Optimized post Gaussian approximation in the background field method.

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Abstract

We have extended the variational perturbative theory based on the background field method to include the optimized expansion of Okopinska and the post Gaussian effective potential of Stansu and Stevenson. This new method provides much simpler way to compute the correction terms to the Gaussian effective action (or potential). We have also renormalized the effective potential in 3 + 1 dimensions by introducing appropriate counter terms in the lagrangian.

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I. INTRODUCTION

The effective action and the effective potential have been used effectively in studying various aspects of quantum field theories. Nonperturbative approximation methods for calculating them have been used to extract essential features of nonlinear quantum field theories, such as properties of the ground state and information about possible phase transition of the system.

One of the most convenient nonperturbative approximations to the effective potential is the Gaussian variational approximation proposed more than 20 years ago [1]. The corresponding Gaussian effective potential (GEP) is known to contain one loop, sum of all daisy and superdaisy graphs of perturbation theory [2] and leading order in $1/N$ expansion.

Lately some procedures to compute the correction terms to the GEP were proposed [3–6]. In our previous paper [6] we have developed a systematic method of the perturbative expansion around the Gaussian effective action based on the background field method (BFM). Although BFM provides one with the simplest method of computing the perturbative corrections to the GEP, one can extend the method to give a better approximation. The point is that in refs. [5,6] the effective mass ($\Omega$) is fixed by the Gaussian gap equation, whereas $\Omega$ is optimized at each stage of approximation in refs. [3,4].

The main purpose of the present article is to extend BFM to include various other post Gaussian methods developed so far. We shall develop an improved post Gaussian effective potential (PGEP) method using BFM, where $\Omega$ is not fixed preliminary.

The paper is organized as follows. In Sec. II we briefly present the basic BFM formulas, which will be necessary for further developments. In Sec. III we derive PGEP using BFM for the $\phi^4$ theory, and we show that with BFM the computation of the higher order corrections is very much simplified compared to other methods. In Sec. IV we consider the renormalization procedure. The summary and conclusions are given in Sec. V.

II. BACKGROUND FIELD METHOD FOR SCALAR FIELDS

For readers convenience we outline here the basic formulas of BFM. It is well known that, BFM is the most effective way of doing the so called loop expansion [7] allowing us to study one - particle irreducible (1PI) Feynman diagrams. The generating functional for disconnected graphs $^1$ is defined by

$$Z[j] = \int \mathcal{D}\phi \exp\{i(S[\phi] + j\phi)\} = \langle 0|0 \rangle^j$$

(2.1)

where $j(x)$ is the external source and $S[\phi]$ is the classical action:

$$S[\phi] = \int d^4x \mathcal{L}[\phi(x), \partial_\mu \phi(x)].$$

(2.2)

In BFM an analogous quantity, in which the classical action is written as a function of the field $\phi$ plus an arbitrary background field $B$, is defined by

$^1$Here and below we use integral convention, e.g. $j\phi \equiv \int d^4x j(x)\phi(x)$.
\[ \tilde{Z}[j, B] = \int D\phi \exp\{i(S[\phi + B] + j\phi)\}, \quad (2.3) \]

where \( \tilde{Z}[j, B] \) depends both on the conventional source \( j \) and on the background field \( B \). Also by analogy with the conventional generator of connected graphs,

\[ W[j] = -i \ln Z[j], \quad (2.4) \]

we define a new generating functional for connected Green’s functions,

\[ \tilde{W}[j, B] = -i \ln \tilde{Z}[j, B]. \quad (2.5) \]

In the conventional approach the vacuum expectation value of the field operator in the presence of external source is defined by

\[ \phi_0(x) \equiv \langle \phi(x) \rangle_j = \frac{\delta W[j]}{\delta j(x)}. \quad (2.6) \]

In BFM it is defined as

\[ \tilde{\phi}_0 = \frac{\delta \tilde{W}[j, B]}{\delta j(x)}, \quad (2.7) \]

and the background field effective action is defined by

\[ \tilde{\Gamma}[\tilde{\phi}_0, B] = \tilde{W}[j, B] - j\tilde{\phi}_0. \quad (2.8) \]

Now it can be easily shown [7] that the conventional effective action can be calculated by the following equation:

\[ \Gamma[\phi_0] = \tilde{\Gamma}[0, B]. \quad (2.9) \]

The background field effective action \( \tilde{\Gamma}[\tilde{\phi}_0, B] \) is just a conventional effective action computed in the presence of background field \( B \). It therefore consists of all \( 1PI \) graphs contributing to Green’s functions. Since \( 1PI \) Green’s functions are generated by taking derivatives of \( \tilde{\Gamma}[\tilde{\phi}_0, B] \) with respect to \( \tilde{\phi}_0 \) it would generate \( 1PI \) Green’s functions in the presence of the background field \( B \). Now \( \tilde{\Gamma}[0, B] \) has no dependence on \( \tilde{\phi}_0 \), so it generates no graphs with external lines. Instead, \( \tilde{\Gamma}[0, B] \) is the sum all \( 1PI \) vacuum graphs in the presence of the \( B \) field. The advantage of the background field method is that many diagrams in the conventional perturbation method are amalgamated into one diagram in BFM due to a new definition of the propagator function [7]. This method can be modified to extract the Gaussian part of the effective action out of the perturbative action part [6]. We will now show that BFM can be used to compute the perturbative expansion of the effective action around the Gaussian effective action without fixing the effective mass.
III. POST - GAUSSIAN EXPANSION OF THE EFFECTIVE POTENTIAL

For technical convenience we shall work throughout in the Euclidian formalism and start with the Lagrangian of the $\phi^4$ theory:

$$\mathcal{L} = \frac{1}{2} \phi (-\partial^2 + m_0^2) \phi + \lambda_0 \phi^4, \quad (3.1)$$

where $m_0$ and $\lambda_0$ are bare mass and bare coupling constant, respectively. Now following refs. [3,4] we rewrite (3.1) in an equal form by introducing a new mass parameter $\Omega_0$:

$$\mathcal{L} = \frac{1}{2} \phi (-\partial^2 + \Omega_0^2) \phi + \frac{1}{2} (m_0^2 - \Omega_0^2) \phi^2 + \lambda_0 \phi^4. \quad (3.2)$$

The generating functional $\tilde{Z}[j, B]$ is then given by Eq. (2.3):

$$\tilde{Z}[j, B] = \int \mathcal{D}\phi \exp\{-\frac{1}{\hbar} \int d^4x [\mathcal{L}(\phi + B) + j \phi]\} = \int \mathcal{D}\phi \exp\{-\frac{1}{\hbar} \int d^4x [\mathcal{L}_0(\phi) + j \phi]\} \exp\{\frac{\hbar}{2} \mathcal{D} j / 2\}, \quad (3.3)$$

where $\mathcal{L}(\phi + B)$ can be separated as follows:

$$\mathcal{L}(\phi + B) = \mathcal{L}_0(\phi) + \mathcal{L}_1(\phi, B),$$

$$\mathcal{L}_0 = \frac{1}{2} \phi (-\partial^2 + \Omega_0^2) \phi,$$

$$\mathcal{L}_1(\phi, B) = v_0(B) + v_1(B) \phi + v_2(B) \phi^2 + v_3(B) \phi^3 + v_4(B) \phi^4,$$

with

$$v_0(B) = \frac{m_0^2 B^2}{2} + \lambda_0 B^4, \quad v_1(B) = B(m_0^2 + 4\lambda_0 B^2),$$

$$v_2(B) = \frac{1}{2} (m_0^2 - \Omega_0^2) + 6\lambda_0 B^2, \quad v_3(B) = 4\lambda_0 B,$$

$$v_4(B) = \lambda_0. \quad (3.5)$$

Now performing explicit Gaussian integration in Eq. (3.3) one obtains

$$\tilde{Z}[j, B] = \exp\{-\frac{1}{\hbar} \int d^4x \mathcal{L}_1(\phi \rightarrow \delta / \delta j, B)\} \int \mathcal{D}\phi \exp\{-\frac{1}{\hbar} \int d^4x \mathcal{L}_0(\phi) + j \phi\} = \det \mathcal{D}^{-1} \cdot \exp\{-\frac{1}{\hbar} \int d^4x \mathcal{L}_1(\phi \rightarrow \delta / \delta j, B)\} \exp\{\hbar j \mathcal{D} j / 2\}, \quad (3.6)$$

where

$$\mathcal{D}^{-1} = (-\partial^2 + \Omega_0^2) \delta_{xy}. \quad (3.7)$$

In accordance with our previous work [6], we introduce the so called primed derivatives:

$$\left(\frac{\delta^2}{\delta j_x}\right)_{\gamma} = \hat{A}_{(2)} = \frac{\delta^2}{\delta j_x}, \quad \hbar \mathcal{D}_{xx},$$

$$\left(\frac{\delta^3}{\delta j_x}\right)_{\gamma} = \hat{A}_{(3)} = \frac{\delta^3}{\delta j_x}, \quad -3\hbar^2 \mathcal{D}_{xx} \mathcal{R}_x,$$

$$\left(\frac{\delta^4}{\delta j_x}\right)_{\gamma} = \hat{A}_{(4)} = \frac{\delta^4}{\delta j_x}, \quad -6\hbar \mathcal{D}_{xx} \frac{\delta^2}{\delta j_x} + 3\hbar^2 \mathcal{D}_{xx}^2, \quad (3.8)$$
where $R_x = \int d^4 y \mathcal{D}_{xy} j(y)$, so that

$$\hat{A}_x^{(n)} \exp\{h j\mathcal{D} j/2\} = h^n R_x^n \exp\{h j\mathcal{D} j/2\}.$$  \hspace{1cm} (3.9)

To isolate the Gaussian approximation we introduce another Green’s function:

$$\exp\{-\frac{1}{\hbar}(v_2 \delta^2/\delta j^2 + v_3 \delta^3/\delta j^3 + v_4 \delta^4/\delta j^4)\} \exp\{h j\mathcal{D} j/2\}
= N_0 \exp\{\frac{1}{\hbar} a(B) \delta/\delta j\} \exp\{-\frac{1}{\hbar}(v_2 \hat{A}^{(2)} + v_3 \hat{A}^{(3)} + v_4 \hat{A}^{(4)})\} \exp\{h jG j/2\},$$  \hspace{1cm} (3.10)

where $N_0$ , $a(B)$ and $G$ are to be determined. By using the definition (3.8) one finds the following solution to the Eq. (3.10):

$$a(B) = -3v_3(B)hG_{xx},$$
$$N_0(B) = (\det G^{-1})^{-1/2}(\det \mathcal{D}^{-1})^{1/2} \exp\{-v_2G_{xx} + 3v_4hG_{xx}^2\},$$  \hspace{1cm} (3.11)

where $G_{xy}$ satisfies the equation:

$$G^{-1}_{xy} = \mathcal{D}^{-1}_{xy} + 12hv_4G_{xx}\delta_{xy}.$$  \hspace{1cm} (3.12)

The last equation looks rather simple in momentum space:

$$\Omega_0^2 = \Omega^2 - 12h\lambda_0 I_0(\Omega),$$  \hspace{1cm} (3.13)

with

$$\mathcal{D}_{xy} = \int \frac{d^4 p \exp[-ip(x-y)]}{(2\pi)^4(p^2 + \Omega_0^2)},$$
$$G_{xy} = \int \frac{d^4 p \exp[-ip(x-y)]}{(2\pi)^4(p^2 + \Omega^2)},$$  \hspace{1cm} (3.14)
$$I_0(\Omega) = \int \frac{d^4 p}{(2\pi)^4(p^2 + \Omega^2)}.$$

As to the determinant factors in Eq. (3.11), the first one can be written in a usual form:

$$(\det G^{-1})^{-1/2} = \exp\{-I_1(\Omega)\}, \hspace{1cm} I_1(\Omega) = \frac{1}{2} \int \frac{d^4 p \ln(p^2 + \Omega^2)}{(2\pi)^4},$$  \hspace{1cm} (3.15)

and the second one will be canceled by the term $(\det \mathcal{D}^{-1})^{-1/2}$ in Eq. (3.6). Thus substituting Eqs. (3.10), (3.11), (3.14) into (3.6) we come to the final expression for $\tilde{Z}[j,B]$ and $\tilde{W}[j,B]$:

$$\tilde{Z}[j,B] = \exp\{-\frac{1}{\hbar}[hI_1(\Omega) + v_0 + hv_2G_{xx} - 3v_4h^2G_{xx}^2]\} I_B,$$
$$\tilde{W}[j,B]/\hbar = \ln \tilde{Z}[j,B] = -I_1(\Omega) - v_0/\hbar - v_2G_{xx} + 3v_4hG_{xx}^2 + \ln I_B,$$  \hspace{1cm} (3.16)

with

$$I_B = \exp\{-\frac{1}{\hbar}(v_1 + 3v_3hG_{xx})\delta/\delta j\} \exp\{-\frac{1}{\hbar}(v_2\hat{A}^{(2)} + v_3\hat{A}^{(3)} + v_4\hat{A}^{(4)})\} \exp\{h jG j/2\}.$$  \hspace{1cm} (3.17)
The effective potential $V_{\text{eff}}$ is defined by

$$V_{\text{eff}}(B) = -\frac{\Gamma(B)}{\int d^4x}$$  \hspace{1cm} (3.18)$$

when $\Gamma(B)$ is independent of momenta. Using Eqs. (2.5)-(2.9) and (3.16)-(3.18) one can represent $V_{\text{eff}}$ as a sum of the GEP, $V_G$, and a correction $\Delta V_G$:

$$V_{\text{eff}}(B) = V_G(B) + \Delta V_G(B).$$  \hspace{1cm} (3.19)$$

The explicit expression for $\Delta V_G$ will be given below. Here we note some points.

1) Due to the special construction of primed derivatives (3.8), the coefficient of $I_B$ in Eq. (3.16) gives rise to the Gaussian effective action, so from Eqs. (2.9), (3.16), (3.18) we immediately obtain the GEP:

$$V_G(B) = h I_1(\Omega) + v_0(B) + v_2 h G_{xx} - 3 v_4 h^2 G^2_{xx}$$

$$= h I_1(\Omega) + v_0(B) + v_2 h I_0(\Omega) - 3 v_4 h^2 I^2_0(\Omega)$$

$$= h I_1(\Omega) + \frac{1}{2} m_0^2 B^2 + \lambda_0 B^4 + h I_0(\Omega)\left[\frac{1}{2} (m_0^2 - \Omega_0^2) + 6 \lambda_0 B^2\right] - 3 \lambda_0 h^2 I^2_0(\Omega)$$

$$= h I_1(\Omega) + \frac{1}{2} m_0^2 B^2 + \lambda_0 B^4 + h I_0(\Omega)\left[\frac{1}{2} (m_0^2 - \Omega^2) + 6 \lambda_0 B^2\right] - 3 \lambda_0 h^2 I^2_0(\Omega).$$  \hspace{1cm} (3.20)$$

Here we used the explicit expressions for $v_0, v_2, v_4$ given by Eqs. (3.5) and (3.13).

2) The correction to the GEP, that is, $\Delta V_G$, in Eq. (3.19) is given by $\ln I_B$. As it was explained in [6] the linear term in the exponent in Eq. (3.17) can be omitted, since it does not contribute to the effective potential. Thus the corrections to the GEP will be given by

$$\Delta V_G(B) = -h \ln I_B = -h \ln \left\{ \exp\left[ -\frac{\delta}{\hbar} (v_2 \dot{A}^{(2)} + v_3 \dot{A}^{(3)} + v_4 \dot{A}^{(4)}) \right] \exp\left[ \hbar j G_j / 2 \right] \right\}_{j=0}$$

$$= -h \ln \left\{ 1 - \frac{\delta (v_2 \dot{A}^{(2)} + v_3 \dot{A}^{(3)} + v_4 \dot{A}^{(4)}) \exp[\hbar j G_j / 2]}{\hbar} \right\}_{j=0} +$$

$$+ \frac{\delta^2 (v_2 \dot{A}^{(2)} + v_3 \dot{A}^{(3)} + v_4 \dot{A}^{(4)})^2 \exp[\hbar j G_j / 2]}{2 \hbar^2} \right\}_{j=0} -$$

$$- \frac{\delta^3 (v_2 \dot{A}^{(2)} + v_3 \dot{A}^{(3)} + v_4 \dot{A}^{(4)})^3 \exp[\hbar j G_j / 2]}{3 \hbar^3} \right\}_{j=0 + \ldots}$$

$$\equiv \delta \Delta V_G^{(1)}(B) + \delta^2 \Delta V_G^{(2)}(B) + \delta^3 \Delta V_G^{(3)}(B) + \ldots.$$  \hspace{1cm} (3.21)$$

Here we have introduced an auxiliary expansion parameter $\delta$ ($\delta = 1$).

The first order term $\Delta V_G^{(1)}(B)$ in this equation will not contribute to the effective potential, i.e., $\Delta V_G^{(1)}(B) = 0$, due to the relation, $\dot{A}_x^{(n)} \exp[\hbar j G_j / 2]_{j=0} = \hbar^n R_x^{(n)} \exp[\hbar j G_j / 2]_{j=0} = 0$ (see Eq. (3.9)).

3) As to the next term of the Eq. (3.21) of order $\delta^2$ the explicit calculations show that, only diagonal terms in the expression $(v_2 \dot{A}^{(2)} + v_3 \dot{A}^{(3)} + v_4 \dot{A}^{(4)})^2$ give a nonvanishing result at $j = 0$:
\[
(\hat{A}^{(2)})^2 \exp \{h jGj/2\} |_{j=0} = 2! \hbar^2 \int d^4 y G_{xy}^2 \\
(\hat{A}^{(3)})^3 \exp \{h jGj/2\} |_{j=0} = 3! \hbar^3 \int d^4 y G_{xy}^3 \\
(\hat{A}^{(4)})^4 \exp \{h jGj/2\} |_{j=0} = 4! \hbar^4 \int d^4 y G_{xy}^4 .
\]

Therefore, to \(\delta^2\) order we obtain the following expression for the first order correction to the Gaussian effective potential,

\[
\Delta V_{G}^{(2)}(B) = -\frac{\hbar}{2} [v_2^2 I_2(\Omega) + h v_3^2 I_3(\Omega) + \hbar^2 v_4^2 I_4(\Omega)] ,
\]

where in accordance with ref. [4], the following integrals are introduced:

\[
\frac{I_2(\Omega)}{2!} \equiv \int d^4 y G_{xy}^2 = \int \frac{d^4 p \ G^2(p)}{(2\pi)^4}, \\
\frac{I_3(\Omega)}{3!} \equiv \int d^4 y G_{xy}^3 = \int \frac{d^4 p \ d^4 q \ G(p) G(q) G(p+q)}{(2\pi)^8},
\]

\[
\frac{I_4(\Omega)}{4!} \equiv \int d^4 y G_{xy}^4 = \int \frac{d^4 p \ d^4 q \ d^4 k \ G(p) G(q) G(k) G(p+q+k)}{(2\pi)^{12}}, \\
G(p) = 1/(p^2 + \Omega^2).
\]

4) The expressions like \{\hat{A}^{(m)} \hat{A}^{(n)} \ldots \} \exp \{h jGj/2\} |_{j=0} can be calculated analytically by using Mathematica. The calculations show the following result in \(\delta^3\) order \(^2\):

\[
\hat{A}_x^{(2)} \hat{A}_y^{(2)} \hat{A}_z^{(2)} \exp \{h jGj/2\} |_{j=0} = 8G_{xy} G_{yz} G_{zx}, \\
\hat{A}_x^{(2)} \hat{A}_y^{(2)} \hat{A}_z^{(4)} \exp \{h jGj/2\} |_{j=0} = 24G_{xy}^2 G_{yz}^2, \\
\hat{A}_x^{(2)} \hat{A}_y^{(3)} \hat{A}_z^{(3)} \exp \{h jGj/2\} |_{j=0} = 36G_{xy}^2 G_{yz}^2 G_{zx}, \\
\hat{A}_x^{(2)} \hat{A}_y^{(4)} \hat{A}_z^{(4)} \exp \{h jGj/2\} |_{j=0} = 192G_{xy} G_{yz}^3 G_{zx}, \\
\hat{A}_x^{(3)} \hat{A}_y^{(3)} \hat{A}_z^{(4)} \exp \{h jGj/2\} |_{j=0} = 216G_{xy}^2 G_{yz}^2 G_{zx}, \\
\hat{A}_x^{(4)} \hat{A}_y^{(4)} \hat{A}_z^{(4)} \exp \{h jGj/2\} |_{j=0} = 1728G_{xy}^2 G_{yz} G_{zx}^2.
\]

Using these equations in (3.21) we get the second order correction (\(\delta^3\) terms) to the GEP,

\[
\Delta V_{G}^{(3)}(B) = \frac{4}{3} v_2^3 G_{xy} G_{yz} G_{zx} + 12v_2^2 v_3 G_{xy}^2 G_{yz}^2 + 18v_2 v_3^2 G_{xy} G_{yz}^2 G_{zx} + 96v_2^2 v_4 G_{xy}^3 G_{yz} G_{zx} + 108v_3^2 v_4 G_{xy}^2 G_{yz} G_{zx}^2 + 288v_3^2 v_4 G_{xy} G_{yz} G_{zx}^2 .
\]

We see that, correction to the GEP in \(\delta^2\), i.e. \(\Delta V_{G}^{(2)}(B)\) consists of only three Feynman diagrams shown in Fig. 1 (a), which we call BFM diagrams. In Fig.1(a) quadratic vertex (marked as \(\diamond\)) represents

\(^2\)All other combinations vanish.
On the other hand, $\Delta V_G^{(2)}(B)$ includes five diagrams found in ref. [4]. These two additional diagrams may be obtained by using (3.13) in (3.27) and further in (3.23). As a result one gets all diagrams of ref. [4] shown in Fig. 1b. One can see that the diagrams of ref. [4] can be obtained from the BFM diagrams by attaching a ring diagram with $I_0(\Omega) = \int d^4p/(2\pi)^4(p^2 + \Omega^2)$ to each $v_2$ vertex of a BFM diagram.

Six BFM diagrams in $\delta^3$ order given by Eq. (3.26), are presented in Fig. 2a. Now using Eqs. (3.13) and (3.27) in (3.26), i.e., by attaching a ring diagram, denoted by a small circle in Fig. 2a to each $v_2$ vertex, one gets seven additional diagrams as is shown in Fig. 2b. After this procedure $\Delta V_G^{(3)}(B)$ will be the same as the one given by Okopinska [3], represented by 13 diagrams. One may conclude that, the application of BFM technique to study corrections to the GEP makes the work much easier than in the formalism used in refs. [4,3]. To illustrate this we represent in Fig. 3 nine BFM diagrams including $v_2$ vertices in $\delta^4$ order. The implementation of a ring diagram, through Eqs. (3.13) and (3.27) gives additional 18 diagrams.

The effective potential in Eq. (3.23) is exactly the same as that given in refs. [5,6], except that the parameter $\Omega$ in Eq. (3.23) is not fixed by the Gaussian part. In fact in accordance with the principle of ”minimal sensitivity” $\Omega$ will be fixed by $dV_{eff}/d\Omega = 0$.

**IV. REGULARIZATION AND RENORMALIZATION**

It is well known that, the renormalization of any field theory can be achieved by introducing appropriate counter terms to the Lagrangian. In this section we apply this procedure to evaluate $V_{eff}$. To do this we rewrite the lagrangian (3.1) including counter terms [9,10]:

$$\mathcal{L} = \frac{1}{2} \phi(-\partial^2 + m^2)\phi + \frac{1}{2}Bm^2\phi^2 + C\lambda\phi^4 - Dm^4 \quad (4.1)$$

where $m$ and $\lambda$ represent renormalized mass and renormalized coupling constant, respectively, and make the following mass substitution:

$$m^2 = \Omega^2 - \delta(\Omega^2 - m^2) \equiv \Omega^2 - \delta\chi \quad (4.2)$$

where $\delta$ is a superficial parameter, well known in the delta expansion method. We then rewrite the Lagrangian in the same way as in $\delta$ expansion [9]

$$\mathcal{L} = \mathcal{L}_0 + \delta(\mathcal{L} - \mathcal{L}_0) \equiv \frac{1}{2}\phi(-\partial^2 + \Omega^2)\phi + \mathcal{L}_{int} \quad (4.3)$$

where

$$\mathcal{L}_{int} = -\frac{1}{2}\delta\chi\phi^2 + \delta\lambda\phi^4 + \mathcal{L}_{cnt} \quad (4.4)$$

and

$$\mathcal{L}_{cnt} = \frac{1}{2}\phi^2B(\Omega^2 - \delta\chi) + \lambda C\phi^4 - D(\Omega^2 - \delta\chi)^2.$$
The idea that the mass substitution (4.2) should be used not only in the standard mass term but also in the counter terms was first recognized in [11] and further developed in [10] where the necessity of the counter term $D$ was also pointed out.

We assume that the parameters $B$, $C$ and $D$ are expanded as

$$
B = B_1\delta + B_2\delta + O(\delta^3)
$$
$$
C = C_1\delta + C_2\delta^2 + O(\delta^3)
$$
$$
D = D_0 + D_1\delta + D_2\delta^2 + O(\delta^3) .
$$

Now using the method outlined in the previous sections one may get the following unrenormalized effective potential up to the $\delta^2$ order

$$
V_{eff}(\phi_0) = V_G(\phi_0) + \Delta V_G(\phi_0),
$$
$$
V_G(\phi_0) = I_1(\Omega) - D_0\Omega^4 + \delta(v_0 + v_2 I_0(\Omega) + 3v_4 I_0^2(\Omega))
$$
$$
\Delta V_G(\phi_0) = -\delta^2\left\{\frac{1}{2}(v_2 + 6v_4 I_0(\Omega))^2 I_2(\Omega) + \frac{1}{2}v_2^2 I_3(\Omega) + \frac{1}{2}v_2^2 I_4(\Omega)
- (u_0 + u_2 I_0(\Omega) + 3u_4 I_0^2(\Omega))\right\},
$$

where

$$
v_0 = \frac{m^2 \phi_0^2}{2} + \lambda \phi_0^4 + \frac{B_1 \Omega^2 \phi_0^2}{2} + C_1 \lambda \phi_0^4 + 2D_0 \Omega^2 \chi - D_1 \Omega^4,
$$
$$
v_2 = -\frac{1}{2}\chi + 6\lambda \phi_0^2 + \frac{B_1 \Omega^2}{2} + 6C_1 \lambda \phi_0^2 ,
$$
$$
v_3 = 4\lambda \phi_0 + 4\lambda \phi_0 C_1 ,
$$
$$
v_4 = \lambda + \lambda C_1 ,
$$
$$
u_0 = \frac{(B_2 \Omega^2 - B_1 \chi)\phi_0^2}{2} + \lambda C_2 \phi_0^4 - D_2 \Omega^4 + 2D_1 \Omega^2 \chi - D_0 \chi^2 ,
$$
$$
u_2 = \frac{(B_2 \Omega^2 - B_1 \chi)}{2} + 6\lambda C_2 \phi_0^2 ,
$$
$$
u_3 = 4\lambda \phi_0 C_2 ,
$$
$$
u_4 = \lambda C_2 ,
$$

and the divergent integrals $I_n(\Omega)$ calculated in the dimensional regularization method in $3 + 1$ dimensions are brought to the Appendix.

We shall evaluate the regularized effective potential \(^3\) order by order in powers of $\delta$.

It is easy to see that, the first divergent term of the GEP , coming from $I_1(\Omega)$ in Eq. (4.6) may be compensated by $D_0$ chosen as:

$$
D_0 = -\frac{1}{32\pi^2\varepsilon} .
$$

As it is seen from Eq. (4.1), possible finite part of the counter term $D$ leads only to shifting the effective potential as a whole and may be neglected. Using explicit expressions for divergent integrals in Eqs. (4.6) and (4.7)the GEP is given by:

$$
V_G(\phi_0) = \lambda(C_1 + 1)\phi_0^4 + M^{(2)}(\Omega)\phi_0^2 + M_{in}^{(0)}(\Omega) + M^{(0)}(\Omega)
$$

where

\(^3\)More exactly $V_{eff}(\phi_0, \Omega^2) - V_{eff}(0, m^2)$
\[ M^{(2)}(\Omega) = \frac{3\Omega^2 \lambda(C_1 + 1)\ln\left(\frac{\Omega^2}{\mu^2}\right)}{8\pi^2} + \frac{[3\lambda(C_1 + 1)(\gamma - 1) + 4B_1\pi^2]\Omega^2 + 4m^2\pi^2}{8\pi^2} \] (4.10)

and \( M^{(0)}(\Omega) \) will be given below. From the first term of Eq. (4.9) one may conclude that, \( C_1 \) should be finite. The requirement of finiteness of \( M^{(2)}(\Omega) \) in Eq. (4.10) leads to:

\[ B_1 = b_0 + \frac{3\lambda(C_1 + 1)}{2\pi^2 \varepsilon} \] (4.11)

where \( b_0 \), as well as \( C_1 \) are finite constant to be determined by suitably chosen renormalization condition. Fortunately, the choice of \( B_1 \) as in Eq. (4.11) will cancel also the logarithmic pole like \( \ln(\Omega^2/\mu^2)/\varepsilon \) coming from \( I_0^2(\Omega) \) in Eq. (4.6). The term \( M^{(0)}_{ln}(\Omega) \) in Eq. (4.9) includes the logarithmic part,

\[ M^{(0)}_{ln}(\Omega) = \frac{1}{256\pi^4} \left\{ \frac{\lambda(C_1 + 1)(\gamma - 1)}{8\pi^2} m^4\ln^2\left(\frac{m^2}{\mu^2}\right) - \frac{m^4}{\mu^2} \ln^2\left(\frac{\Omega^2}{\mu^2}\right) \right\} \] (4.12)

which is now finite, and the nonlogarithmic part

\[ M^{(0)}(\Omega) = \frac{1}{256\pi^4} \left\{ (\Omega^4 - m^4)(\gamma - 1)[3\lambda(C_1 + 1)(\gamma - 1) + 8b_0\pi^2] + 2\pi^2(3m^4 + \Omega^4) + 8m^2\Omega^2\pi^2(\gamma - 1) - 4\pi^2\gamma(m^4 + \Omega^4) + (m^4 - \Omega^4)[256\pi^4 D_1 + \frac{16b_0\pi^2}{\varepsilon} + \frac{12\lambda(C_1 + 1)}{\varepsilon^2}] \right\} + \frac{1}{8\pi^2}(\gamma - 1) \] (4.13)

which becomes finite when \( D_1 \) is fixed as:

\[ D_1 = -\frac{1}{16} b_0 - \frac{3}{64} \frac{\lambda(C_1 + 1)}{\pi^4 \varepsilon^2}. \] (4.14)

Now, after having removed all the singularities one comes to the following finite effective potential in \( \delta \) order,
\[ V^{(1)}_{\text{eff}}(\phi_0) = \frac{1}{2} m^2 \phi_0^2 + \lambda (C_1 + 1) \phi_0^4 + \frac{\Omega^2 \phi_0^2 \beta \lambda (C_1 + 1) \ln \left( \frac{\Omega^2}{\mu^2} \right) + 3 \lambda (C_1 + 1) + 4 b_0 \pi^2}{8 \pi^2} \]

\[ = \frac{3 \lambda (C_1 + 1) \left[ \Omega^2 \ln \left( \frac{\Omega^2}{\mu^2} \right) - m^2 \ln \left( \frac{m^2}{\mu^2} \right) \right]}{256 \pi^4} \]

\[ + \frac{1}{128 \pi^4} \left[ ((3 \lambda (C_1 + 1) + 4 b_0 \pi^2 - 2 \pi^2) \Omega^4 + 4 m^2 \pi^2 \Omega^2) \ln \left( \frac{\Omega^2}{\mu^2} \right) - m^4 ((3 \lambda (C_1 + 1) + 4 b_0 \pi^2) + 2 \pi^2 (3 m^4 + \Omega^4) \right] \]

\[ + \frac{1}{256 \pi^4} \left[ (\Omega^4 - m^4) (\gamma - 1) (3 \lambda (C_1 + 1) + 8 b_0 \pi^2) + 2 \pi^2 (3 m^4 + \Omega^4) \right] \]

\[ + 8 m^2 \Omega^2 \pi^2 (\gamma - 1) - 4 \pi^2 \gamma (m^4 + \Omega^4) \].

(4.15)

Note that, \( V^{(1)}_{\text{eff}}(\phi_0) \) includes two finite constants \( b_0 \) and \( C_1 \) to be determined by the suitably chosen renormalization conditions. If we employ the minimal subtraction (MS) scheme these constants may be neglected, since in the MS scheme the counter terms are supposed to remove only the singularities in \( \varepsilon \). On the other hand, they may be determined by an intermediate renormalization scheme [12]:

\[ \frac{d^2 V_{\text{eff}}(\phi_0)}{d \phi_0^2} |_{\phi_0 = 0} - m^2 = 0, \quad \frac{1}{4!} \frac{d^4 V_{\text{eff}}(\phi_0)}{d \phi_0^4} |_{\phi_0 = 0} - \lambda = 0, \]

which is commonly used for fixing the parameters of the effective potential. In this case they may naturally depend on \( \ln (m^2/\mu^2) \). From the first condition in Eq. (4.16) the finite counter term \( b_0 \) is easily determined as

\[ b_0 = -\frac{3}{4} \frac{\lambda (C_1 + 1) (\gamma + \ln \left( \frac{m^2}{\mu^2} \right) - 1)}{\pi^2}, \]

(4.17)

while \( C_1 \) can be determined in principle from the second condition of Eq. (4.16) which turned out to be a nonlinear equation. We shall come back to this point later.

Now passing to the regularization of \( V_{\text{eff}} \) in next order of \( \delta \) we rewrite it explicitly as:

\[ \Delta V_G(\phi_0) = K^{(4)}(\Omega) \phi_0^4 + [K_{\ln}^{(2)}(\Omega) + K^{(2)}(\Omega)] \phi_0^2 + K^{(0)}(\Omega) + K_{\ln}^{(0)}(\Omega) \]

(4.18)

where

\[ K^{(4)}(\Omega) = \frac{\lambda [9 \lambda (C_1 + 1)^2 (\gamma + \ln \left( \frac{\Omega^2}{\mu^2} \right) + 4 C_2 \pi^2)]}{4 \pi^2 (\gamma + \ln \left( \frac{\Omega^2}{\mu^2} \right) + 4 C_2 \pi^2) - 9 \lambda^2 (C_1 + 1)^2}{2 \pi^2 \varepsilon} \]

(4.19)
and \(K^{(2)}(\Omega), K^{(0)}(\Omega)\) will be given below. Finiteness of \(K^{(4)}(\Omega)\) determines \(C_2\) as

\[ C_2 = \tilde{C}_2 + \frac{9\lambda(C_1 + 1)^2}{2\pi^2\varepsilon} \tag{4.20} \]

where \(\tilde{C}_2\) is a finite constant. The term \(K^{(2)}(\Omega)\) is given by:

\[
K^{(2)}_{\ln}(\Omega) = \frac{45}{32} \lambda^2 \Omega^2 (C_1 + 1)^2 \ln^2\left(\frac{\Omega^2}{\mu^2}\right) \\
- \frac{3}{32\pi^4} \lambda \ln\left(\frac{\Omega^2}{\mu^2}\right) \left\{3\lambda(C_1 + 1)^2 \Omega^2 \ln\left(\frac{m^2}{\mu^2}\right) \right\} \tag{4.21}
\]

\[
+ \left[(36\lambda - 27\lambda\gamma)C_1^2 + (72\lambda + 4\pi^2 - 54\lambda\gamma)C_1 + 4\pi^2 + 36\lambda - 4\tilde{C}_2\pi^2 - 27\lambda\gamma\right] \Omega^2 \\
- 4\pi^2(C_1 + 1)m^2 \right\} - \frac{9}{8} \lambda^2 \Omega^2 (C_1 + 1)^2 \ln\left(\frac{\Omega^2}{\mu^2}\right) \pi^2 \varepsilon
\]

Here we see that there is a dangerous pole term proportional to \(\ln(\Omega^2/\mu^2)/\varepsilon\) which can be removed only by the condition

\[ C_1 = -1, \tag{4.22} \]

since there is no infinite counter term in (4.21) to compensate this pole. Now the nonlogarithmic term in (4.18), \(K^{(2)}(\Omega)\), is simply given by:

\[
K^{(2)}(\Omega) = \frac{1}{8} \left[4B_2\pi^2 + 3\lambda\tilde{C}_2(-1 + \gamma)\right] \Omega^2 - \frac{3\lambda\tilde{C}_2\Omega^2}{4\pi^2\varepsilon} \tag{4.23}
\]

and may be regularized by

\[ B_2 = b_2 + \frac{3\lambda\tilde{C}_2}{2\pi^2\varepsilon} \tag{4.24} \]

where \(b_2\) is a finite constant. One can see that the choice \(C_1 = -1\) will remove logarithmic pole terms in \(K^{(0)}_{\ln}(\Omega)\) also:

\[
K^{(0)}_{\ln}(\Omega) = -\frac{3}{256} \lambda\tilde{C}_2 \left[m^4\ln^2\left(\frac{m^2}{\mu^2}\right) - \Omega^4 \ln^2\left(\frac{\Omega^2}{\mu^2}\right)\right] \\
+ \frac{1}{128} \frac{(4\Omega^4 b_2\pi^2 + 3\lambda\tilde{C}_2\Omega^4\gamma - 3\lambda\tilde{C}_2\Omega^4 + 2\Omega^4\pi^2 + 2m^4\pi^2 - 4\Omega^2 m^2\pi^2)\ln\left(\frac{\Omega^2}{\mu^2}\right)}{\pi^4} \tag{4.25}
\]

\[-\frac{1}{128} \frac{m^4(4b_2\pi^2 - 3\lambda\tilde{C}_2 + 3\lambda\tilde{C}_2\gamma)\ln\left(\frac{m^2}{\mu^2}\right)}{\pi^4} \]
However, the nonlogarithmic term in Eq. (4.18), \( K^{(0)}(\Omega) \), includes a simple pole term:

\[
K^{(0)}(\Omega) = (m^4 - \Omega^4)[-8\pi^2(-1 + \gamma)b_2 - 3\lambda(-1 + \gamma)^2\tilde{C}_2 + 256D_2\pi^4] + 4\pi^2\gamma(m^2 - \Omega^2)^2
\]

\[\frac{1}{16} \frac{b_2(\Omega^4 - m^4)}{\pi^2\varepsilon} - \frac{3}{64} \frac{\lambda\tilde{C}_2(\Omega^4 - m^4)}{\pi^4\varepsilon^2}\]

(4.26)

which can be made finite by the choice of \( D_2 \) as

\[
D_2 = -\frac{1}{16} \frac{b_2}{\pi^2\varepsilon} - \frac{3}{64} \frac{\lambda\tilde{C}_2}{\pi^4\varepsilon^2}.
\]

(4.27)

Now the effective potential has been successfully regularized up to \( \delta^2 \) order and is given by

\[
\tilde{V}_{\text{eff}}^{(2)}(\tilde{\phi}_0) = \frac{1}{256\pi^4}(256\lambda\tilde{C}_2\tilde{\phi}_0^4\pi^4 + 32\pi^2(4\pi^2(1 + b_2x) + 3x\lambda\tilde{C}_2(\ln(x\bar{\mu}) + \gamma - 1))\tilde{\phi}_0^2
\]

\[+ 3\lambda\tilde{C}_2(x^2\ln(x\bar{\mu})^2 - \ln(\bar{\mu})^2) + (6\lambda\tilde{C}_2(1 - \gamma) - 4\pi^2(2b_2 + 1))\ln(\bar{\mu})
\]

\[+ ((6\gamma - 6)\lambda x^2\tilde{C}_2 + 4\pi^2(1 + 2b_2x^2))\ln(x\bar{\mu}) + 3\lambda(\gamma - 1)^2(x^2 - 1)\tilde{C}_2
\]

\[+ 2(x - 1)\pi^2(4b_2(\gamma - 1)(x + 1) + x - 3))
\]

(4.28)

where we introduced dimensionless parameters \( x, \bar{\mu}, \tilde{\phi}_0 \) defined by

\[
\Omega^2 = xm^2, \quad \mu^2 = m^2/\bar{\mu}, \quad \phi_0 = m\tilde{\phi}_0,
\]

(4.29)

and presented the effective potential in the units of \( m^4 \). The effective potential includes two extra parameters \( \Omega \) and \( \mu \). In practical calculations the former may be determined by the principle of minimal sensitivity (PMS) [13] or alternatively, by the fastest apparent convergence (FAC) conditions (see e.g. [11]) while the latter may be fixed by a renormalization scheme.

Below we shall use the PMS which is natural for the renormalization scheme given in Eqs. (4.16). In accordance with the PMS the optimum value of \( \Omega \) will be given by the equation, \( dV_{\text{eff}}/d\Omega = 0 \), so, differentiating (4.28) with respect to \( x \) leads to the following 'gap' equation:

\[
16\pi^2 x[4b_2\pi^2 + 3\lambda\tilde{C}_2\ln(x) + 3\lambda\tilde{C}_2\gamma + 3\lambda\tilde{C}_2\ln(\bar{\mu})]\tilde{\phi}_0^2
\]

\[+ 3\lambda\tilde{C}_2 x^2(\ln^2(\bar{\mu}) + \ln^2(x)) + 2\pi^2[2b_2x^2(2\gamma - 1) + (x - 1)^2]
\]

\[+ [6\lambda\tilde{C}_2 x^2\ln(\bar{\mu}) + (6x^2\lambda - 3x^2\lambda)\tilde{C}_2 + 8x^2b_2\pi^2]\ln(x)
\]

\[\quad + x^2(6\lambda\tilde{C}_2\gamma - 3\lambda\tilde{C}_2 + 8b_2\pi^2)\ln(\bar{\mu}) + 3\gamma x^2\lambda(\gamma - 1)\tilde{C}_2 = 0
\]

(4.30)
which determines $\Omega$ as a function of $\phi_0$. Note that, this dependence, or, more exactly, $d\Omega/d\phi_0$ should be explicitly used when calculating high order total derivatives of $V_{\text{eff}}$ with respect to $\phi_0$, for example, in Eqs. (4.16). The first condition in Eqs. (4.16) gives:

$$3\lambda \tilde{C}_2(\ln(\bar{\mu}) - 1 + \gamma) + 4b_2\pi^2 = 0$$

(4.31)

which determines $b_2$ as

$$b_2 = -\frac{3 \lambda \tilde{C}_2(\ln(\bar{\mu}) - 1 + \gamma)}{4\pi^2}$$

(4.32)

and the second condition

$$[2(\gamma + \ln(\bar{\mu}))^2 - 4\ln(\bar{\mu}) + 4 - 4\gamma] \tilde{C}_2 - 2(\gamma + \ln(\bar{\mu}))^2 - 1 + 4\gamma + 4\ln(\bar{\mu}) = 0$$

(4.33)

determines $\tilde{C}_2$ as:

$$\tilde{C}_2 = \frac{12(\gamma + \ln(\bar{\mu}))^2 - 4\ln(\bar{\mu}) - 4\gamma + 1}{2(\gamma + \ln(\bar{\mu}))^2 + 2 - 2\ln(\bar{\mu}) - 2\gamma}.$$ 

(4.34)

Now substituting this into (4.30) and considering the point $\phi_0 = 0$, $x = 1$, (i.e. $\Omega = m$ at $\phi_0 = 0$) we get :

$$2\ln^4(\bar{\mu}) + 8\lambda(-1 + \gamma)\ln^3(\bar{\mu}) + \lambda(-24\gamma + 12\gamma^2 + 11)\ln^2(\bar{\mu})$$

$$+ 2\lambda(-1 + \gamma)(2\gamma - 1)(2\gamma - 3)\ln(\bar{\mu}) + \lambda(2\gamma^2 - 4\gamma + 1)(-1 + \gamma)^2 = 0.$$ 

(4.35)

One can easily check that this equation is satisfied for

$$\ln(\bar{\mu}) = \ln(m^2/\mu^2) = 1 - \gamma$$

(4.36)

which leads to the following values for the finite counter terms:

$$b_2 = 0, \quad \tilde{C}_2 = -\frac{1}{2}.$$ 

(4.37)

Finally, the renormalized effective potential up to $\delta^2$ order is given by:

$$\tilde{V}_{\text{eff}}^{(2)}(\bar{\phi}_0) = -\frac{1}{2}\lambda \bar{\phi}_0^4 + \frac{[-3x\lambda\ln(x) + 3x\lambda + 16\pi^2] \bar{\phi}_0^2}{32\pi^2}$$

$$- \frac{8\pi^2(x - 1) - \ln(x)(3x^2\lambda + 8\pi^2)}{512\pi^4},$$

(4.38)

with $x$ being the solution of the gap equation,

$$3x\lambda[\ln(x) + 1] [\ln(x) + 16\bar{\phi}_0^2\pi^2] - 4\pi^2(x - 1)^2 = 0$$

(4.39)

Comparing Eqs. (4.38), (4.39) with the results of the ”precarious” version of renormalization obtained by Stancu and Stevenson (Eqs. (3.30) and (3.21) of [4]), one can see that their mathematical forms are different. One of the reasons of this difference is that, in ref. [4] the finite part of two and three loop integrals $I_3$ and $I_4$ were not given while in the present work we used the explicit form of the finite part (see Appendix). The second reason may be hidden in the fact that, to make the effective potential finite the most authors use the procedure ”regularization after renormalization” while in the present work we first regularized the potential, i.e. made it finite, and then applied the renormalization conditions - Eqs. (4.16). Note that, the necessity that the regularization should be carried out before imposing the gap equation was pointed out also in refs. [10,14].
V. CONCLUSIONS

We have developed a systematic method for computing the correction terms to the Gaussian effective action. They are given by the expansion of:

\[
\exp\{-\frac{\delta}{\hbar}[v_2 \hat{A}^{(2)} + v_3 \hat{A}^{(3)} + v_4 \hat{A}^{(4)}]\} \exp \{\hbar \dot{g} / 2\}
\]

in powers of \(\delta\). Although our final result for unrenormalized PGEP is the same as that of refs. [3] and [4], the method proposed in the present work has some advantages.

1) In refs. [3,4] the expansion parameter \(\delta\) is introduced directly into the Lagrangian at the very beginning of calculations. Instead, by using BFM we preliminary extracted the Gaussian part of the effective potential and then introduced an expansion parameter \(\delta\) in Eq. (3.17). This seems to be a more natural way, since in accordance with general rules of quantum mechanics exponent of any operator \(\hat{A}\) should be understood as a Taylor expansion, i.e.: \(\exp(\hat{A}) = \{1 + \delta \hat{A} + (\delta \hat{A})^2 / 2! + (\delta \hat{A})^3 / 3! + \ldots\}|_{\delta=1}\).

2) In the present approach Gaussian part of the effective action is extracted by introducing the primed derivatives (3.8). We have shown that, in our method the computation of correction terms to the GEP is greatly simplified. In fact, at each order of \(\delta\) it gives BFM diagrams (three in \(\delta^2\) order and six in \(\delta^3\) order) with a quadratic vertex \(v_2 = \frac{1}{2}(m_0^2 - \Omega_0^2) + 6\lambda_0 \phi^2\), cubic vertex \(4\lambda_0 \phi\) and quartic vertex \(\lambda_0\). These diagrams do not include a ring diagram \((G_{xx} I_0)\) explicitly, which is included there through the relation \(\Omega_0^2 = \Omega^2 - 12\bar{h}\lambda_0 I_0(\Omega)\), giving rise to the ones (five in \(\delta^2\) order and 13 in \(\delta^3\) order) appearing in the methods used in ref.s [3,4].

3) In the present method it is easier to solve equations (2.7) -(2.9) of BFM than similar equations in the method used in [4]. In fact, in the method of refs. [3,4] one has to solve the equations \(\Gamma[\phi] = W[J] - J\phi, \phi = \delta W / \delta J\) to define \(\phi\) and \(J\) at each order of \(\delta\). Although the corresponding solutions are simple in \(\delta^2\) order \(^4\), \(J = \delta(v_1 + 3v_0 I_0) + O(\delta^2)\), they are very complicated at higher orders. On the other hand, we have shown that due to the advantages of BFM one may simply put \(J = 0\) at each order of \(\delta\). Thus using the method proposed in the present paper it will be much simpler to calculate higher order corrections to GEP in general.

We have shown that the variational perturbation expansion of \(\phi^4\) scalar theory in four dimensions can be successfully renormalized by introducing appropriate counter terms in such a way that the whole Lagrangian has the same polynomial form as the bare one.

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\(^4\) Eq. (2.29) of ref. [4].
APPENDIX: EXPLICIT EXPRESSION FOR DIVERGENT INTEGRALS.

Here we bring explicit expressions for the divergent integrals defined as:

\[ I_0(\Omega) = \int \frac{d^4p}{(2\pi)^4(p^2 + \Omega^2)} , \quad I_1(\Omega) = \frac{1}{2} \int \frac{d^4p \ln(p^2 + \Omega^2)}{(2\pi)^4} , \]

\[ I_2(\Omega) = \frac{1}{2!} \int \frac{d^4p G^2(p)}{(2\pi)^4} , \quad G(p) = 1/(p^2 + \Omega^2) , \]

\[ I_3(\Omega) = \frac{1}{3!} \int \frac{d^4p d^4q G(p)G(q)G(p + q)}{(2\pi)^8} , \]

\[ I_4(\Omega) = \frac{1}{4!} \int \frac{d^4p d^4q d^4k G(p)G(q)G(k)G(p + q + k)}{(2\pi)^{12}} , \]

in four dimension. Integrals \( I_0(\Omega) \), \( I_1(\Omega) \) and \( I_2(\Omega) \) are quite simple and may be found elsewhere \[8\].

\[ I_0(\Omega) = -\frac{1}{8} \frac{\Omega^2}{\pi^2 \varepsilon} + \frac{\Omega^2[-1 + \gamma_0 + \ln(\frac{1}{4} \frac{\Omega^2}{\mu^2 \pi^2})]}{16\pi^2} \]

\[ I_2(\Omega) = \frac{1}{4} \frac{\Omega^2}{\pi^2 \varepsilon} - \frac{\gamma_0 + \ln(\frac{1}{4} \frac{\Omega^2}{\mu^2 \pi^2})}{\pi^2} \]

\[ I_1(\Omega) = I_1(m) + \frac{1}{2}(\Omega^2 - m^2)I_0(m) - \frac{1}{8}(\Omega^2 - m^2)^2I_2(m) \]

\[ + \frac{m^4[2x^2\ln(x) - 2x + 2 - 3(x - 1)^2]}{128\pi^2} \]

where \( x = \Omega^2/m^2 \), \( \varepsilon \) is defined as \( d = 4 - \varepsilon \), \( d \)-space - time dimension, \( \gamma_0 = 0.5777215 \) and \( \mu \) is an arbitrary constant with mass dimension which usually appears in dimensional regularization scheme.

The two loop \( I_3(\Omega) \) and three loop integrals \( I_4(\Omega) \) have been accurately calculated in refs. \[15\] including finite parts:
\[ I_3(\Omega) = -\frac{3}{256\pi^2} \Omega^2 \left( 24 \frac{1}{\varepsilon^2} - 24\gamma_0 - 24 \ln\left(\frac{1}{4\pi\mu^2}\right) + 36 \langle \frac{1}{4\pi\mu^2} \rangle + 12 \ln^2\left(\frac{1}{4\pi\mu^2}\right) \right. \]
\[ + (24\gamma_0 - 36) \ln\left(\frac{1}{4\pi\mu^2}\right) + 12\bar{A} - 36\gamma_0 + 12\gamma_0^2 + \pi^2 + 42 \right) \]

\[ I_4(\Omega) = \frac{3}{512\pi^6} \Omega^4 \left( 16 \frac{1}{\varepsilon^3} + \frac{92}{3} - 24\gamma_0 - 24 \ln\left(\frac{1}{4\pi\mu^2}\right) \right. \]
\[ + \frac{35 - 46\gamma_0 + 18\gamma_0^2 + \pi^2 + (36\gamma_0 - 46) \ln\left(\frac{1}{4\pi\mu^2}\right) + 18 \ln^2\left(\frac{1}{4\pi\mu^2}\right) \right. \]
\[ \left. + \ln\left(\frac{1}{4\pi\mu^2}\right) \left( -\frac{105}{2} + 69\gamma_0 - 27\gamma_0^2 - \frac{3}{2} \pi^2 \right) + \ln^2\left(\frac{1}{4\pi\mu^2}\right) \left( \frac{69}{2} - 27\gamma_0 \right) - 9 \ln^3\left(\frac{1}{4\pi\mu^2}\right) \right) \]

where \( \bar{A} = -1.171953 \). Note that in the main part of the text we used notation \( \mu^2 \to 4\pi\mu^2 \).
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Fig. 1(a). BFM Feynman diagrams contributing to the effective potential in $\delta^2$ order (see Eq. (3.23)). Solid lines represent the propagator $G(p) = 1/(p^2 + \Omega^2)$. The vertices $v_2 = (m_0^2 - \Omega_0^2)/2 + 6\lambda_0\phi^2$, $v_3 = 4\lambda_0\phi$ and $v_4 = \lambda_0$ are marked by diamonds, circles and squares, respectively.

Fig.1(b). Feynman diagrams contributing to the effective potential in $\delta^2$ order in the method of refs. [3,4] (see Eq. (4.1)). These are obtained by an attachment of the ring diagram, $I_0$, denoted here by the small circle, to the each two body vertex $v_2$ of the FIG.1(a). Solid lines represent the propagator $G(p) = 1/(p^2 + \Omega^2)$. The vertices $u_2 = (m_0^2 - \Omega_0^2)/2 + 6\lambda_0\phi^2$, $v_3 = 4\lambda_0\phi$, $v_4 = \lambda_0$ and $u_4 = 6\lambda_0$ are marked by diamonds, circles, filled and opened squares, respectively.

Fig.2(a). The same as in FIG.1(a). but in $\delta^3$ order.

Fig.2(b). Feynman diagrams contributing to the effective potential in $\delta^3$ order of ref. [3]. These are obtained by attachment of a ring diagram, $I_0$, denoted here by the small circle, to the each two body vertex $v_2$ of the Fig.2(a). The notations are the same as in FIG.1(b).

Fig.3. The same as in FIG.1(a) but in $\delta^4$ order. Note that, only those, including quadratic vertex $v_2$ are presented. Implementation of a ring diagram to each $v_2$ vertex with all possible ways will give 18 additional diagrams.
FIG. 2a.
FIG. 3.