Damping rates for moving particles
in hot $QCD$

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Abstract

Using a program of perturbative resummation I compute the damping rates for fields at nonzero spatial momentum to leading order in weak coupling in hot $QCD$. Sum rules for spectral densities are used to simplify the calculations. For massless fields the damping rate has an apparent logarithmic divergence in the infrared limit, which is cut off by the screening of static magnetic fields (“magnetic mass”). This demonstrates how at high temperature even perturbative quantities are sensitive to nonperturbative phenomenon.
I. Introduction

For an asymptotically free theory such as QCD, at high temperature perturbation theory is a reasonable first approximation. Even if initially there are no bare masses, in an interacting plasma mass scales small relative to the temperature $T$ are generated radiatively. With $g$ the QCD coupling constant, elementary diagramatic techniques show that quasiparticles acquire thermal “masses” of order $gT$ at one loop order. For example, both time dependent electric and magnetic fields are screened by a thermal gluon mass $m_g \sim gT$, as are static electric fields. Static magnetic fields are not screened to this order, since the plasma is one of (colored) electric charge. It is expected that static magnetic fields are screened nonperturbatively by a “magnetic mass” [1].

The thermal masses are related to the real part of the pole in the quasiparticle propagator. The imaginary part of the pole is proportional to the damping rate, $\gamma$, and determines how rapidly a system near equilibrium approaches it. Unlike the screening lengths, which are easy to compute, even to lowest order in $g$ the damping rates can only be computed consistently after the resummation of an infinite set of diagrams, termed hard thermal loops [2-6]. While infinite, the entire series of hard thermal loops can be succinctly expressed in terms of simple effective actions [7]. This resummation program has been applied to a variety of problems [3-5, 8-16].

The damping rates are inevitably of order $g^2 T$, but they depend in an interesting fashion on how fast the quasiparticle is moving through the thermal medium [2]. As a consequence of Landau damping, resummation produces an effective gluon that propagates below the light cone, with damping dominated by scattering off of spacelike effective gluons. If the incident field is initially at rest, both the recoil field and the effective gluon carry nonzero energy and momentum. While the calculations are involved, the effective gluon only probes energies and momenta of order $gT$, and so the damping rate is just some pure number times $g^2 T$ [4,5,8,9].

If the initial field is in motion, however, the effective gluon can be emitted at ninety degrees relative to the incident (and final) direction. When the field is moving sufficiently fast, the effective gluon carries almost zero energy and spatial momentum, and yet still contributes to the damping rate. In this way, transverse effective gluons probe the static magnetic sector, and so are sensitive to the presence of a magnetic mass. For example, in an $SU(N)$ color gauge theory the damping rate for a gluon
moving with momenta of order $T$ is shown in sec. IV to be

$$\gamma_t = \frac{g^2 N T}{8\pi} \left( \ln \left( \frac{m_q^2}{m_{mag}^2 + 2m_{mag}\gamma_t} \right) + 1.09681... \right); \quad (1.1)$$

for technical reasons this expression only holds in the limit where $m_{mag} \gg \gamma_t$.

Over large distances the behavior of static magnetic fields is controlled by a purely bosonic effective gauge theory in three dimensions, with a (dimensional) coupling constant $= g^2 T$. Assuming that bosonic $QCD$ in three dimensions has a mass gap, the magnetic mass must be a number times $g^2 T$, $m_{mag} \sim g^2 T$, up to a possible factor of $\sqrt{\ln(1/g^2)}$ [13]. Thus the damping rate in (1.1) is of order $\gamma_t \sim g^2 T \ln(1/g^2)$. It was first shown in ref. [2] that factors of $\ln(1/g^2)$ are generic to damping rates at nonzero velocity: here I consider more carefully how the logarithm is cut off, and try to evaluate the constant under the logarithm. Thus while (1.1) holds for $m_{mag} \gg \gamma_t$, unless $m_{mag}$ is some very large number times $g^2 T$, this limit probably does not apply: more likely, $m_{mag} \leq \gamma_t$. I present this result to demonstrate that it is possible, at least in principle, to compute the constant under the logarithm.

The formula in (1.1) does not apply to hot $QED$. In hot $QED$ the behavior of static magnetic fields is determined by bosonic $QED$ in three dimensions. This is a free theory, so $m_{mag}^{(1)} = 0$, and the damping rate for fast fermions, analogous to (1.1), is logarithmically divergent. (The damping rate for fast photons is not very interesting: it is finite and of order $e^3 T$.) I suspect that the damping rate for fast fermions in hot $QED$ is finite and of order $e^2 T \ln(1/e)$, but this requires separate analysis beyond that presented here. Amusingly, as suggested originally in ref. [2], the calculation in hot $QED$ is more difficult than that for hot $QCD$, at least when $m_{mag} \gg \gamma_t$ [16].

Notice that the damping rate appears in the argument of the logarithm. Lebedev and Smilga [14] first pointed out that it is necessary to include $\gamma_t$ self consistently, which is how it enters into the right hand side of (1.1). Their calculations indicated that $\gamma_t$ alone suffices to cut off the logarithmic divergence, even if $m_{mag} = 0$. Recently, Baier, Nakkagawa, and Niegawa [16] argued that while the damping rate must be included self consistently, that if $\gamma_t$ is determined from the position of the singularity in the propagator, then without the magnetic mass, by itself $\gamma_t$ does not cut off the logarithmic divergence. My result in (1.1) accords with their arguments. This happens because the analytic structure of the propagator is rather complicated, with an unexpected branch cut appearing off the physical sheet, near the pole in the
quasiparticle propagator. The pole has an imaginary part $\gamma_t$; the branch cut begins at a point that is separated by an amount $m_{mag}$ from this pole. The restriction that $m_{mag} \gg \gamma_t$ arises because it is much simpler to treat the case when the branch cut and the pole are far from each other than if they are close.

Linde first argued that a magnetic mass renders the free energy sensitive to non-perturbative effects at four loop order [1]. The above shows that for fast fields the damping rates are sensitive to such effects even at leading order. I confess that while I introduce the magnetic mass in a plausible fashion, it is at best a caricature of nonperturbative effects. Moreover, the appearance of the magnetic mass does not imply that calculations are fruitless: perhaps a marriage of perturbation theory (as in (1.1)) and lattice gauge theory (to determine $m_{mag}$) can be arranged.

In sec. II I discuss I derive some necessary sum rules and thereby introduce the magnetic mass. In sec. III the damping rate for a slow, heavy fermion is computed. For kinematic reasons this damping rate does not probe very small momenta, of order $g^2T$, and so is insensitive to the magnetic mass or to the details of analytic continuation. The damping rate for fast, massless quarks and gluons is computed in sec. IV. In sec. V the Ward identities are used to compute the leading logarithmic dependence in the damping rates of quarks and gluons traveling with momenta greater than order $g^2T$. An appendix discusses how the the sum rules of sec. II can be used to compute the term of order $g^3$ in the free energy.

For calculational ease all damping rates are computed in Coulomb gauge. I appeal to general proofs of gauge invariance in refs. [3] and [17] to establish that this result is independent of the choice of gauge. Recently, Baier, Kunstatter, and Schiff [18] observed that naive calculation in covariant gauges appear to violate these general proofs. The dilemma was resolved by Rebhan [19], who demonstrated that an infrared regulator is required to treat the mass shell singularities which arise in covariant gauges; see, also, refs. [5], [20], and [18]. I avoid these delicacies by sticking with Coulomb gauge, but note that explicit calculations in other gauges may well require the introduction of infrared regulators.

II. Sum rules

The conventions and notation of ref. [3] are followed. For an $SU(N)$ gauge theory with $N_f$ flavors of massless quarks in the fundamental representation, the effective
gluon mass induced by the thermal medium is

\[ m_g^2 = \left( N + \frac{N_f}{2} \right) g^2 T^2 \left( \frac{9}{2} \right) . \] (2.1)

In Coulomb gauge the only nonzero components of the gluon propagator \( \ast \Delta^{\mu \nu} \) are \( \ast \Delta^{00}(K) = \ast \Delta_t(K) \) and \( \ast \Delta^{ij}(K) = (\delta^{ij} - \hat{k}^i \hat{k}^j) \ast \Delta_t(K) \). Here the gluon four momentum is \( K = (k^0, \vec{k}) \) and \( \hat{k} = \vec{k}/k \); analytic continuation to real energies, with \( k^0 = i\omega \), is implicit. When the momentum is soft, with \( \omega \) and \( k \) of order \( m_g \), the effective plasmon and transverse gluon propagators are given by the tree term, plus the corresponding hard thermal loop \([21,22]\)

\[ \ast \Delta_t^{-1}(K) = k^2 - 3 m_g^2 Q_1 \left( \frac{ik^0}{k} \right) , \] (2.2)

\[ \ast \Delta_t^{-1}(K) = K^2 - \frac{3}{5} m_g^2 \left( Q_3 \left( \frac{ik^0}{k} \right) - Q_1 \left( \frac{ik^0}{k} \right) - \frac{5}{3} \right) . \] (2.3)

The \( Q_n \) are Legendre functions of the second kind.

The propagator determines the spectral densities of the effective fields. For the transverse field,

\[ \ast \Delta_t(k^0, k) = \int_0^{1/T} d\tau e^{ik^0 \tau} \int_{-\infty}^{+\infty} d\omega \ast \rho_t(\omega, k) \left( 1 + n(\omega) \right) e^{-\omega \tau} , \] (2.4)

where \( n(\omega) = 1/(e^\omega - 1) \) is the Bose–Einstein statistical distribution function. The transverse spectral density is given by

\[ \ast \rho_t(\omega, k) = Im \ast \Delta_t(-i\omega + 0^+, k)/\pi , \] (2.5)

and is a sum of pole and cut terms,

\[ \ast \rho_t(\omega, k) = Z_t(k) \left( \delta(\omega - E^t_k) + \delta(\omega + E^t_k) \right) + \beta_t(\omega, k) \vartheta(k^2 - \omega^2) ; \] (2.6)

\( \vartheta(x) \) is the step function, \( \vartheta(x) = 1 \) for \( x > 0, = 0 \) for \( x < 0 \). The delta functions in (2.6) represent the propagation of transverse gluons as quasiparticles with energy \( \omega = E^t_k \) and residue \( Z_t(k) \). The spectral density also includes the contribution of a cut below the light cone, \( |\omega| \leq k \), with spectral weight \( \beta_t(\omega, k) \). This cut is the result of Landau damping in a thermal distribution. The plasmon spectral density, \( \ast \rho_p(\omega, k) \), is defined similarly from \( \ast \Delta_\ell(K) \), and determines the plasmon mass shell \( E^\ell_k \), residue
$Z_\ell(k)$, and the plasmon cut, $\beta_\ell(\omega, k)$. Complete expressions for these quantities are given in refs. [21] and [23]. For example, about zero momentum

$$
\rho_\ell(\omega, 0) = \left( -\frac{k^2}{m_g^2} \right) \rho_\ell(\omega, k) = \frac{1}{2m_g} \left( \delta(\omega - m_g) + \delta(\omega + m_g) \right).
$$

That is, at rest a transverse gluon and the plasmon are degenerate in energy, $E_\ell^0 = E_0^t = m_g$. The residue for the plasmon is a bit unusual — it is proportional to $-1/k^2$ — but this is innocuous [23]. The mass shells for transverse gluons and for the plasmon split away from zero momentum, and are determined numerically as the solution of transcendental equations.

For later purposes, note that about zero energy the contribution of the cut terms to the transverse and plasmon spectral densities are

$$
\beta_\ell(\omega, k) \sim_0 \frac{1}{\pi} \text{Im} \frac{1}{k^2 - 3\pi m_g^2 \omega i/(4k)},
$$

and

$$
\beta_\ell(\omega, k) \sim_0 \frac{1}{\pi} \text{Im} \frac{1}{k^2 + 3m_g^2 + 3\pi m_g^2 \omega i/(2k)}.
$$

The imaginary terms in each propagator are both due to Landau damping, proportional to $m_g^2 \omega/k$ as $\omega \to 0$. The real terms are given by limits of the propagators at zero frequency. Thus for the plasmon term $k^2 + 3m_g^2$ enters in the denominator, with $3m_g^2$ the static electric mass squared. For the transverse term only $k^2$ appears, since at this order static magnetic fields are not screened.

In sec. III only the expressions in (2.8) and (2.9) are required. In sec.’s IV and V we need integrals of the spectral densities with respect to powers of $\omega$. These integrals can be evaluated by means of sum rules [23,10,24].

The derivation of sum rules is an elementary exercise in complex analysis. While familiar, I go through several examples in order to emphasize the relevant physics. The essential point is to turn the integral over $\omega$ into a contour integral in the plane of complex $k^0$; to avoid confusion I relabel complex $k^0$ as $z$. For example,

$$
\int_{-\infty}^{+\infty} \omega \rho_\ell(\omega, k) d\omega = \frac{1}{2\pi i} \oint \Delta_\ell(z, k) dz.
$$

The contour $C$ runs counter clockwise around the imaginary $z$ axis. Since there are no intervening poles, the contour can then be deformed into a loop at infinity. For large $z$, the hard thermal loop $\delta \Pi(z, k)$ falls off as $m_g^2/z^2$, and the effective propagator
behaves as the bare one, \( \Delta_t(z,k) \sim 1/z^2 \). Thus (2.10) is the same as for free field theory,

\[
\int_{-\infty}^{+\infty} \omega \, \rho_t(\omega,k) \, d\omega = 1. \tag{2.11}
\]

At zero spatial momentum this sum rule is dominated by the pole terms in the spectral density, and is easily checked by using (2.7).

The sum rule in (2.11) is familiar as a consequence of the equal time commutation rules. It is only valid to lowest order in \( g^2 \), when the effective propagator includes just the hard thermal loop. For example, if the effective propagator included the full gluon self energy at one loop order, then the right hand side of (2.11) is modified by terms for the (ultraviolet divergent) wave function renormalization constant of the gluon. This is distinct from the finite renormalization constant, \( Z_t(k) \), above.

Two other sum rules are needed for what follows. One is a relation for the plasmon spectral density:

\[
\int_{-\infty}^{+\infty} \frac{d\omega}{\omega} \, \rho_t(\omega,k) = -\frac{1}{2\pi i} \oint_{C-\mathcal{O}} \frac{1}{z} \Delta_t(z,k) \, dz
\]

\[
= -\frac{1}{k^2} + \Delta_t(0,k) = -\frac{1}{k^2} + \frac{1}{k^2 + 3m_g^2}. \tag{2.12}
\]

The contour in the complex plane is now \( C - \mathcal{O} \), where \( \mathcal{O} \) is a circle about the origin. The modification of contour is required because of the factor of \( 1/z \) in the integrand: this factor generates a pole in \( z \) whose contribution must be included. The result on the right hand side is a sum of two terms. The first results from deforming \( C \) into the circle at infinity. As for the transverse density, the contribution from the hard thermal loop vanishes at large \( z \), and so the integral over \( C \) gives the same result as in free field theory, \(-1/k^2\). Secondly, there is the contribution from \( \mathcal{O} \); there the residue of the integrand at \( z = 0 \) is just the value of the plasmon propagator at zero frequency, \( \Delta_t(0,k) = 1/(k^2 + 3m_g^2) \). About zero momentum this sum rule is dominated by the pole terms, (2.7).

The last sum rule required is superficially similar to that for the plasmon density:

\[
\int_{-\infty}^{+\infty} \frac{d\omega}{\omega} \, \rho_t(\omega,k) = -\frac{1}{2\pi i} \oint_{C-\mathcal{O}} \frac{1}{z} \Delta_t(z,k) \, dz
\]

\[
= \Delta_t(0,k) \equiv \frac{1}{k^2 + m_{mag}^2}. \tag{2.13}
\]
The contour at infinity does not contribute because $^*\Delta_t(z,k)$ falls off as $1/z^2$ at large $z$. For the contour about the origin, $O$, the residue of the integrand is equal to the value of the transverse propagator at zero frequency, $^*\Delta_t(0,k)$.

Using the effective propagator of (2.3), which includes just the hard thermal loop, (2.13) equals $^*\Delta_t(0,k) = 1/k^2$. Unlike the two previous sum rules, about zero spatial momentum (2.13) is dominated not by the pole term, (2.7), but by the cut term in the spectral density, (2.8) [23].

In (2.13) I extend this relation, and introduce the magnetic mass, by defining the magnetic mass as the position of a presumed pole in the static transverse propagator, $^*\Delta_t(0,k) = 1/(k^2 + m_{mag}^2)$. This is merely a crude parametrization of the complicated physics which is responsible for the dynamical generation of a finite correlation length for static magnetic fields. As written, the magnetic mass represents the effects of a single glueball; surely there is a entire tower of glueball states, none of which need show up simply as a pole in the transverse propagator. A better approach would be to relate the quantities which enter into the damping rate to the vacuum expectation values of gauge invariant operators, which could then be computed by lattice gauge theory. At present this noble goal is beyond my means.

In the calculations of the hard damping rate, kinematics typically restricts the integral over the gluon spectral densities to lie below the light cone. For example, for the sum rule of (2.13),

$$
\int_{-k}^{+k} \frac{d\omega}{\omega} ^*\rho_t(\omega,k) = \frac{1}{k^2 + m_{mag}^2} - \frac{2}{E_k} Z_t(k) .
$$

That is, the sum rule allows one to exchange an integral over the cut in the spectral density, $^*\beta_t(\omega,k)$, for a function of $E_k$ and $Z_t(k)$.

To incorporate $m_{mag}$, the limiting form of the spectral density in (2.8) becomes

$$
^*\beta_t(\omega,k) \sim_0 \frac{1}{\pi} m \frac{1}{k^2 + m_{mag}^2 - 3\pi m_{mag}^2 \omega i/(4k)} ,
$$

Using just (2.15), the integral $\int_{-k}^{+k} d\omega \rho_t(\omega,k)/\omega \sim 1/(k^2 + m_{mag}^2)$. For $k \sim m_{mag}$ this is the dominant term on the right hand side of (2.14); the pole term contributes $\sim 1/m_{mag}^2$, which is smaller by order $g^2$. This shows how the sum rule in (2.13) is dominated by the cut term at small momenta. When $k \gg m_{mag}$ the magnetic mass is negligible, and we recover the full sum rule.
III. Damping rate for a slow, heavy fermion

For a heavy fermion of momentum $P$ and mass $M$, the bare inverse propagator is $\Delta_f^{-1}(P) = -i \slashed{P} + M$. To leading order the fermion self energy is

$$\Sigma_f(P) = -g^2 C_f \text{tr} \left( \gamma^\mu \gamma^\nu \gamma^\mu \Delta_f(P - K) \right).$$

(3.1)

For a fermion in the fundamental representation the Casimir constant $C_f = (N^2 - 1)/(2N)$; $\text{tr}$ represents the integral over the loop four momentum $K$. In (3.1) I have replaced the bare propagator by an effective propagator, $\ast \Delta_f$. This is defined as follows. For the bare propagator, the spectral density is

$$\rho_f(\omega, k) = (-\omega \gamma^0 + i \vec{k} \cdot \vec{\gamma} + M) \frac{1}{2 \omega} \left( \delta(\omega - E_k^M) - \delta(\omega + E_k^M) \right),$$

(3.2)

where $E_k^M = \sqrt{k^2 + M^2}$ is the fermion energy. To include the effects of damping I replace the sharp delta function in the spectral density by a Breit–Wigner form, with width $\gamma_f$:

$$\rho_f(\omega, k) = (-\omega \gamma^0 + i \vec{k} \cdot \vec{\gamma} + M) \frac{\gamma_f}{2\pi \omega_f} \left( \frac{1}{(\omega_f - E_k^M)^2 + \gamma_f^2} - \frac{1}{(\omega_f + E_k^M)^2 + \gamma_f^2} \right).$$

(3.3)

By the properties of the delta function this reduces to (3.2) as $\gamma_f \to 0$. The damping rate $\gamma_f$ turns out to be of order $g^2 T$; this is $g$ times the natural scale for the gluon spectral densities, which is set by the thermal gluon mass, $m_g \sim gT$. This inclusion of higher order effects in a hard propagator extends the program of resummation outlined in ref. [3]. I discuss later why it is valid to include these higher order effects, and not others, after computing $\gamma_f$. Lebedev and Smilga [14] were the first to introduce the damping rate in this way.

To simplify the computations I assume that the particle’s motion is nonrelativistic, with a velocity $v = \vec{p}/M \ll 1$. The case of relativistic motion is treated following the analysis of sec. IV. In (3.1) the Saclay method [3] is used to perform the sum over $k^0$. The damping rate is proportional to the the imaginary part of the self energy on the mass shell. This is a sum of two terms, from the plasmon and transverse spectral densities:

$$\text{Disc} \Sigma_f(iE_p^M, p) = 2i a_t (\gamma^0 - 1) - i a_t (\gamma^0 + 1),$$

(3.4)

where

$$a_{t,t} = \frac{g^2 \pi C_f T}{2} \int \frac{d^3 k}{(2\pi)^3} \int_{-\infty}^{+\infty} \frac{d\omega}{\omega} \int_{-\infty}^{+\infty} d\omega_f \frac{\gamma_f}{\omega_f^2 + \gamma_f^2} \ast \rho_{t,t}(\omega, k) \delta(\omega + \omega_f - E_p^M + E_{p-k}^M).$$

(3.5)
Several approximations have been made to reach (3.4) and (3.5). The term of leading order in \( g \) is given by replacing the Bose–Einstein statistical distribution function for the gluon by \( n(\omega) \sim T/\omega \). Further, in most instances the fermion spectral parameter, \( \omega_f \), can be replaced by its average value, equal to the energy on the mass shell. At nonrelativistic velocities this energy is just the mass: \( \omega_f \sim E_{p-k}^M \sim M \). The only instance where this is not allowed is in energy denominators: there, after taking the discontinuity of the self energy, one of the energy denominators produces the delta function for energy conservation in (3.5). (The other energy denominators don’t contribute, since the spectral density for the soft gluon only has support for \( \omega \) and \( k \) of order \( m_g \).) For that term the spectral parameter \( \omega_f \) is redefined as \( \omega_f \rightarrow \omega_f + E_{p-k}^M \).

Including the self energy, the renormalized fermion propagator is

\[
-i \mathcal{P} + M - \Sigma_f
\]

The pole in the renormalized propagator is shifted from the bare mass shell to \( (E_{p-k}^M + i\gamma_f, \vec{p}) \), where \( \gamma_f \) is the damping rate,

\[
\gamma_f = -2 a_t + v^2 a_t.
\]

For simplicity, assume that the velocity, while small, is larger than \( g, 1 \gg v \gg g \), so that \( E_{p-k}^M \sim M + (\vec{p} - \vec{k})^2/(2M) \sim M - pk\cos\theta/M \). The delta function for energy conservation is used to fix the angle between \( \vec{p} \) and \( \vec{k} \), \( \cos\theta = \frac{\omega + \omega_f}{\sqrt{\gamma_f^2 + \omega_f^2}} \).

\[
a_{\ell,t} = \frac{g^2 C_{f} T}{8 v \pi^2} \int_0^\infty k^2 dk \int_{-\infty}^{+\infty} d\omega \frac{\omega_f}{\omega^2} \int_{-\infty}^{+\infty} d\omega_f \frac{\gamma_f}{\omega_f^2 + \gamma_f^2} \* \rho_{\ell,t}(\omega, k) \vartheta(v k - |\omega + \omega_f|). \tag{3.7}
\]

The step function \( \vartheta \) enters to ensure that \( |\cos\theta| \leq 1 \). I assume that the constraint is satisfied separately by \( |\omega| \leq v k \) and \( |\omega_f| \leq v k \), so that the integrals over \( \omega_f \) and \( \omega \) decouple; this is justified following (3.17). The integral over \( \omega_f \) gives

\[
\int_{-v k}^{v k} d\omega_f \frac{\gamma_f}{\omega_f^2 + \gamma_f^2} = 2 \tan^{-1}\left(\frac{v k}{\gamma_f}\right). \tag{3.8}
\]

The integral over the plasmon spectral density in \( a_{\ell} \) is

\[
\int_{-v k}^{v k} d\omega \frac{\* \rho_{\ell}(\omega, k)}{\omega} \sim -\frac{3m_g^2 v}{(k^2 + 3m_g^2)^2}. \tag{3.9}
\]

Since \( v k \) in (3.9) is small relative to \( k \), the integral in (3.9) is just \( 2 v k \) times the limit of \( \* \rho_{\ell}(\omega, k)/\omega \) as \( \omega \to 0 \), (2.9). The remaining integral over \( k \) is finite, dominated by momenta \( k \sim m_g \sim g T \). Then I can take \( \gamma_f \sim 0 \) in (3.8), so that

\[
a_{\ell} = -\frac{g^2 C_{f} T}{16 \pi}. \tag{3.10}
\]
This term is negative, and so from (3.6) a positive contribution to the damping rate.
While various assumptions were made about the velocity to obtain (3.10), it is not
difficult to go back to (3.5) and show that one obtains identically the same result
even for a field at rest, (5) of ref. [2].

For the contribution of the transverse gluons, again since the integral runs only
from $-vk$ to $vk$, for small velocities only the limiting form in (2.8) is required; the
integral over it gives

$$
\int_{-vk}^{vk} \frac{d\omega}{\omega} \rho_t(\omega, k) = \frac{2}{\pi k^2} \tan^{-1}\left(\frac{3\pi v m_g^2}{4k^2}\right). \tag{3.11}
$$

The magnetic mass, as enters in (2.15), has been neglected; see the discussion fol-
lowing (3.17). Corrections to (3.11) are proportional to the velocity. After rescaling
$k \rightarrow \gamma_f k/v$, (3.5) becomes

$$
a_t = \frac{g^2 C_f T}{2\pi^3 v} \int_0^{\infty} \frac{dk}{k} \tan^{-1}(k) \tan^{-1}\left(\frac{c^2}{k^2}\right), \tag{3.12}
$$

where $c$ is a pure number,

$$
c^2 = \frac{3\pi^2 m_g^2 v^3}{4\gamma_f^2}. \tag{3.13}
$$

In this instance it is necessary to keep $\gamma_f \neq 0$, as otherwise (3.12) develops a loga-
rithmic divergence. Assume that the velocity lies in the range

$$
1 \gg v \gg g^{2/3}. \tag{3.14}
$$

Then the parameter $c$ is large, since with $m_g \sim gT$ and $\gamma_f \sim g^2 T$, $c^2 \sim m_g^2 v^3/\gamma_f^2 \sim v^3/g^2 \gg 1$. It is then straightforward to compute the integral in (3.12). The integrand
behave like $1/k$ only for $c \gg k \gg 1$ and is otherwise well behaved. Up to corrections
of order $\ln(c)/c$,

$$
a_t = \frac{g^2 C_f T}{16\pi v} \ln(c^2). \tag{3.15}
$$

Altogether, (3.6), (3.10), (3.13), and (3.15) give

$$
\gamma_f = \frac{g^2 C_f T}{8\pi} \left(1 + \frac{v}{2} \ln\left(\frac{3\pi m_g^2 v^3}{4\gamma_f^2}\right)\right). \tag{3.16}
$$

At small velocities, inside the logarithm I can replace $\gamma_f$ by its value at zero velocity
to obtain

$$
\gamma_f = \frac{g^2 C_f T}{8\pi} \left(1 + \frac{v}{2} \ln\left(\frac{16\pi^3 (N + N_f/2) v^3}{3C_f^2 g^2}\right)\right) + \ldots. \tag{3.17}
$$
The coefficient of the logarithm agrees with previous results [2, 16].

I now justify separating the constraints on \( \omega \) and \( \omega_f \) in (3.7). For the term involving the longitudinal spectral density there is no question; \( \gamma_f \) can be sent to zero at the outset. For the transverse spectral density, the integral over \( \omega \) is dominated by \( \omega \sim vk \); from the right hand side of (3.11), the relevant scale of momenta is \( k \sim \sqrt{vmg} \), so the dominant frequencies are \( \omega \sim \sqrt{v} g T \). For \( v \gg g^{2/3} \), then, \( \omega \gg g^2 T \), and so the scale for \( \omega \) is much greater than that for \( \omega_f \), which is \( \omega_f \sim \gamma_f \sim g^2 T \). This separation in scales produces the logarithm in (3.16) and (3.17), and justifies treating the constraints separately. This is not allowed if (3.14) is not obeyed. For smaller velocities, \( v \leq g^{2/3} \), the scales in \( \omega \) and \( \omega_f \) do mix; ultimately there will be no logarithm, with the term from the transverse spectral density vanishing smoothly as \( v \to 0 \). Larger velocities, \( v \gg 1 \), puts us in the relativistic regime, which is the subject of the next section.

Also, since the dominant momenta for the transverse density are \( k \sim \sqrt{vmg} \gg g^{1/3} m_g \sim m_{mag}/g^{2/3} \), it is permissible to neglect the magnetic mass, taking (2.8) for the limiting form of the spectral density instead of (2.15).

I have been somewhat careless on one other point. The correct damping rate is given by evaluating the imaginary part of the self energy at the position of the pole in the propagator. Including damping, this pole is off the physical sheet, at \( \omega_{pole} = E_p^M + i\gamma_f \). Instead, I evaluated the imaginary part at \( \omega = E_p^M \), and assumed that the continuation to \( \omega_{pole} \) is trivial. Baier, Nakagawa, and Niegawa [16] have recently argued this continuation can produce a nonzero contribution to the damping rate.

For velocities which satisfy (3.14), though, these subtleties can be overlooked. Suppose I were to evaluate the self energy not just for \( \omega = E_p^M \), but for for \( \omega = E_p^M + \delta E \), with \( \delta E \) of order \( g^2 T \). This alters energy conservation, so that in (3.7) \( |\omega| \) and \( |\omega_f| \) must be \( \leq vk + \delta E \). For \( \delta E \sim g^2 T \), however, this change is negligible, since \( vk \sim \sqrt{v} g T \gg g^2 T \) if \( v \gg g^{2/3} \). Thus the continuation from \( \delta E = 0 \) to \( \delta E = i\gamma_f \) does not affect the result for the damping to leading order in \( g \).

Consequently, for a slow, heavy fermion, due to kinematic reasons there is no sensitivity to the magnetic mass (contrary to what I claimed in ref. [2]) nor to details of analytic continuation (unlike ref. [16], at least for velocities as in (3.14)). Both of these effects do enter for a field moving at relativistic velocities.
Why is it that corrections from the imaginary part of the self energy must be included, and not those from the real part? While there are certainly corrections to the mass shell of order \( g^2 T \), at hard momenta these are independent of the spatial momentum. By energy conservation in (3.5), however, at this order all that enters into the damping rate is the difference in energies, \( E_p^M - E_{p-k}^M \) — so a constant shift in the mass shell cancels out. Similarly, consider the contribution of those higher loop diagrams which can be represented as vertex corrections to the diagram at lowest order. Assuming that the vertex correction is fixed by the appropriate Ward identity, if the self energies are slowly varying functions of momenta, the vertex corrections are even better behaved, and so will not generate the type of logarithmic divergences found above. (These arguments do not apply at soft momenta, where the self energies, and so the vertex corrections, depend nontrivially on momenta.) Nevertheless, it would be well worth checking these naive arguments by explicit calculation at two loop order.

IV. Damping rates for fast fields

Since it enters accompanied by an overall factor of the velocity, the logarithmic sensitivity found for the damping rate of a slow, heavy field is relatively innocuous. The damping rate of a fast particle is much more sensitive to small momenta. I first consider the case of a transverse gluon at “hard” momenta, \( p \sim T \).

For a transverse gluon the bare propagator is \( \Delta_t^{-1}(P) = (p^0)^2 + p^2 \). About the mass shell \( E^t_P = p \), the bare propagator behaves as \( \Delta_t^{-1} \sim -Z_t^{-1}(p)(\omega - E^t_P) \), where \( Z_t(p) = 1/(2p) \) is the residue for a hard field. Following Baier, Nakkagawa, and Niegawa [16], the damping rate is determined from the discontinuity of the self energy at the position of the singularity in the propagator. To take this into account, I evaluate the function \( \Gamma_t(\delta E) \) for real \( \delta E \), and then analytically continue in \( \delta E \) to determine the damping rate from \( \gamma_t = \Gamma_t(i\gamma_t) \). Implicitly, this definition assumes that the propagator has a true pole at \( \omega = E^t_P + i\gamma_t \), which I shall show is justified in hot QCD when \( m_{mag} \neq 0 \). In hot QED, it is necessary to define the
By resumming the procedure of ref. [3], to determine the damping rate at hard momentum requires the evaluation of the self energy which differs only slightly from the usual one loop diagram. Kinematics requires that only one line in the loop is soft, so bare vertices can be used. With bare vertices, the only contribution to the discontinuity is from the diagram with three gluon vertices. (The quark loop does not contribute to this order because there is no enhancement from Bose–Einstein statistics.) After projecting out the transverse piece of the gluon self energy,

\[ \Gamma_t(\delta E) = g^2N Z_t^{-1}(p) \text{tr} \left( (1 - (\hat{k} \cdot \hat{p}))^2 *\Delta_t(K) - *\Delta_t(K) \right) \Delta_t(P - K). \]  

(4.2)

For the soft field, with spectral parameter \( \omega \), the spectral densities are those of sec. II. For the hard field, with spectral parameter \( \omega_h \), the bare spectral density is replaced by

\[ \rho_t(\omega, p - k) = Z_t(p - k) \frac{\gamma_t}{\pi} \left( \frac{1}{(\omega_t - E_{p-k}^t)^2 + \gamma_t^2} + \frac{1}{(\omega_t + E_{p-k}^t)^2 + \gamma_t^2} \right). \]  

(4.3)

After doing the sum over \( k^0 \) and retaining only terms of leading order in \( g \),

\[ \Gamma_t(\delta E) = g^2 N T \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{+\infty} \frac{d\omega}{\omega} \int_{-\infty}^{+\infty} d\omega_t \frac{\gamma_t}{\omega_t^2 + \gamma_t^2} \]

\[ \left( (1 - (\hat{k} \cdot \hat{p}))^2 \rho_t(\omega, k) - \rho_t(\omega, k) \right) \delta(\omega + \omega_t - E_{p}^t - \delta E + E_{p-k}^t). \]  

(4.4)

The spectral parameter for the hard field has been shifted by \( \omega_t \to \omega_t + E_{p-k}^t \). With the mass shells for ultrarelativistic fields, the delta function for energy conservation fixes the angle between \( \hat{p} \) and \( \hat{k} \) as \( \cos \theta = (\omega + \omega_t - \delta E)/k \).

Again I assume that the integrals over \( \omega \) and \( \omega_t \) decouple. The integral over \( \omega_t \) is peaked about \( \omega_t \sim \gamma_t \sim g^2 T \), so there the effects of \( \delta E \neq 0 \) must be included. The integral over the longitudinal spectral density, as in (2.12), is perfectly well behaved, allowing me to set \( \gamma_t, \delta E, \) and \( m_{\text{mag}} \) to zero; this is also valid for the integral over \( \omega \rho_t(\omega, k) \), as in (2.11). The integral over \( \rho_t(\omega, k)/\omega \), as in (2.13), is in principal sensitive to \( \omega_f, \delta E, \) and \( m_{\text{mag}} \) when \( k \sim m_{\text{mag}} \). I now make the further assumption, however, that

\[ m_{\text{mag}} \gg \gamma_t. \]  

(4.5)
The limits of integration over $\omega$ run properly over $\pm k - \omega_t + \delta E$. When (4.5) holds, however, even for $k \sim m_{mag}$ I can neglect the effects of $\omega_t \sim \delta E \sim \gamma_t$, and just let the integral over $\omega$ run from $\pm k$, as in the sum rule of (2.14).

The integral over $\omega_t$ is

$$
\int_{-k+\delta E}^{k+\delta E} d\omega_t \frac{\gamma_t}{\omega_t^2 + \gamma_t^2} = \tan^{-1}\left(\frac{k + \delta E}{\gamma_t}\right) + \tan^{-1}\left(\frac{k - \delta E}{\gamma_t}\right). \tag{4.6}
$$

This is similar to (3.8), except that here $v = 1$ and the shift in the mass shell, $\delta E$, enters.

The integrals over the soft spectral functions are done using the sum rules of (2.11), (2.12), and (2.13). As in (2.14) these sum rules are used to trade an integral from $\pm k$ for terms which involve the right hand side of the sum rule and pole terms. In this way $\Gamma_t = \Gamma_t^{\text{sing}} + \Gamma_t^{\text{reg}}$ is written as a sum of a singular term,

$$
\Gamma_t^{\text{sing}}(\delta E) = \frac{g^2 N T}{4\pi} \int_0^\infty k \, dk \left( -\frac{1}{k^2 + m_g^2} \right.
+ \frac{2}{\pi} \left( \tan^{-1}\left(\frac{k + \delta E}{\gamma_t}\right) + \tan^{-1}\left(\frac{k - \delta E}{\gamma_t}\right) \right) \left. \frac{1}{k^2 + m_{mag}^2} \right)
$$

and a regular term,

$$
\Gamma_t^{\text{reg}} = \frac{g^2 N T}{4\pi} \int_0^\infty k \, dk \left( \frac{1}{k^2 + m_g^2} - 2Z_t(k) \frac{E_t^k}{E_t^l} \right.
- \frac{1}{k^2} \left(1 - 2Z_t(k)E_t^l\right)
+ \frac{1}{k^2} \left(\frac{3m_g^2}{k^2 + 3m_g^2} + \frac{2k^2Z_t(k)}{E_t^l(E_t^k)}\right) \right) = 1.09681\ldots. \tag{4.7}
$$

So that each integral is finite at large momentum, a term proportional to $k/(k^2 + m_g^2)$ is subtracted from the integrand of the singular term, and added to that for the regular term. After doing so, each term is separately finite and well behaved for both small and large momentum. The singular term is sensitive to momenta of order $k \sim g^2 T$, so there the dependence on $m_{mag}$, $\gamma_t$, and $\delta E$ must be retained. The regular term depends only upon momenta of order $k \sim m_g$, and so up to corrections of order $g$, it is a pure number. This number, $\Gamma_t^{\text{reg}} \simeq 1.09681\ldots$, was determined by numerical integration.

The analytic form of the singular term was computed in the following manner. At zero magnetic mass it is easy to show that

$$
\Gamma_t^{\text{sing}}(\delta E, m_{mag} = 0) = \frac{g^2 N T}{8\pi} \ln\left(\frac{m_g^2}{\gamma_t^2 + \delta E^2}\right). \tag{4.8}
$$
Next, I compute the derivative of $\Gamma_t^{\text{sing}}$ with respect to the magnetic mass squared:

$$\frac{\partial \Gamma_t^{\text{sing}}(\delta E)}{\partial m_{\text{mag}}^2} = \frac{g^2 NT}{8\pi^2} \left( -\frac{\gamma_t}{\pi} \right) \int_{-\infty}^{+\infty} \frac{dk}{(k^2 + m_{\text{mag}}^2)((k + \delta E)^2 + \gamma_t^2)} . \quad (4.10)$$

This is a standard loop integral in one dimension, and can be done using the Feynman parametrization for the denominators. After doing the integrals over $dk$ and the Feynman parameter, a relatively complicated form for $\partial \Gamma_t^{\text{sing}}/\partial m_{\text{mag}}^2$ results. Doing the integral over $m_{\text{mag}}$, and knowing $\Gamma_t^{\text{sing}}$ reduces to (4.9) for $m_{\text{mag}} = 0$, gives a simple result,

$$\Gamma_t^{\text{sing}}(\delta E) = \frac{g^2 NT}{8 \pi} \ln \left( \frac{m_y^2}{(m_{\text{mag}} + \gamma_t)^2 + \delta E^2} \right) . \quad (4.11)$$

Having computed for real $\delta E$, the analytic continuation of (4.11) to complex $\delta E$ is evident. There are branch points at

$$\delta E = \pm i(\gamma_t + m_{\text{mag}}) , \quad (4.12)$$

which is off the physical sheet. The damping rate is evaluated at the pole in the propagator, at $\delta E = i\gamma_t$. In the limit when $m_{\text{mag}} \gg \gamma$ this pole is well separated from the branch point, and the damping rate is just

$$\gamma_t = \Gamma_t^{\text{sing}}(i\gamma_t) + \Gamma_t^{\text{reg}} = \frac{g^2 NT}{8\pi} \left( \ln \left( \frac{m_y^2}{m_{\text{mag}}^2 + 2m_{\text{mag}}\gamma_t} \right) + 1.09681... \right) , \quad (4.13)$$

which is the result quoted in (1.1).

The same manipulations can be carried through for the damping rate of a (massless) quark field at hard momentum. I denote the damping rate of the standard mode, for which the chirality equals its helicity, as $\gamma_+$. In the end the only change is in an overall factor for the Casimir of the representation:

$$\gamma_+ = \frac{g^2 C_f T}{8\pi} \left( \ln \left( \frac{m_y^2}{m_{\text{mag}}^2 + 2\gamma_+ m_{\text{mag}}} \right) + 1.09681... \right) . \quad (4.14)$$

These damping rates are the only ones of significance at hard momenta. At low momenta the gauge field and the quark fields each have collective modes, the plasmon and plasmino, respectively. But the residues of these fields are exponentially small for hard momenta, so these fields can be neglected.
What of the damping rate of a fast fermion in hot QED, where the magnetic mass vanishes? In this instance the above approximations are inconsistent: the position of the branch point in the propagator, as in (4.10), coincides with the position of what is supposed to be a pole. Thus the spectral density for the hard fermion is not the pole of a Breit-Wigner form, but a branch point. Presumably the damping rate, defined as the imaginary part of the position of the branch point in the propagator, is gauge invariant and of order $e^2 T \ln(1/e)$. The question of gauge dependence is now more involved. For instance, at zero temperature it is known that the fermion propagator has a branch point singularity at the electron mass; while the position of the branch point is gauge invariant, the strength of the singularity is not [25].

The analysis of hot QCD in what is probably the realistic case of $m_{\text{mag}} \leq \gamma_t$ is even more involved. Then the propagator has a pole at $p + i \gamma_t$, and a branch cut beginning at $p + i(m_{\text{mag}} + \gamma_t)$. The spectral density for the hard field must now include the effects of both the pole and the nearby branch cut.

V. Damping rates for light fields

The logarithm in the damping rate of a fast field arises from a very limited kinematic region: in the one loop diagram, one line is very near the mass shell, while the other line carries almost zero momentum. In this section I analyze the same kinematic regime for fields moving at momenta comparable to the scale of the thermal mass. I just calculate the coefficient of the logarithm, which is relatively simple to compute.

According to the effective expansion [3], for soft momenta the bare propagator is replaced by one which includes the hard thermal loop, with the effective plasmon and transverse propagators are those of (2.2) and (2.3). For instance, about the mass shell $\omega = E_p^t$, the effective transverse propagator behaves

$$^* \Delta_t^{-1}(i\omega, p) \sim - Z_t^{-1}(p) (\omega - E_p^t),$$

where $Z_t(p)$ is the residue. The leading corrections to the damping rate are then determined by an effective gluon self energy, $^* \Pi^{\mu\nu}$, from which the transverse term $^* \Pi_t$ is extracted as usual. Using (5.1), the pole of the corrected transverse propagator, $^* \Delta_t^{-1} - ^* \Pi_t$, then determines the damping rate for a soft transverse gluon, $\gamma_t(p)$ to be

$$\gamma_t(p) \sim Z_t(p) \text{Disc} \ ^* \Pi_t(iE_p^t, p).$$
This is similar to the function introduced in (4.1). Since I don’t compute the constant under the logarithm, the subtleties of the previous section can be ignored, and it suffices to evaluate the discontinuity on the effective mass shell, without including the damping rate. The damping rates for the plasmon, and the quark modes, are defined similarly, and given at the end of this section.

From (4.21)-(4.25) of ref. [3], the effective gluon self energy is a sum of three terms,

$$\Pi^{\mu\nu}(P) = \Pi_3^{\mu\nu}(P) + \Pi_4^{\mu\nu}(P) + \Pi_{gh}^{\mu\nu}(P),$$

where $\Pi_3$ is the graph with three gluon vertices,

$$\Pi_3^{\mu\nu}(P) = \frac{g^2N}{2} tr_{soft} \Gamma^{\sigma\mu\lambda}(−P + K, P, −K) \Delta^{\lambda\lambda'}(K)$$

and $\Pi_4$ involves the four gluon vertex,

$$\Pi_4^{\mu\nu}(P) = -\frac{g^2}{2} tr_{soft} \Gamma^{\mu\nu\lambda\sigma}(P, -P, K, -K) \Delta^{\lambda\sigma}(K).$$

These two terms are the same as in the bare expansion, except that bare propagators and vertices are everywhere replaced by effective quantities (the propagators are all in Coulomb gauge). Lastly, there is the ghost loop, $\Pi_{gh}$; because there are no hard thermal loops in ghost amplitudes, this loop equals that in the bare expansion. In Coulomb gauge the ghost loop has zero discontinuity, and so doesn’t contribute to the damping rate. The subscripts on the trace in (5.4) and (5.5) indicate that the dominant term is given by the integral over soft momenta.

The effective vertices which appear in $\Pi$ are nontrivial functions of momenta, and so the discontinuity of such diagrams is far more complicated than in the bare expansion. For example, with a bare vertex the tadpole diagram has no discontinuity. In contrast, the discontinuity of $\Pi_4$ is nonzero, because the vertex itself has a discontinuity from Landau damping.

Assume that the leading logarithmic behavior of the damping rate at nonzero momentum arises from the same kinematic regime as for a fast field. Then while the diagram with a four gluon (effective) vertex, $\Pi_4$, does contribute to the discontinuity, it can’t generate a logarithm, since after cutting through one line and the vertex, there is no soft line left to integrate over. Thus the only diagram to contribute is that with three gluon vertices, $\Pi_3$. If the gluon with momentum $K$ is very soft, with
$K \sim 0$, then the other gluon, with momentum $P - K$, is very near its mass shell. (Of course I have to multiply by two, since the two gluons could be interchanged: $K$ could be near $P$, so $P - K$ is very soft.) Now much of what makes computation in the effective expansion so involved is the momentum dependence of the effective vertices. Under the assumption of very soft $K$, however, the Ward identities can be used to avoid having to compute any vertices whatsoever. Similar Ward identities have also been obtained by Weldon [24].

The effective three gluon vertex satisfies the Ward identity

$$K^\lambda \ast \Gamma^{\mu\nu\lambda}(P, -P - K, K) = \ast \Delta_{\mu\nu}^{-1}(P + K) - \ast \Delta_{\mu\nu}^{-1}(P) , \quad (5.6)$$

Hence as $K \to 0$,

$$\ast \Gamma^{\mu\nu\lambda}(P, -P, 0) = \frac{\partial}{\partial P^\lambda} \ast \Delta_{\mu\nu}^{-1}(P) . \quad (5.7)$$

At small $K$ (5.7) can be used to compute the effective three gluon vertices which appear in $\ast \Pi_{3g}$, (5.4). Even more labor can be saved by recognizing that only part of (5.7) contributes.

I introduce polarization vectors, $e^i_a(\hat{p})$. These are defined as transverse to $\hat{p}$,

$$p^i e^i_a(\hat{p}) = 0 , \quad (5.8)$$

and an orthonormal set,

$$e^i_a(\hat{p}) e^j_b(\hat{p}) = \delta_{ab} . \quad (5.9)$$

They can be combined to form a projection operator in $\hat{p}$ as

$$P^{ij}(\vec{p}) = \delta^{ij} - \hat{p}^i \hat{p}^j = \sum_{a=1,2} e^i_a(\hat{p}) e^j_a(\hat{p}) . \quad (5.10)$$

The advantage of using the polarization vectors is that by sandwiching the effective self energy between the $e^i_a$’s, from (5.9) I automatically project onto the transverse part. Further, for the gluon of momentum $P - K$ in the loop, in Coulomb gauge the propagator can be approximated as

$$\ast \Delta^{ij}(P - K) = P^{ij}(\vec{p} - \vec{k}) \ast \Delta_{t}(P - K) \sim P^{ij}(\vec{p}) \ast \Delta_{t}(P - K) . \quad (5.11)$$

Then (5.10) is used to write $P^{ij}(\vec{p})$ as a sum over polarization vectors $e^i_a(\hat{p})$.

In this way, all I need of the Ward identity in (5.7) are the terms which survive after sandwiching it between polarization operators. Using the behavior of the transverse propagator in (5.1),

$$e^i_a(\hat{p}) \ast \Gamma^{ijk}(P, -P, 0) e^j_b(\hat{p}) = \delta_{ab} \hat{p}^k Z^{-1}_t(p) v^t(p) . \quad (5.12)$$
Here \( v_t(p) \) is the group velocity for a transverse gluon of momentum \( p \) on the effective mass shell,
\[
v_t(p) = \frac{\partial E^t_t}{\partial p}.
\] (5.13)

Of course at hard momenta \( v_t(p) \to 1 \).

Computing in the kinematic regime when one transverse gluon is very soft, and the other gluon is transverse and very near its mass shell, it is then trivial to mimic the calculations of the previous sections to extract the leading logarithm in the damping rate. For the gluon with spectral parameter \( \omega_t \), near its mass shell I replace the delta functions for the pole term by smeared Breit-Wigner forms with width \( \gamma_t \). If the soft gluon has spectral parameter \( \omega \) the contribution to the damping rate from a very soft transverse gluon is
\[
(1 - (\hat{k} \cdot \hat{p})^2) \delta_t(\omega, k) \delta(\omega + \omega_t - E^t_t - E^t_{p-k}) .
\] (5.14)

This is almost the same integral as for the damping rate at hard momentum (4.4), except that each vertex contributes a factor of the group velocity, and the mass shell is now that for an effective field. Expanding the delta function for energy conservation in small \( k \) fixes the angle between \( \hat{p} \) and \( \hat{k} \) to be \( \cos \theta = (\omega + \omega_t)/v_t(p)k \): notice the extra factor of one over the group velocity, which is like the nonrelativistic case in sec. III. Using the delta function to integrate over \( \theta \) gives one factor of \( 1/|v_t(p)| \), so that in all
\[
\gamma_{t,\ell}(p) \sim \frac{g^2NT}{8\pi} v_{t,\ell}(p) \ln \left( \frac{1}{g^2} \right) + \ldots .
\] (5.15)

I approximate the argument of the logarithm as \( \ln(m_g^2/m_{mag}^2) \sim \ln(1/g^2) \). As indicated in (5.15), the damping rate of the plasmon can be computed similarly: it is given by replacing \( v_t(p) \) with the plasmon group velocity, \( v_\ell(p) = \partial E_\ell^t/\partial p \).

For the quark field there are two modes: at positive energy, the standard mode has chirality equal to helicity, and while for the plasmino, its chirality is equal to minus its helicity [26]. Denoting the mass shells by \( E^\pm_p \), in analogy to the result at hard momenta, (4.14), the damping rate of the quark field is
\[
\gamma_\pm(p) \sim \frac{g^2C_fT}{8\pi} |v_\pm(p)| \ln \left( \frac{1}{g^2} \right) + \ldots ,
\] (5.16)
where \( v_\pm (p) = \partial E_\pm / \partial p \) are the group velocities for the quark modes. In this instance I explicitly write the absolute value of the group velocity. This was not necessary before, since the mass shells for the transverse, plasmon, and standard quark modes are each monotonically increasing in \( p \), so the group velocities are always positive. The plasmino mass shell, however, decreases from zero momentum, reaches a minimum at \( p = p_c \), and then increases, so \( v_- (p) \) is negative for \( p_c > p > 0 \). Similarly, \( \gamma_- (p_c) \) in (5.16) vanishes at \( p = p_c \) since \( v_- (p_c) = 0 \).

The terms in (5.15) and (5.16) apply for momenta \( p \gg g^2 T \). For momenta \( p \sim T \), all group velocities \( v \rightarrow 1 \), and the results of the previous section are recovered. The restriction that the momenta be greater than \( g^2 T \) is necessary because I assumed that I could approximate \( E_p - E_{p-k} \sim v(p) \cos \theta \), which is incorrect for \( p \ll m_g \). For instance, if one computes as above, but now at exactly zero momentum, \( p = 0 \), one finds no term as \( \ln (1/g^2) \) in \( \gamma(p) \). The crossover scale at which a \( \ln (1/g^2) \) appears in the damping rate is set by the width of the spectral density for a field near its mass shell, which is of order \( \sim g^2 T \). This restriction is of no concern for the gluon fields, since \( v_t, \ell (p) \sim p/m_g \) as \( p \rightarrow 0 \). For the quark fields, however, \( |v_\pm (p)| \rightarrow 1/3 \) as \( p \rightarrow 0 \), and this caveat is important: there is no \( \ln (1/g^2) \) in \( \gamma_\pm (0) \), as indicated by the naive extrapolation of (5.16).

This is consistent with explicit calculations at zero momentum. The damping rates of both gluons [4] and quarks [5,9] at zero momentum have been computed: both results are a pure number times \( g^2 T \), with no terms as \( \ln (1/g^2) \). Similarly, the damping rate of the plasmino at the minimum in its dispersion relation, \( p = p_c \), is surely a pure number times \( g^2 T \).

It would take some effort to compute the constants under the logarithm in the damping rates of (5.15) and (5.16). Not only does the other diagram, \( \ast \Pi_4g \), enter, but for both diagrams the full form of the vertices are required. Also, as in sec. IV, the effects of analytic continuation to the pole in the propagator must be included.

**Appendix:** **The plasmon pressure and sum rules**

van Weert and collaborators [27] raised the following question: the order \( g^3 \) term in free energy arises entirely from the plasmon through the static electric mass. The transverse modes do not contribute because static transverse fields are not screened perturbatively. On the other hand, for time dependent fields both the plasmon and
transverse fields are screened by thermal masses \( m_g \). So why don’t transverse fields contribute to the free energy at order \( g^3 \)? In accord with an analysis by Toimela [28], in this appendix I show how the sum rules can be used to demonstrate the cancellation of the transverse pressure at order \( g^3 \).

The terms of order \( g^3 \) in the free energy arise from the summation of “ring” diagrams [29], as the free energy of effective fields. For the transverse gluon, per color degree of freedom the two transverse modes give a free energy

\[
{^\ast F}_t = \text{tr} \left( \ln( {^\ast \Delta}_t^{-1}(K)) - \ln(K^2) \right) ;
\] (a.1)

as before, \( \text{tr} \) denotes the integral over the four momentum \( K \). The effective propagator \( {^\ast \Delta}_t^{-1}(K) = K^2 - \delta \Pi_t(K) \), where \( \delta \Pi_t(K) \) is the hard thermal loop in the transverse gluon self energy. From (2.1) and (2.3), \( \delta \Pi_t(K) \) is \( g^2T^2 \) times a function of momentum. Then it is easy to compute the derivative of \( {^\ast F}_t \) with respect to \( g^2 \):

\[
\frac{\partial {^\ast F}_t}{\partial g^2} = \frac{1}{g^2} \text{tr} \left( \frac{- \delta \Pi_t(K)}{K^2 - \delta \Pi_t(K)} \right) = \frac{1}{g^2} \text{tr} \left( 1 - K^2 {^\ast \Delta}_t(K) \right) .
\] (a.2)

The sum over \( k^0 \) is done by using the spectral representation of the effective propagator,

\[
\text{tr} \left( K^2 {^\ast \Delta}_t(K) \right) = \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{+\infty} d\omega \left( 1 + n(\omega) \right) \left( -\omega^2 + k^2 \right) {^\ast \rho_t}(\omega, k) .
\] (a.3)

The dominant term is given by approximating the Bose–Einstein statistical distribution function \( n(\omega) \sim T/\omega \),

\[
\text{tr} \left( K^2 {^\ast \Delta}_t(K) \right) \sim T \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{+\infty} d\omega \left( -\omega + \frac{k^2}{\omega} \right) {^\ast \rho_t}(\omega, k) .
\] (a.4)

Now use the sum rules of (2.11) and (2.13) to do the \( \omega \) integrals,

\[
\text{tr} \left( K^2 {^\ast \Delta}_t(K) \right) \sim T \int \frac{d^3k}{(2\pi)^3} \left( 1 - \frac{k^2}{k^2 + m^2_{mag}} \right) .
\] (a.5)

Hence if the magnetic mass is ignored, the contribution of transverse gluons to the pressure cancels identically. With \( m_{mag} \sim g^2T \), up to a possible \( \sqrt{\ln(1/g^2)} \) [13], the first nonvanishing contribution to the transverse free energy is of order \( {^\ast F}_t \sim g^6T^4 \). This is where the effects of the magnetic mass are expected to arise in the free energy [1].
Why did this cancellation occur? The arguments of van Weert et al [27] considered only the contributions of pole terms to the spectral density. This is the dominant contribution to the sum rule of (2.11). But the effective spectral density of (2.6) also includes cuts from Landau damping, and it is these which dominate the sum rule of (2.13). For $m_{mag} = 0$ in $\Phi_t$, the contribution of the cuts identically cancels that of the pole terms.

The analogous contribution of the plasmon to the free energy is, per color degree of freedom,

$$
\Phi^*_t = \frac{1}{2} \text{tr} \left( \ln(\Delta_t^{-1}(K)) - \ln(k^2) \right),
$$

where $\Delta_t^{-1}(K) = k^2 - \delta \Pi_t(K)$, with $\delta \Pi_t(K)$ the hard thermal loop in the plasmon part of the gluon self energy, (2.2). The derivative of this term with respect to $g^2$ is

$$
\frac{\partial \Phi^*_t}{\partial g^2} = \frac{1}{2g^2} \text{tr} \left( 1 - k^2 \Delta_t(K) \right).
$$

Then

$$
\text{tr}(-k^2 \Delta_t(K)) \sim T \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{+\infty} \frac{d\omega}{\omega} \Phi_t(\omega, k).
$$

Using the sum rule of (2.12),

$$
\Phi_t^* \sim \frac{T}{2} \int \frac{d^3k}{(2\pi)^3} \ln(k^2 + 3m_g^2).
$$

Remembering that the static electric mass squared $m_{el}^2 = 3m_g^2$, (a.9) is exactly equal to the free energy as computed in the imaginary time formalism, including only the effects of the term with $k^0 = 0$. The result for $\Phi_t^*$ is a term of order $g^2$ (this is part of the free energy at two loop order) plus the term of interest, $= -Tm_{el}^3/(12\pi) \sim g^3T^4$.

Thus sum rules demonstrate the equivalence with the results of the imaginary time formalism.

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References


[13] Infrared divergences which signal the appearance of the magnetic mass first arise in the gluon self energy at two loop order. As these are two loop diagrams in an effectively three dimensional theory, they may contain a single logarithm:

$$m_{\text{mag}}^2 \sim g^4 T^2 \ln \left( \frac{T}{g^2 T} \right),$$

by which $m_{\text{mag}} \sim g^2 T \sqrt{\ln(1/g^2)}$. Probably, however, gauge invariance (transversality of the gluon self energy) requires the logarithms in individual diagrams to cancel in the sum, with $m_{\text{mag}}^2 \sim g^4 T^2$. 

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