Is Geodetic Precession of Massive Neutrinos the solution to the Solar Neutrino Problem?

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Abstract

We examine a recent suggestion by Milburn that slightly massive electron neutrinos, produced left-handed at the core of the sun, suffer geodetic precession adequate to render them right-handed (and therefore sterile) in sufficient numbers to solve the solar neutrino problem. In that light, we perform a complete, general-relativistic calculation of the geodetic spin precession of an ultrarelativistic particle in the Schwarzschild metric. We conclude that the effect is negligible, in disagreement with Milburn’s analysis.
I. INTRODUCTION

Recently, Milburn [1] suggested a solution to the solar neutrino problem which does not require any assumptions beyond the existence of a small, but non-zero, mass for the electron neutrinos, consistent with experimental limits \((m_\nu \leq 10 \text{ eV})\). These neutrinos are produced in the core of the sun with typical values for \(\gamma = E_\nu/m_\nu\) of the order of \(10^4\)–\(10^6\). If produced eccentrically, they will suffer a small bending due to the gravitational pull of the sun. Milburn argued that these neutrinos undergo a Thomas precession of their spin given by the same formula relating the spin precession angle of charged particles in an accelerator to their bending angle [2],

\[
\theta_p = -\gamma \theta_b ,
\]

where in the case at hand \(\theta_b\) is the angle of gravitational bending. The smallness of \(\theta_b\) is overcompensated by the magnitude of \(\gamma\), which for reasonable values of the parameters, results in an adequate repolarization of the originally left-handed neutrinos, turning them into right-handed, and hence sterile for the purposes of weak interactions.

Despite its ingenious simplicity, this suggestion is wrong, inasmuch as it is based on the aforementioned formula. Spin precession calculations have been carried out in the special cases of circular orbits or arbitrary orbits with small velocities (in the sense of the PPN formalism) [3]. Since both special cases are clearly inapplicable to our current application, we perform, in this letter, the complete general-relativistic calculation of the spin precession of a particle in the Schwartzschild geometry. We obtain,

\[
\theta_p = -\frac{1}{2\gamma} \theta_b ,
\]

which demonstrates that for ultrarelativistic particles, \(\theta_p \ll \theta_b\), and therefore unable to provide us with a satisfactory rate of left- to right-handed neutrino conversion.
II. THE CALCULATION

We consider a test particle approaching a body of mass $M$. Its original direction is along $\phi = 0$, while its final direction is along the asymptote $\phi = \pi + \theta_b$. The Schwartzschild metric is

$$ds^2 = B(r)dt^2 - A(r)dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2,$$

(2.1)

where

$$B(r) = A(r)^{-1} = 1 - 2m/r,$$

(2.2)

with $m = GM/c^2$. In the remaining analysis we use units with $G = 1 = c$. Instead of considering this coordinate frame, it is useful to construct an orthonormal frame (vierbein), $e^{\alpha}_\mu$, and its inverse $e^\mu_\alpha$, where

$$g_{\mu\nu} = e^\alpha_\mu e^\beta_\nu \eta_{\alpha\beta}.$$

(2.3)

This frame is given by

$$e^0_t = \sqrt{B}, \quad e^1_r = 1/\sqrt{B}, \quad e^2_\theta = r, \quad e^3_\phi = r \sin \theta.$$

(2.4)

To separate orthonormal indices from coordinate indices, we will use letters from the beginning of the greek alphabet ($\alpha, \beta, \gamma, \ldots$) for the former, while the latter shall be denoted by greek letters from the middle of the alphabet ($\mu, \nu, \ldots$). Then the components of any vector $V$ can be expressed as

$$V^\mu = e^\alpha_\mu V^\alpha, \quad V^\alpha = e^\mu_\alpha V^\mu.$$

(2.5)

In particular, the four-velocity of the particle can be expressed as

$$u^\mu = (\dot{t}, \dot{r}, \dot{\theta}, \dot{\phi}), \quad u^\alpha = (\sqrt{B}\dot{t}, \dot{r}/\sqrt{B}, r\dot{\theta}, r \sin \theta \dot{\phi}),$$

(2.6)

where $\dot{\cdot} = \frac{d}{ds}$ is the derivative with respect to the proper time of the particle. Covariant derivatives are given in terms of the connection coefficients $\omega^\beta_\alpha\gamma$. The only non-vanishing such coefficients are
\[ \omega_{0}^{0} = \frac{d}{dr} \sqrt{B}, \]  
\[ \omega_{2}^{1} = \omega_{3}^{1} = -\omega_{1}^{2} = -\omega_{3}^{2} = -\frac{1}{r} \sqrt{B}, \]  
\[ \omega_{3}^{2} = -\omega_{2}^{3} = r^{-1} \cot \theta. \]  

(2.7)  
(2.8)  
(2.9)

The spin vector of the particle will in general be Fermi-Walker transported along an arbitrary path \( x^\beta(s) \) under external non-gravitational forces [3],

\[ u^\alpha \nabla_\alpha S^\beta = -S^\gamma a_\gamma u^\beta, \]  

(2.10)

where \( a^\beta = \frac{Du^\beta}{ds} \) is the four-acceleration. However, in the absence of non-gravitational forces the four-acceleration is zero and the particle will move along a geodesic, while its spin will be parallel-transported,

\[ u^\alpha \nabla_\alpha S^\beta = 0. \]  

(2.11)

Thomas precession is a purely kinematical effect which comes about when there is a non-zero four-acceleration, for instance in an electron moving around a nucleus or in an accelerator. For a particle moving along a geodesic, the Thomas precession is identically zero. Therefore, no direct analogy with special relativistic electrodynamics can be drawn and the reasoning leading to (1.1) is incomplete.

However, the spin will still experience geodetic precession. Eq. (2.11) above becomes in components,

\[ \frac{dS^\alpha}{ds} + \omega_\beta^\alpha \gamma u^\beta S^\gamma = 0. \]  

(2.12)

Explicitly,

\[ \frac{dS^0}{ds} = -(\sqrt{B})' u^0 S^1, \]  
\[ \frac{dS^1}{ds} = -(\sqrt{B})' u^0 S^0 + \frac{\sqrt{B}}{r} u^2 S^2 + \frac{\sqrt{B}}{r} u^3 S^3, \]  
\[ \frac{dS^2}{ds} = -\frac{\sqrt{B}}{r} u^2 S^1 + \frac{\cot \theta}{r} u^3 S^3, \]  
\[ \frac{dS^3}{ds} = -\frac{\sqrt{B}}{r} u^3 S^1 - \frac{\cot \theta}{r} u^3 S^2. \]  

(2.13)  
(2.14)  
(2.15)  
(2.16)
For motion in the equatorial plane, $\theta = \pi/2$ and $u^2 = 0$. Furthermore the spin is a spacelike vector, which in the rest frame of the particle has vanishing zeroth component. Given that the four-velocity is a timelike vector with vanishing spatial components in the rest-frame of the particle implies that the spin is orthogonal to the four-velocity. This allows us to express $S^0$ as a linear combination of the spatial components of $S$, namely

$$S^0 = \frac{1}{u^0}(u^1 S^1 + u^3 S^3). \quad (2.17)$$

Substituting into Eq. (2.14) gives three independent equations

$$\frac{dS^1}{ds} = -(\sqrt{B})' u^1 S^1 + \left( \frac{\sqrt{B}}{r} - (\sqrt{B})' \right) u^3 S^3,$$

$$\frac{dS^2}{ds} = 0,$$

$$\frac{dS^3}{ds} = -\sqrt{B} \frac{u^3}{r} S^1. \quad (2.18)$$

We can simplify further the equations of motion of the spin components, in view of the non-uniqueness of this orthonormal basis $\{e^\alpha \mu\}$. We can use the freedom of choosing an appropriate basis, in which the spin precession is evident. This is achieved by going to the rest frame of the particle. The basis corresponding to the rest-frame $\{\bar{e}^\alpha \mu\}$ is constructed through a local Lorentz transformation with the following properties: (a) the resulting zeroth component of the spin vanishes and (b) the four velocity takes on the form $(1, \vec{0})$. Explicitly,

$$\bar{e}_\beta = e_\alpha L^\alpha _\beta , \quad (2.19)$$

with

$$u_\alpha L^\alpha _i = 0 , \quad u_\alpha L^\alpha _0 = 1. \quad (2.20)$$

Then, $\bar{e}_i$ ($i = 1, 2, 3$) are orthogonal to the four-velocity, and we are guaranteed to get $\bar{S}^0 = 0$ when we resolve $S$ along the new basis,

$$S = S^\alpha e_\alpha = \bar{S}^i \bar{e}_i. \quad (2.21)$$

To construct the requisite Lorentz transformation we note that the vanishing of the $\alpha = 0$ component of
\[ S_\alpha = S_\gamma L^\gamma_\alpha, \quad (2.22) \]
together with \( S \cdot u = 0 \), gives
\[ \frac{L^i_0}{L^0_0} = \frac{u^i}{u^0}. \quad (2.23) \]

Keeping in mind that a Lorentz transformation matrix satisfies
\[ \eta_{\alpha\delta} L^\alpha_\beta L^\delta_\gamma = \eta_{\beta\gamma}, \quad (2.24) \]
we obtain
\[ L^\alpha_0 = u^\alpha. \quad (2.25) \]

One can similarly obtain the remaining entries to arrive at
\[ L^\alpha_\beta = \begin{pmatrix} u^0 & u^1 & 0 & u^3 \\ u^1 & 1 + \frac{(u^1)^2}{1+u^0} & 0 & \frac{u^1u^3}{1+u^0} \\ 0 & 0 & 1 & 0 \\ u^3 & \frac{u^1u^3}{1+u^0} & 0 & 1 + \frac{(u^3)^2}{1+u^0} \end{pmatrix}. \quad (2.26) \]

The equations of motion for the spin components in the rest frame can be inferred from (2.18) with the aid of the equation for the vanishing of the four-acceleration,
\[ \frac{du^\alpha}{ds} + \omega^\beta_\alpha u^\beta u^\gamma = 0. \quad (2.27) \]

We obtain
\[ \frac{dS^1}{ds} = \tilde{S}_3 \frac{(r - 2m + (r + 3m)u^0)u^3}{r^2\sqrt{B}(1 + u^0)}, \]
\[ \frac{dS^2}{ds} = 0, \quad (2.28) \]
\[ \frac{dS^3}{ds} = -\tilde{S}_1 \frac{(r - 2m + (r + 3m)u^0)u^3}{r^2\sqrt{B}(1 + u^0)}. \]

Next we replace the derivatives with respect to proper time with derivatives with respect to coordinate time. Recalling that
\[ u^0 = \sqrt{Bt}, \quad u^3 = r\dot{\phi}, \] (2.29)

Eq. (2.28) obtains

\[ \frac{d\vec{S}}{dt} = \vec{\Omega} \times \vec{S}, \] (2.30)

where we dropped the bars for notational simplicity, and \( \Omega^1 = 0 = \Omega^3 \), while

\[ \Omega^2 \equiv \Omega = \left( \frac{r - 3m}{r} + \frac{m}{r(1 + u^0)} \right) \frac{1}{\sqrt{1 - \frac{2m}{r}}} \frac{d\phi}{dt}. \] (2.31)

How can we turn the knowledge of \( \vec{\Omega} \) into an expression for the geodetic precession angle over the whole trajectory of the particle? First, we recall that the Schwarzschild geometry admits two Killing vectors, namely \( \frac{\partial}{\partial t} \) and \( \frac{\partial}{\partial \phi} \). These two isometries correspond to two constants of the motion,

\[ E \equiv (1 - \frac{2m}{r})t, \quad L \equiv r^2\dot{\phi}. \] (2.32)

For a particle coming in from infinity,

\[ E = t_{r \to \infty}. \] (2.33)

Asymptotically, the Schwarzschild metric becomes Minkowskian, \( ds^2 = dt^2 - d\vec{x}^2 \), hence

\[ E = (1 - \frac{d\vec{x}^2}{dt^2})^{-\frac{1}{2}} \bigg|_{r \to \infty} \equiv \gamma. \] (2.34)

We therefore obtain,

\[ u^0 = \frac{\gamma}{\sqrt{1 - \frac{2m}{r}}}. \] (2.35)

Denoting

\[ S_\pm = S^1 \pm iS^3, \] (2.36)

\[ e(\phi) = \Omega/\frac{d\phi}{dt}, \] (2.37)

the spin precession equations may be recast in the form
\[
\frac{dS_\pm}{d\phi} = \mp ie(\phi)S_\pm,
\]
with solution
\[
S_\pm(\phi_f) = S_\pm(\phi_i) e^{\mp \int_{\phi_i}^{\phi_f} d\phi e(\phi)}.
\]

In the absence of gravity \( e(\phi) = 1 \), and if \( \phi_i = 0 \), then \( \phi_f = \pi \), whereas in the presence of gravity \( e(\phi) \neq 1 \) and \( \phi_f = \pi + \theta_b \). The total geodetic deviation is
\[
\Delta S_\pm = S_\pm(\pi + \theta_b) - S_\pm^{(0)}(\pi)
= S_\pm(\pi + \theta_b) + S_\pm(0).
\]

An ultrarelativistic particle follows approximately the same geodesic as light, which to first order in \( m/b \) is given by
\[
u \equiv \frac{1}{r} = \frac{1}{b} \sin \phi + \frac{3m}{2b^2} \left( 1 + \frac{1}{3} \cos 2\phi \right),
\]
where \( b \) is the impact parameter. With the aid of Eqs. (2.35) and (2.41),
\[
e(\phi) = \frac{1 - 3mu}{\sqrt{1 - 2mu}} + \frac{mu}{\gamma + \sqrt{1 - 2mu}},
= 1 - mu(2 - \frac{1}{1 + \gamma}) + \cdots.
\]

Substituting into Eq. (2.40) gives
\[
\frac{\Delta S_\pm}{S_\pm(0)} = e^{\mp i(\pi + \frac{1}{2}\theta_b/\gamma)} + 1
= \pm i \frac{\theta_b}{2\gamma} + \cdots
\]
This clearly agrees with the expectation that for light \( (\gamma \to \infty) \), there is no excess spin precession.

**III. DISCUSSION**

We have shown that geodetic precession of massive, initially left-handed, neutrinos moving along geodesics in the Schrödinger geometry, cannot produce enough sterile neutrinos.
to solve the solar neutrino puzzle. In fact, the answer is $O(\infty / \gamma) = O(\frac{\gamma}{E})$, and conforms with conventional field-theoretic experience, in which helicity flips vanish in the massless fermion (or ultrarelativistic) limit. However, one may argue that our analysis is steeped in the spirit of general relativity and may rely on some special features which may not be shared by some other interaction (i.e. one which does not satisfy the principle of equivalence). In such a case, it could be imagined that a small deflection due to a non-gravitational “fifth force” interaction could be amplified to yield the requisite sterilizing effect suggested by Milburn. But how can an interaction produce counter-intuitive $O(\frac{E}{\gamma})$ helicity-flip probabilities? Milburn’s argument suggests that kinematics, not dynamics, can yield this effect. After all, in the case of electrons deflected at SLAC [2], the vector couplings of the QED lagrangian do not even connect left- and right-handed spinors. But if such kinematical effects held for other interactions, why would gravity (which, in many respects, can be modelled by graviton exchange amplitudes) yield such a dramatically different answer? It is, therefore, worth asking whether our result carries over to other interactions, within the scope of Milburn’s suggestion.

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REFERENCES

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