, \] (B.9)

where the legs for \( q, k \) are amputated and the scalar vertex is defined by \( \Gamma_S \equiv 1 \).

The RHS of Eq. (B.8) can be evaluated by Fig. 11 which reads
\[
Z_m c_{\bar{\psi}\psi}(p; g_R, m_R, \mu) \left[ 1 - iV''(\sigma_{\text{sol}})D_{\sigma\sigma}(0) \right] \Gamma_S(-q^2)
\]

11 This agrees with Ref. [23] except for the factor \( 1/(D - 2) \) in the \( S\chi SB \) vacuum.
\[ \frac{1}{V''_{(cl)}} = \frac{G}{N} = \frac{(4\pi)^{D/2}\Gamma(D/2)}{4N} \frac{g}{\Lambda^{D-2}}. \]

\section*{C Renormalization Group Functions from the Solution of Ladder SD Gap Equation}

Let us consider the RG of \textit{bare} parameters which was first discussed by Miransky \cite{8} in the analysis of the ladder SD equation of dynamical mass of fermion in the quenched QED. In this argument the bare parameters of the theory are required to depend on the cutoff \( \Lambda \) so as to fix the fermion dynamical mass \( M_d \), with such a flow of bare parameters being identified as the RG evolution.

Solving the ladder SD gap equation of the gauged NJL model, Kondo, Mino and Yamawaki \cite{12} and independently Appelquist, Soldate, Takeuchi and Wijewardhana \cite{13} obtained the critical line

\[ g = g^* \equiv \frac{1}{4}(1 + \omega)^2, \quad \omega = \sqrt{1 - \alpha/\alpha_c}, \]  

(B.11)
and the scaling relation near the critical line:

\[
\frac{M_d}{\Lambda} \sim \left\{ \begin{array}{ll}
\frac{1}{\sqrt{\Lambda}} \left( \frac{1}{g^*} - 1 \right) & (0 < \alpha < \alpha_c) \\
\exp \left[ 1 + \delta_0 \left( \frac{1}{g^*} - 1 \right) \right] & (\alpha = \alpha_c)
\end{array} \right. \\
\exp \left[ \delta \left( \frac{n\pi + \tan^{-1} \omega' + \tan^{-1} \left( \frac{\omega'/2}{g - (1 - \omega^2)/4} \right)}{\omega'} \right) \right] & (\alpha > \alpha_c)
\right\}
\]

with \( n = 1 \) being the ground state solution, where \( g^* \) is defined by \( g^* \equiv (1 - \omega^2)/4 \).

The usual SD equation [11] is obtained from the stationary condition of \( V[\Sigma, \Sigma_5] \) Eq.(2.24):

\[ 0 = \frac{\delta}{\delta \Sigma(p^2_E)} V[\Sigma, \Sigma_5], \quad (C.3) \]

\[ 0 = \frac{\delta}{\delta \Sigma_5(p^2_E)} V[\Sigma, \Sigma_5], \quad (C.4) \]

which read

\[
\Sigma(p^2_E) = m_0 + g \frac{1}{\Lambda^2} \int_0^\Lambda dp^2_E \frac{p^2_E \Sigma(p^2_E)}{p^2_E + \Sigma^2 + \Sigma_5^2} + \int_0^\Lambda dk^2_E \frac{k^2_E \Sigma(k^2_E)}{k^2_E + \Sigma^2 + \Sigma_5^2} K(p^2_E, k^2_E), \quad (C.5a)
\]

\[
\Sigma_5(p^2_E) = \frac{g}{\Lambda^2} \int_0^\Lambda dp^2_E \frac{p^2_E \Sigma(p^2_E)}{p^2_E + \Sigma^2 + \Sigma_5^2} + \int_0^\Lambda dk^2_E \frac{k^2_E \Sigma_5(k^2_E)}{k^2_E + \Sigma^2 + \Sigma_5^2} K(p^2_E, k^2_E). \quad (C.5b)
\]

Since \( \Sigma_5 \) can be rotated away by the chiral symmetry, it is sufficient to consider the SD equation

\[
\Sigma(p^2_E) = m_0 + g \frac{1}{\Lambda^2} \int_0^\Lambda dp^2_E \frac{p^2_E \Sigma(p^2_E)}{p^2_E + \Sigma^2} + \int_0^\Lambda dk^2_E \frac{k^2_E \Sigma(k^2_E)}{k^2_E + \Sigma^2} K(p^2_E, k^2_E), \quad (C.6)
\]
which is equivalent to the differential equation
\[
\left[ p_E^2 \left( \frac{d}{dp_E^2} \right)^2 + 2 \frac{d}{dp_E^2} + \frac{3C_F}{4\pi} \frac{\alpha}{p_E^2 + \Sigma(p_E^2)} \right] \Sigma(p_E^2) = 0, \tag{C.7}
\]
with IR BC:
\[
\lim_{p_E^2 \to 0} p_E^4 \frac{d}{dp_E^2} \Sigma(p_E^2) = 0, \tag{C.8}
\]
and UV BC:
\[
\left[ 1 + \left( 1 + \frac{g}{3C_F\alpha/4\pi} \right) p_E^2 \frac{d}{dp_E^2} \right] \Sigma(p_E^2) \bigg|_{p_E^2 = \Lambda^2} = m_0. \tag{C.9}
\]
Irrespective of presence or absence of the explicit fermion mass \(m_0\), the solution of this differential equation exhibits the same asymptotic behavior in the high energy region:
\[
\Sigma(p_E^2) = M \left[ c_1 \left( \frac{p_E^2}{M^2} \right)^{(1-\omega)/2} + d_1 \left( \frac{p_E^2}{M^2} \right)^{-(1+\omega)/2} + \mathcal{O}\left( \left( \frac{p_E^2}{M^2} \right)^{-3(1-\omega)/2-1} \right) \right], \tag{C.10}
\]
where this time \(M\) is not a field but a constant having the dimension of mass and gives the mass scale of the solution. For details on the behavior of this solution, see section 2.

First of all, we calculate the mass renormalization constant \(Z_m \text{ á la Miransky} [8]\). From the UV BC, the equation of state is obtained:
\[
\frac{m_0}{\Lambda} = - \left( \frac{M}{\Lambda} \right)^2 \left[ \frac{g - g^*}{g^*} C_1 \left( \frac{M}{\Lambda} \right)^{-\omega} + \frac{g - \tilde{g}^*}{g^*} D_1 \left( \frac{M}{\Lambda} \right)^{\omega} \right]. \tag{C.11}
\]
Then we obtain
\[
\frac{\partial m_0}{\partial M} = - \left( \frac{M}{\Lambda} \right) \left[ (2 - \omega) \frac{g - g^*}{g^*} C_1 \left( \frac{M}{\Lambda} \right)^{-\omega} + (2 + \omega) \frac{g - \tilde{g}^*}{g^*} D_1 \left( \frac{M}{\Lambda} \right)^{\omega} \right]. \tag{C.12}
\]
In the chiral limit \(m_0 = 0\), the dynamical fermion mass \(M_d\) obeys the scaling law Eq.(C.2)
\[
\frac{M_d}{\Lambda} = \left[ - \frac{\tilde{g}^* C_1}{g^* D_1 \left( g - \tilde{g}^* \right)^{\frac{1}{2}}}, \quad (0 < \alpha < \alpha_c) \right]. \tag{C.13}
\]
Then the renormalization constant in the region $0 < \alpha < \alpha_c$ is obtained as

$$Z_m \equiv \frac{\partial m_0}{\partial M} |_{M=M_d} = 2 \frac{M}{\Lambda} \sqrt{-\frac{\omega^2 C_1 D_1}{\tilde{g}^* g^*(g - \tilde{g}^*)(g - g^*)},} \quad (C.14)$$

Thus we obtain the mass renormalization constant:

$$Z_m = \begin{cases} 
2 A \sqrt{\frac{(g - \tilde{g}^*)(g - g^*)}{1 - \omega^2}} \frac{M_d}{\Lambda} & (0 < \alpha < \alpha_c) \\
2 A_0 (g - \frac{1}{4}) \frac{M_d}{\Lambda} & (\alpha = \alpha_c) \\
2 A' \sqrt{\frac{(g - \bar{g}^c)(g - g^c)}{1 + \omega'^2}} \frac{M_d}{\Lambda} & (\alpha > \alpha_c)
\end{cases} \quad (C.15)$$

where $g^c \equiv (1 + i\omega')^2 / 4$ and $\tilde{g}^c \equiv (1 - i\omega')^2 / 4$.

Since the bare chiral condensation $\langle \bar{\psi}\psi \rangle$ is given by

$$\langle \bar{\psi}\psi \rangle \equiv \frac{N}{4\pi^2} \int_0^\Lambda^2 dp_x^2 \frac{p_x^2 \Sigma(p_x^2)}{p_x^2 + \Sigma^2(p_x^2)}$$

$$= \begin{cases} 
- \frac{N A}{4\pi^2 \sqrt{1 - \omega^2}} \frac{\Lambda M_d^2}{[(g - \tilde{g}^*)(g - g^*)]^{1/2}} & (0 < \alpha < \alpha_c) \\
- \frac{N A_0 \Lambda M_d^2}{4\pi^2 (g - 1/4)} & (\alpha = \alpha_c) \\
- \frac{N A'}{4\pi^2 \sqrt{1 + \omega'^2}} \frac{\Lambda M_d^2}{[(g - \bar{g}^c)(g - g^c)]^{1/2}} & (\alpha > \alpha_c)
\end{cases} \quad (C.16)$$

the renormalized condensate $\langle (\bar{\psi}\psi)_R \rangle$ near the critical line is calculated as

$$\langle (\bar{\psi}\psi)_R \rangle = Z_m \langle \bar{\psi}\psi \rangle = \begin{cases} 
- \frac{N A^2}{2\pi^2 (1 - \omega^2)} M_d^3 & (0 < \alpha < \alpha_c) \\
- \frac{N A_0^2}{2\pi^2} M_d^3 & (\alpha = \alpha_c) \\
- \frac{N A'^2}{2\pi^2 (1 - \omega'^2)} M_d^3 & (\alpha > \alpha_c)
\end{cases} \quad (C.17)$$
This shows that $(\langle \bar{\psi} \psi \rangle_R )$ is $g$-independent and the RG flow can be identified with the fixed-$\alpha$ line (upward direction)[12, 14, 15].

Once the RG flow is so identified, the $\beta$ function of bare four-fermion coupling was explicitly calculated from Eq.(C.2) in the gauged NJL model [20, 21]:

$$\beta_g(g, \alpha) \equiv \Lambda \frac{\partial g}{\partial \Lambda} \mid_{\alpha,M_d} = -2(g - \tilde{g}^*)(g - g^*), \quad (g > g^*).$$

(C.18)

The anomalous dimension is obtained [10]:

$$\gamma_m(g, \alpha) \equiv -\Lambda \frac{\partial \ln Z_m}{\partial \Lambda} \mid_{\alpha,M_d} = -\Lambda \frac{\partial \ln m_0}{\partial \Lambda} \mid_{\alpha,M_d} = 1 + \omega, \quad (g = g^*).$$

(C.19)

Now we calculate the anomalous dimension above the critical line $g > g^*$ from the solution of the SD equation. The scaling law leads to

$$g - \tilde{g}^* = \frac{1}{1 - F}(g^* - \tilde{g}^*), \quad g - g^* = \frac{F}{1 - F}(g^* - \tilde{g}^*),$$

(C.20)

with $F$ being defined by

$$F \equiv -\frac{g^* D_1}{\tilde{g}^* C_1} \left( \frac{M_d}{\Lambda} \right)^{2\omega}.$$

Then we get

$$\frac{\partial \ln(g - \tilde{g}^*)}{\partial \ln \Lambda} = -\frac{\Lambda}{M_d} \frac{\partial \ln(1 - F)}{\partial(\frac{\Lambda}{M_d})} = -2\omega \frac{F}{1 - F} = -2\omega \frac{g - g^*}{g^* - \tilde{g}^*} = -2(g - g^*).$$

(C.21)

Similarly, we get

$$\frac{\partial \ln(g - g^*)}{\partial \ln \Lambda} = \frac{\Lambda}{M_d} \frac{\partial[\ln F - \ln(1 - F)]}{\partial(\frac{\Lambda}{M_d})} = -2\omega - 2\omega \frac{F}{1 - F} = -2\omega - 2(g - g^*).$$

(C.22)

Accordingly, we obtain the anomalous dimension:

$$\gamma_m \equiv -\frac{\partial \ln Z_m}{\partial \ln \Lambda} = 1 - \frac{1}{2} \frac{\partial}{\partial \ln \Lambda} [\ln(g - \tilde{g}^*) + \ln(g - g^*)]$$

$$= 1 + \omega + 2(g - g^*) = 2g + \frac{\alpha}{2\alpha_c}.$$ 

(C.23)

This is shown to be valid also in the region $\alpha > \alpha_c$ and coincides with the anomalous dimension numerically obtained in the gauged NJL-model with running gauge coupling.
D Renormalization Group of Bare Parameters

In the previous appendix, we have defined and calculated the RG function of bare parameters. It is evident, however, that such a definition of RG function is applicable only in the $S\chi SB$ vacuum of strong coupling phase $g > g^*$ where fermion acquires non-vanishing dynamical mass.

Thus, we define here the RG of the bare parameters by using the effective potential expressed in terms of mass scale of the fermion $M$. This enables us to obtain the RG functions also in the symmetric phase.

We first consider the case of $m_0 = 0$. The effect of explicit chiral symmetry breaking term will be discussed later. In section 2, we have obtained the effective potential:

$$-\frac{8\pi^2}{N} \frac{V(M)}{\Lambda^4} = \left(\frac{1}{g^*} - \frac{1}{g}\right) C_1 \left[C_1 \left(\frac{M}{\Lambda}\right)^{4-2\omega} + \frac{2 + \omega}{2} D_1 \left(\frac{M}{\Lambda}\right)^4\right] + \left(\frac{1}{\tilde{g}^*} - \frac{1}{g}\right) D_1 \left[D_1 \left(\frac{M}{\Lambda}\right)^{4+2\omega} + \frac{2 + \omega}{2} C_1 \left(\frac{M}{\Lambda}\right)^4\right]. \quad (D.1)$$

Actually, the stationary condition of Eq.(D.1) $\partial V/\partial M = 0$ has a nontrivial solution $M = M_d \neq 0$ for strong coupling region $g > g^*$ and leads to the correct scaling relation:

$$\left(\frac{M_d}{\Lambda}\right)^{2\omega} = -\frac{C_1}{D_1} \frac{1}{\tilde{g}^* - \frac{1}{g}}. \quad (D.2)$$

Let us define the RG flow of the bare four-fermion coupling $g$ so as to make Eq.(D.1) independent of $\Lambda$: \footnote{It is evident, however, that the effective potential Eq.(D.1) cannot be made $\Lambda$-independent for all the region of $M$. Thus, we need to specify the value of $M$ to define the RG flow of bare parameters. The plausible choice of $M$ is the value of the stationary condition. In this sense, the RG flow of the bare parameters depends on the choice of vacuum.}

$$0 = -\frac{8\pi^2}{N\Lambda^3} \frac{dV(M)}{d\Lambda} = 2\omega \left\{ \left(\frac{1}{g^*} - \frac{1}{g}\right) C_1^2 \left(\frac{M}{\Lambda}\right)^{4-2\omega} - \left(\frac{1}{\tilde{g}^*} - \frac{1}{g}\right) D_1^2 \left(\frac{M}{\Lambda}\right)^{4+2\omega} \right\}$$
\[ + \beta_g \left( \frac{M}{\Lambda} \right)^4 \left[ C_1 \left( \frac{M}{\Lambda} \right)^{-\omega} + D_1 \left( \frac{M}{\Lambda} \right)^{\omega} \right]^2, \]  
(D.3)

where \( \beta_g \) is defined by \( \beta_g(g, \alpha) = \Lambda \frac{\partial g}{\partial \alpha} \). Namely, the \( \beta \) function reads

\[
\beta_g(g, \alpha) = -2\omega g^2 \frac{1}{g^* - g} \left[ C_1 \left( \frac{M}{\Lambda} \right)^{-\omega} + D_1 \left( \frac{M}{\Lambda} \right)^{\omega} \right]^2. \quad \text{(D.4)}
\]

We first consider the symmetric vacuum \( M = 0 \). In this case, Eq.(D.4) becomes

\[
\beta_g(g, \alpha) = 2\omega g \left( 1 - \frac{g}{g^*} \right), \quad \text{(D.5)}
\]

which corresponds to the phase-independent RG evolution of the renormalized four-fermion coupling discussed in section 5.

Another choice of \( M \) is the solution of the gap equation Eq.(D.2) in the strong coupling region \( g > g^* \), which gives

\[
\beta_g(g, \alpha) = -2(g - g^*)(g - g^*). \quad \text{(D.6)}
\]

Eq.(D.6) agrees with the original definition of the RG flow of the bare parameters Eq.(C.18).

These two results Eq.(D.5) and Eq.(D.6) give the same result for the region \( g \simeq g^* \) where \( M \) is sufficiently smaller than cutoff \( \Lambda \). On the other hand, the deviation becomes significant when the four-fermion coupling \( g \) is far beyond the critical line \( g^* \) and \( M/\Lambda \) in Eq.(D.2) is not negligible.

For the anomalous dimension \( \gamma_m \), we need to evaluate the explicit chiral symmetry breaking term proportional to the bare mass of fermion \( m_0 \) in the effective potential:

\[
V_{\text{explicit}}(M) \equiv -\frac{N \Lambda^2 m_0 \sigma}{4\pi^2 g}. \quad \text{(D.7)}
\]
The result is:

\[-\frac{8\pi^2 V_{\text{explicit}}(M)}{N\Lambda^4} = \frac{2m_0}{\Lambda g} \left[ C_1 \left( \frac{M}{\Lambda} \right)^{2-\omega} + D_1 \left( \frac{M}{\Lambda} \right)^{2+\omega} \right]. \tag{D.8} \]

The anomalous dimension \(\gamma_{m_0} = -\Lambda \partial m_0 / \partial \Lambda\) can be obtained in the same way as the \(\beta\) function; \(\frac{\partial V_{\text{explicit}}}{\partial \Lambda} = 0\). We find

\[
\gamma_m(g, \alpha) = 1 + \omega \left[ \left( \frac{2g}{g^*} - 1 \right) C_1^2 \left( \frac{M}{\Lambda} \right)^{-2\omega} - \left( \frac{2g}{g^*} - 1 \right) D_1^2 \left( \frac{M}{\Lambda} \right)^{2\omega} \right. \\
\left. \left[ C_1 \left( \frac{M}{\Lambda} \right)^{-\omega} + D_1 \left( \frac{M}{\Lambda} \right)^{\omega} \right]^2 \right]. \tag{D.9} \]

Again, the definition of the anomalous dimension \(\gamma_m\) depends on the choice of \(M\). The symmetric vacuum \(M = 0\) gives the anomalous dimension

\[
\gamma_m(g, \alpha) = 1 - \omega + 2\omega g g^*/g^* \tag{D.10} \]

which takes the same form as Eq.(5.22), the anomalous dimension for the renormalized coupling in the symmetric renormalization. It should be noted that the anomalous dimension \(\gamma_m\) is continuous across the critical line \(g = g^*\). On the other hand, using the value of \(M\) in \(S\chi SB\) vacuum Eq.(D.2), we find

\[
\gamma_m(g, \alpha) = 2g + \frac{\alpha}{2\alpha_c}. \tag{D.11} \]

Eq.(D.11) for \(g > g^*\) can also be derived by use of the SD gap equation, as shown in Appendix C. Eq.(D.10) and Eq.(D.11) give the same anomalous dimension Eq.(C.19) at \(g = g^*\).
References


[22] K. Yamawaki, the second reference of Ref.3.


Figure Captions

1. Critical line in \((\alpha, g)\) plane. It separates the spontaneously broken phase \((S\chi_{SB})\) and the unbroken phase \((\chi_{\text{Sym}})\) of the chiral symmetry.

2. The lowest order diagram in the two-particle irreducible part \(\kappa^{2\text{PI}}[S]\) of the CJT potential of the gauged NJL model written in terms of auxiliary field Eq.(2.2). The wavy line and the solid line with shaded blob represent the bare gauge boson propagator \(D_{\mu\nu}\) and the full fermion propagator \(S\), respectively.

3. The lowest order diagram in the two-particle irreducible part of the CJT potential of the gauged NJL model (without auxiliary fields), Eq.(2.1).

4. Momentum assignment of Yukawa type vertex \(\Gamma_S(p, q)\). Dashed line represents auxiliary field propagator.

5. Amputated multi-point Green function at zero momentum of \(\sigma\). External fermion lines are amputated. The shaded blob stands for the induced \(\sigma\) vertex \((V^{(n)}(\sigma))\).

6. Auxiliary field propagator. The solid line with shaded blob represents the full fermion propagator.

7. The second derivative of the auxiliary field propagator. A slash represents once derivative with respect to \(q_\mu\).

8. Self-consistent equation for the Yukawa-type vertex \(\Gamma_S(p, q)\).

9. Another form for the second derivative of the auxiliary field propagator.

10. Four-point fermion Green function: (a) \(\sigma\)-exchange diagram, (b) pure ladder diagram.

11. \((\bar{\psi}\psi)\) inserted Green function.

12. RG functions in the symmetric and the \(\bar{M}\)-dependent renormalizations: (a) \(\beta\) function, (b) anomalous dimension. Solid line is for the symmetric renormalization \((\bar{M} = 0)\), while dotted line is for \(\bar{M} = M_d\).