United Nations Educational Scientific and Cultural Organization
and
International Atomic Energy Agency
THE ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

**ON A p -LAPLACE NEUMANN PROBLEM WITH
ASYMPTOTICALLY ASYMMETRIC PERTURBATIONS**

Mohssine Alif¹

The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy

and

Pierpaolo Omari²

*Dipartimento di Scienze Matematiche, Università di Trieste,
Via A. Valerio 12/1, I-34127 Trieste, Italy.*

Abstract

We study the solvability of the quasilinear elliptic problem

$$\begin{cases} -\Delta_p u = f(u) - h(x) & \text{in } \Omega, \\ \partial u / \partial \nu = 0 & \text{on } \partial \Omega, \end{cases}$$

in the case where f lies in some sense below the first nontrivial branch of the Dancer–Fučík spectrum Σ of the Neumann p -Laplacian. The interaction with Σ is controlled imposing a condition on a primitive of f and assuming the existence of upper and lower solutions without any special ordering.

MIRAMARE – TRIESTE

March 2001

¹E-mail: malif@ictp.trieste.it

²E-mail: omari@univ.trieste.it

1 Introduction and statements

Let us consider the elliptic problem

$$\begin{cases} -\Delta_p u = f(u) - h(x) & \text{in } \Omega, \\ \partial u / \partial \nu = 0 & \text{on } \partial \Omega. \end{cases} \quad (1.1)$$

Throughout we assume that $\Omega (\subset \mathbb{R}^N)$ is a bounded domain, with a $C^{1,1}$ boundary $\partial \Omega$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the p -Laplacian, with $p > 1$, $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function and $h \in L^\infty(\Omega)$. By a solution of (1.1) we mean a function $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, such that

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla w = \int_{\Omega} (f(u) - h) w,$$

for every $w \in W^{1,p}(\Omega)$. Known regularity results [15] then imply that $u \in C^{1,\sigma}(\overline{\Omega})$, for some $\sigma > 0$.

The aim of this paper is to improve and extend to the p -Laplacian, with $p > 1$ arbitrary, a result obtained in [11] for the Laplacian, in the case where the function f exhibits an asymmetric behaviour at infinity and may interact with the Dancer–Fučík spectrum of the involved differential operator. This is made possible also in the light of two recent contributions in this area given in [5] and in [7], respectively. We recall that the Dancer–Fučík spectrum Σ of the Neumann p -Laplacian is defined as the set of all $(\lambda_+, \lambda_-) \in \mathbb{R}^2$ such that

$$\begin{cases} -\Delta_p u = \lambda_+ |u|^{p-2} u^+ - \lambda_- |u|^{p-2} u^- & \text{in } \Omega, \\ \partial u / \partial \nu = 0 & \text{on } \partial \Omega \end{cases} \quad (1.2)$$

admits a nontrivial solution. In [5] (see also [3], for further information about the Neumann problem) it was proved that Σ contains, besides the trivial lines $\mathbb{R} \times \{\lambda_1\}$ and $\{\lambda_1\} \times \mathbb{R}$, $\lambda_1 = 0$ being the principal eigenvalue, a first nontrivial curve $\mathcal{C}$ which passes through (λ_2, λ_2) , λ_2 denoting the second eigenvalue. This curve is continuous, decreasing and symmetric with respect to the diagonal. Moreover, if $p \leq N$ it is asymptotic to the lines $\mathbb{R} \times \{\lambda_1\}$ and $\{\lambda_1\} \times \mathbb{R}$, while if $p > N$ is asymptotic to the lines $\mathbb{R} \times \{\bar{\lambda}\}$ and $\{\bar{\lambda}\} \times \mathbb{R}$, where $\bar{\lambda} > 0$ is defined by

$$\bar{\lambda} = \inf \left\{ \int_{\Omega} |\nabla u|^p \mid u \in W^{1,p}(\Omega), \int_{\Omega} |u|^p = 1, \min u = 0 \right\}. \quad (1.3)$$

On the other hand, in [7, Th. 8.2] the solvability of the Dirichlet problem for a quasilinear equation involving the p -Laplacian was proved in the presence of a pair of upper and lower solutions, possibly without any order. In our main result (Theorem 1.1 below) the interaction of f with the principal eigenvalue λ_1 is controlled by means of a pair of upper and lower solutions and the interaction with the first nontrivial branch $\mathcal{C}$ by an asymptotic condition similar to that considered in [16] and [11] and involving a primitive F of f .

In this frame the following definition of upper and lower solutions is adopted. We say that a function $\alpha \in W^{1,p}(\Omega) \cap C^0(\overline{\Omega})$ is a lower solution of (1.1) if

$$\int_{\Omega} |\nabla \alpha|^{p-2} \nabla \alpha \nabla w \leq \int_{\Omega} (f(\alpha) - h) w, \quad (1.4)$$

for every $w \in W^{1,p}(\Omega)$, with $w(x) \geq 0$ a.e. in Ω . An upper solution β is defined similarly by reversing the inequality in (1.4).

Theorem 1.1 *Let $(\lambda_+, \lambda_-) \in \mathcal{C}$ be given. Assume*

$$\limsup_{s \rightarrow -\infty} \frac{f(s)}{|s|^{p-2}s} \leq \lambda_- \quad \text{and} \quad \limsup_{s \rightarrow +\infty} \frac{f(s)}{|s|^{p-2}s} \leq \lambda_+ \quad (1.5)$$

and

$$\liminf_{s \rightarrow -\infty} \frac{pF(s)}{|s|^p} < \lambda_- \quad \text{or} \quad \liminf_{s \rightarrow +\infty} \frac{pF(s)}{|s|^p} < \lambda_+. \quad (1.6)$$

Moreover, suppose that there exist a lower solution α and an upper solution β . Then, problem (1.1) has at least one solution.

We point out that one of the difficulties, which must be overcome for proving Theorem 1.1, consists in the extension to the Neumann problem of Lemma 5.3 in [5], which is given in Lemma 2.3 below. We feel that this result has an independent interest. Indeed, the arguments developed in its proof might be used also in different contexts, like for instance to carry over to the Neumann problem Theorem 5.1 in [5], or Theorem 2.1 in [6].

The simplest way of applying Theorem 1.1 is to look for constant upper and lower solutions. It is obvious that any $\alpha \in \mathbb{R}$ such that $f(\alpha) \geq \text{ess sup } h$ is a lower solution and any $\beta \in \mathbb{R}$ such that $f(\beta) \leq \text{ess inf } h$ is an upper solution. Accordingly, we immediately get the following characterization of nonresonance, which generalizes [11, Th. 1.1] to the quasilinear setting.

Corollary 1.1 *Assume (1.5) and (1.6). Then, problem (1.1) has at least one solution for any given h if and only if the range of f is $\mathbb{R}$.*

Another condition which implies the existence of upper and lower solutions is the following Landesman–Lazer condition: *there exists $r > 0$ such that*

$$f(s) \text{sign}(s) \geq \frac{1}{\text{meas } \Omega} \int_{\Omega} h \quad \text{for every } s, \text{ with } |s| \geq r. \quad (1.7)$$

We stress that, when compared with the semilinear case ($p = 2$), already discussed for instance in [13], [12], [7], the construction of upper and lower solutions using (1.7) appears in the quasilinear setting rather more delicate (cf. Lemma 2.3 below).

Using (1.7) we obtain the following corollary, which yields a partial extension of Theorem 1.3 in [2] to unbounded nonlinearities.

Corollary 1.2 *Assume (1.5), (1.6) and (1.7). Then, problem (1.1) has at least one solution.*

Now, we confine ourselves to the case $p > N$ and briefly consider the situation where the two-sided control imposed by (1.5) and (1.6) is replaced by the following one-sided condition

$$\limsup_{s \rightarrow -\infty} \frac{f(s)}{|s|^{p-2}s} < \bar{\lambda} \quad \text{or} \quad \limsup_{s \rightarrow +\infty} \frac{f(s)}{|s|^{p-2}s} < \bar{\lambda}, \quad (1.8)$$

$\bar{\lambda}$ being defined by (1.3).

Proposition 1.1 *Let $p > N$ be and assume (1.8). Moreover, suppose that there exist a lower solution α and an upper solution β . Then, problem (1.1) has at least one solution.*

This statement yields a partial extension of Theorem 7 in [18]. Note that in Proposition 1.1 no condition at all is assumed on f at $+\infty$, or at $-\infty$. It would be interesting to obtain analogous results if $p \leq N$. At the moment we are able to do this only in the semilinear case ($p = 2$), or when Ω is a ball and h is radially symmetric. In this frame, condition (1.8) must however be replaced by the stronger assumption

$$\limsup_{s \rightarrow -\infty} \frac{f(s)}{|s|^{p-2}s} \leq 0 \quad \text{or} \quad \limsup_{s \rightarrow +\infty} \frac{f(s)}{|s|^{p-2}s} \leq 0$$

and a further growth restriction on f involving the “critical” exponent $\frac{N(p-1)}{N-p}$ is required. More general results can be obtained in the one-dimensional case ($N = 1$) on the line of [10].

2 Proofs

We start with some preliminary results. The first concerns the solvability of the problem

$$\begin{cases} -\Delta_p u = g(x, u) & \text{in } \Omega, \\ \partial u / \partial \nu = 0 & \text{on } \partial \Omega, \end{cases} \quad (2.1)$$

with $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ a L^∞ -Carathéodory function, in the presence of upper and lower solutions without any special ordering. This is essentially an adaptation to the Neumann problem of [7, Th. 8.2].

Lemma 2.1 *Assume that, for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$,*

$$g(x, s) = -q(x, s)|s|^{p-2}s^- + r(x, s), \quad (2.2)$$

where $q, r : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ are L^∞ -Carathéodory functions satisfying, for some constant $M > 0$,

$$0 \leq q(x, s) \leq M \quad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbb{R} \quad (2.3)$$

and

$$\lim_{|s| \rightarrow +\infty} \frac{r(x, s)}{|s|^{p-1}} = 0 \quad \text{uniformly a.e. in } \Omega. \quad (2.4)$$

Moreover, suppose that there exist a lower solution α and an upper solution β of (2.1) such that

$$\min(\beta - \alpha) < 0.$$

Then, problem (2.1) has at least one solution $u \in C^{1,\sigma}(\overline{\Omega})$ satisfying

$$\min(u - \alpha) \leq 0 \quad \text{and} \quad \max(u - \beta) \geq 0.$$

Proof of Lemma 2.1. We begin observing that, for each $h \in L^\infty(\Omega)$, there exists a unique solution $u \in C^{1,\sigma}(\overline{\Omega})$, with $\sigma > 0$ independent of h , of the problem

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = h(x) & \text{in } \Omega, \\ \partial u / \partial \nu = 0 & \text{on } \partial \Omega, \end{cases} \quad (2.5)$$

which further satisfies $-\|h^-\|_\infty^{\frac{1}{p-1}} \leq u(x) \leq \|h^+\|_\infty^{\frac{1}{p-1}}$ in Ω . (The easy proof of these facts is omitted.)

Hence, we can define a solution operator $T : C^0(\overline{\Omega}) \rightarrow C^0(\overline{\Omega})$ associated to problem (2.1), which sends any function $u \in C^0(\overline{\Omega})$ onto the unique solution $v \in C^{1,\sigma}(\overline{\Omega})$ of

$$\begin{cases} -\Delta_p v + |v|^{p-1}v = g(x, u) & \text{in } \Omega, \\ \partial v / \partial \nu = 0 & \text{on } \partial \Omega. \end{cases}$$

The regularity theory and the compact embedding of $C^{1,\sigma}(\overline{\Omega})$ into $C^1(\overline{\Omega})$ imply that T is completely continuous. Moreover, the fixed points of T are precisely the solutions of (2.1).

In the sequel, we say that a lower solution α is strict if there is no solution u such that $\min(u - \alpha) = 0$. Similarly, an upper solution β is said to be strict if there is no solution u such that $\max(u - \beta) = 0$.

Claim 1. Assume that γ and δ are strict lower and upper solutions of (2.1), satisfying $\min(\delta - \gamma) > 0$, and introduce the open set

$$\mathcal{O}_{\gamma,\delta} = \{u \in C^0(\overline{\Omega}) \mid \min(u - \gamma) > 0 \text{ and } \max(u - \delta) < 0\}.$$

Then, $\deg(I - T, \mathcal{O}_{\gamma,\delta}) = 1$ and, hence, problem (2.1) has at least one solution $u \in \mathcal{O}_{\gamma,\delta} \cap C^{1,\sigma}(\overline{\Omega})$.

The proof of this result follows standard lines; we refer e.g. to [7, Th. 8.1] where the case of a quasilinear elliptic problem, involving the Dirichlet p -Laplacian, is discussed.

Now, we are in position to prove Lemma 2.1. Let us define, for each n , a L^∞ -Carathéodory function $g_n : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ by setting, for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$,

$$g_n(x, s) = \begin{cases} 1 & \text{if } s \leq -n - 1, \\ (s + n + 1)g(x, s) - (s + n) & \text{if } -n - 1 < s < -n, \\ f(x, s) & \text{if } |s| \leq n, \\ (n + 1 - s)g(x, s) - (s - n) & \text{if } n < s < n + 1, \\ -1 & \text{if } s \geq n + 1. \end{cases}$$

Testing against $(u - n - 1)^+$ and $(u + n + 1)^-$, it is easily seen that, for each n , any solution u of the problem

$$\begin{cases} -\Delta_p u = g_n(x, u) & \text{in } \Omega, \\ \partial u / \partial \nu = 0 & \text{on } \partial \Omega \end{cases} \quad (2.6)$$

satisfies $\|u\|_\infty \leq n + 1$. Hence, it follows in particular that any constant $A < -n - 1$ is a strict lower solution and any constant $B > n + 1$ is a strict upper solution of (2.6).

Claim 2. There exist an integer $\bar{n}$ and a constant R such that, for every $n \geq \bar{n}$, any solution u of (2.6), with $\min(u - \alpha) \leq 0$ and $\max(u - \beta) \geq 0$, satisfies

$$\|u\|_\infty < R. \quad (2.7)$$

Proof of Claim 2. Assume, by contradiction, that for each k there is an integer n_k and a solution u_k of (2.6), with $n = n_k$, such that

$$\min(u_k - \alpha) \leq 0 \quad \text{and} \quad \max(u_k - \beta) \geq 0 \quad (2.8)$$

and

$$\|u_k\|_\infty \geq k.$$

Define, for each k , the function

$$v_k = \frac{u_k}{\|u_k\|_\infty},$$

which satisfies

$$\int_{\Omega} |\nabla v_k|^{p-2} \nabla v_k \nabla w = \int_{\Omega} \frac{g_{n_k}(x, u_k)}{\|u_k\|_\infty^{p-1}} w$$

for every $w \in W^{1,p}(\Omega)$. We can write

$$g_{n_k}(x, s) = -q_k(x, s)(s^-)^{p-1} + r_k(x, s)$$

for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$, where q_k and r_k are L^∞ -Carathéodory functions satisfying

$$0 \leq q_k(x, s) \leq M$$

for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$ and

$$\lim_{|s| \rightarrow +\infty} \frac{r_k(x, s)}{|s|^{p-1}} = 0$$

uniformly with respect to k and to a.e. $x \in \Omega$. Since the sequence $\left(\frac{g_{n_k}(\cdot, u_k)}{\|u_k\|_\infty^{p-1}} \right)_k$ is uniformly bounded in $L^\infty(\Omega)$, by the regularity theory and the compact embedding of $C^{1,\sigma}(\bar{\Omega})$ into $C^1(\bar{\Omega})$, we have, possibly passing to a subsequence, that $(v_k)_k$ converges in $C^1(\bar{\Omega})$ to some function v , which satisfies $\|v\|_\infty = 1$ and

$$\int_{\Omega} |\nabla v|^{p-2} \nabla v \nabla w = - \int_{\Omega} q_0 (v^-)^{p-1} w \quad (2.9)$$

for every $w \in W^{1,p}(\Omega)$. Here, $q_0 \in L^\infty(\Omega)$ is the weak* limit, possibly of a subsequence, of $(q_k(\cdot, u_k))_k$, with

$$0 \leq q_0(x) \leq M \quad \text{for a.e. } x \in \Omega.$$

Now, we want to prove that v is constant in Ω . Testing (2.9) against $w = -1$, we get $\int_{\Omega} q_0 (v^-)^{p-1} = 0$, which implies that $v^-(x) = 0$ on the set where $q_0(x) > 0$. Testing against

$w = v$, we obtain $\int_{\Omega} |\nabla v|^p = -\int_{\Omega} q_0(v^-)^p = 0$ and hence $\nabla v = 0$ in Ω . Therefore, we conclude that $v = 1$ or $v = -1$ in Ω . Accordingly, we have that

$$u_k(x) \rightarrow +\infty \quad \text{or} \quad u_k(x) \rightarrow -\infty$$

uniformly in Ω . This contradicts (2.8). Hence, Claim 2 is proved. $\blacksquare$

Claim 3. There exists an integer $\bar{n}$ such that, for every $n > \bar{n}$, problem (2.6) has at least one solution u_n satisfying

$$\min(u_n - \alpha) \leq 0 \quad \text{and} \quad \max(u_n - \beta) \geq 0.$$

Proof of Claim 3. Let us define the open set

$$\mathcal{O} = \{u \in C^0(\bar{\Omega}) \mid \min(u - \alpha) < 0 \quad \text{and} \quad \max(u - \beta) > 0\}.$$

Let $\bar{n}$ be an integer satisfying $\bar{n} > \max\{\|\alpha\|_{\infty}, \|\beta\|_{\infty}\}$. Fix any integer $n > \bar{n}$ and set $B = n + 2$ and $A = -B$. As already observed, A and B are strict lower and upper solutions of (2.6). Assume also that there is no solution u of (2.6) with $u \in \partial\mathcal{O}$, or equivalently that α and β are strict lower and upper solutions of (2.6); indeed, otherwise we are done. We have

$$\mathcal{O} \cap \mathcal{O}_{A,B} = \mathcal{O}_{A,B} \setminus (\bar{\mathcal{O}}_{A,\beta} \cup \bar{\mathcal{O}}_{\alpha,B})$$

and, by Claim 1,

$$\begin{aligned} \deg(I - T_n, \mathcal{O} \cap \mathcal{O}_{A,B}) &= \deg(I - T_n, \mathcal{O}_{A,B}) - \deg(I - T_n, \mathcal{O}_{A,\beta}) \\ &\quad - \deg(I - T_n, \mathcal{O}_{\alpha,B}) = -1. \end{aligned}$$

Here, $T_n : C^0(\bar{\Omega}) \rightarrow C^0(\bar{\Omega})$ is the solution operator associated to problem (2.6), which sends any function $u \in C^0(\bar{\Omega})$ onto the unique solution $v \in C^{1,\sigma}(\bar{\Omega})$ of

$$\begin{cases} -\Delta_p v + |v|^{p-1}v = g_n(x, u) & \text{in } \Omega, \\ \partial v / \partial \nu = 0 & \text{on } \partial\Omega. \end{cases}$$

Accordingly, there exists at least one solution $u_n \in \mathcal{O}$ of (2.6) for each $n > \bar{n}$. This concludes the proof of Claim 3. $\blacksquare$

By Claim 2, we have that, for each $n > \max\{\bar{n}, R\}$, every solution $u_n \in \bar{\mathcal{O}}$ of (2.6) is a solution of (2.1) too. Hence, Claim 3 yields the conclusion of Lemma 2.1. $\blacksquare$

The second preliminary result extends to the Neumann problem Lemma 5.3 in [5], stated and proved in the Dirichlet case. We stress that, although the idea of the proof is substantially the same as in [5], there is an essential step, involving the consideration of a mixed problem on a nonsmooth domain, which requires some care.

Lemma 2.2 *Let $(\lambda_+, \lambda_-) \in \mathcal{C}$ be given and let $p_0, q_0 \in L^\infty(\Omega)$ satisfy*

$$0 \leq p_0(x) \leq \lambda_+ \quad \text{and} \quad 0 \leq q_0(x) \leq \lambda_- \quad \text{a.e. in } \Omega. \quad (2.10)$$

Suppose that

$$\text{meas} \{x \in \Omega \mid p_0(x) > 0\} > 0 \quad \text{and} \quad \text{meas} \{x \in \Omega \mid q_0(x) > 0\} > 0. \quad (2.11)$$

Then, any nontrivial solution u of

$$\begin{cases} -\Delta_p u = p_0(x)|u|^{p-2}u^+ - q_0(x)|u|^{p-2}u^- & \text{in } \Omega, \\ \partial u / \partial \nu = 0 & \text{on } \partial\Omega \end{cases} \quad (2.12)$$

changes sign in Ω and

$$p_0(x) = \lambda_+ \quad \text{a.e. on } \{x \in \Omega \mid u(x) > 0\} \quad (2.13)$$

and

$$q_0(x) = \lambda_- \quad \text{a.e. on } \{x \in \Omega \mid u(x) < 0\} \quad (2.14)$$

(i.e. u is an eigenfunction associated to (λ_+, λ_-)).

Proof of Lemma 2.2. Let $u \in C^1(\overline{\Omega})$ be a nontrivial solution of (2.12), which therefore satisfies

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla w = \int_{\Omega} p_0 (u^+)^{p-1} w - \int_{\Omega} q_0 (u^-)^{p-1} w \quad (2.15)$$

for every $w \in W^{1,p}(\Omega)$.

First, we prove that u changes sign in Ω . Indeed, suppose, by contradiction, that e.g. $u(x) \geq 0$ in Ω . Testing (2.15) against $w = 1$, we get

$$\int_{\Omega} p_0 u^{p-1} = 0$$

which implies that $p_0(x) = 0$ a.e. on $\{x \in \Omega \mid u(x) > 0\}$. Hence, testing (2.15) against $w = u$, we deduce that $\int_{\Omega} |\nabla u|^p = 0$ and therefore u is a positive constant. Accordingly, $p_0(x) = 0$ a.e. in Ω , which is in contradiction with (2.11).

Second, we show that (2.13) holds; condition (2.14) follows from a totally similar argument. Assume, by contradiction, that

$$\text{meas} \{x \in \Omega \mid u(x) > 0 \text{ and } p_0(x) < \lambda_+\} > 0. \quad (2.16)$$

We begin stating the following result, whose proof is postponed.

Claim. There exists a function $v \in W^{1,p}(\Omega)$, which changes sign on subsets of Ω of positive measure and satisfies

$$\frac{\int_{\Omega} |\nabla v^+|^p}{\int_{\Omega} |v^+|^p} < \lambda_+ \quad \text{and} \quad \frac{\int_{\Omega} |\nabla v^-|^p}{\int_{\Omega} |v^-|^p} < \lambda_-. \quad (2.17)$$

Once this Claim is proved, the remainder of the argument proceeds exactly as in the Dirichlet case (cf. [5, Lemma 5.3]). However, we give a brief sketch for the sake of completeness. Let us consider, for each $s \geq 0$, a functional $J_s : W^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$J_s(u) = \int_{\Omega} |\nabla u|^p - s \int_{\Omega} (u^+)^p$$

and let $\tilde{J}_s$ be its restriction to $S = \{u \in W^{1,p}(\Omega) \mid \int_{\Omega} |u|^p = 1\}$. It is proved in [5] (see also [3]) that $\mathcal{C}$ consists of the points $(s + c(s), c(s))$, for all $s \geq 0$, and of their symmetric with respect to the diagonal, where

$$c(s) = \inf_{\gamma \in \Gamma} \max_{u \in \gamma[-1,1]} \tilde{J}_s(u).$$

Here, $\Gamma = \{\gamma \in C^0([-1,1], S) \mid \gamma(-1) = -\varphi_1 \text{ and } \gamma(1) = \varphi_1\}$ and $\varphi_1 = (1/\text{meas } \Omega)^{\frac{1}{p}}$ is the first eigenfunction.

Possibly replacing u with $-u$, one can suppose that $\lambda_+ \geq \lambda_-$. Then, putting $s = \lambda_+ - \lambda_-$, one has $\lambda_- = c(s)$. Using the above Claim, it is possible, as in [5], to build a path $\gamma \in \Gamma$ such that

$$\max_{u \in \gamma[-1,1]} \tilde{J}_s(u) < \lambda_-, \quad (2.18)$$

thus getting a contradiction with the definition of $c(s)$. Indeed, γ can be constructed as follows. One starts joining $\frac{v}{\|v\|_p}$ to $\frac{v^+}{\|v^+\|_p}$ by a path γ_1 , $\frac{v^+}{\|v^+\|_p}$ to $\frac{v^-}{\|v^-\|_p}$ by a path γ_2 and $\frac{-v^-}{\|v^-\|_p}$ to $\frac{v}{\|v\|_p}$ by a path γ_4 . These paths are defined by convex combinations on S , i.e.

$$\gamma_1(t) = \frac{tv + (1-t)v^+}{\|tv + (1-t)v^+\|_p}, \quad \gamma_2(t) = \frac{tv^+ + (1-t)v^-}{\|tv^+ + (1-t)v^-\|_p}, \quad \gamma_4(t) = \frac{-tv^- + (1-t)v}{\|-tv^- + (1-t)v\|_p},$$

for $t \in [0, 1]$. Using (2.17), one verifies that $\tilde{J}_s(\gamma_i(t)) < \lambda_-$, for each $i = 1, 2, 4$ and every t , and, moreover, that $\tilde{J}_s\left(\frac{v^-}{\|v^-\|_p}\right) < \lambda_- - s$. Then, one observes, as in [5], that any component of the set $\{u \in S \mid \tilde{J}_s(u) < \lambda_- - s\}$ contains a critical point of $\tilde{J}_s$ and φ_1 and $-\varphi_1$ are the only critical points of $\tilde{J}_s$ at levels $< \lambda_- - s$. Taking into account these facts, it is possible to join $\frac{v^-}{\|v^-\|_p}$ to φ_1 , or to $-\varphi_1$, by a continuous path γ_3 in such a way that $\tilde{J}_s(\gamma_3(t)) < \lambda_- - s$, for every t . Assume, for instance, that φ_1 is reached in this way. Then, one considers the path $-\gamma_3$ which joins $-\varphi_1$ to $\frac{-v^-}{\|v^-\|_p}$. Again, one has $\tilde{J}_s(-\gamma_3(t)) < \lambda_-$, for every t . Accordingly, it is proved that one can define a continuous path γ on S from $-\varphi_1$ to φ_1 , which satisfies (2.17).

Proof of the Claim. If $\int_{\Omega} |\nabla u^-|^p < \lambda_- \int_{\Omega} |u^-|^p$, then we simply set $v = u$, which, by (2.16), satisfies (2.17). On the contrary, if $\int_{\Omega} |\nabla u^-|^p = \lambda_- \int_{\Omega} |u^-|^p$, we resort to the following more complicated construction.

From (2.16), it follows that there exists $\varepsilon_0 > 0$ such that $\text{meas}\{x \in \Omega \mid u(x) > \varepsilon_0 \text{ and } p_0(x) < \lambda_+\} > 0$. Denote by Π_{ε_0} a connected component of $\{x \in \Omega \mid u(x) > \varepsilon_0\}$ such that $\text{meas}\{x \in \Pi_{\varepsilon_0} \mid p_0(x) < \lambda_+\} > 0$ and, for every $\varepsilon \in]0, \varepsilon_0[$, let Π_{ε} be a connected component of $\{x \in \Omega \mid u(x) > \varepsilon\}$ such that $\Pi_{\varepsilon_0} \subset \Pi_{\varepsilon}$. Hence, we have, in particular, $\text{meas}\{x \in \Pi_{\varepsilon} \mid p_0(x) < \lambda_+\} > 0$. Define $\Pi_0 = \bigcup_{\varepsilon \in]0, \varepsilon_0[} \Pi_{\varepsilon}$. Clearly, Π_0 is open and connected, $u(x) > 0$ in Π_0 and $\text{meas}\{x \in \Pi_0 \mid p_0(x) < \lambda_+\} > 0$. Define, for each $\varepsilon \in [0, \varepsilon_0]$,

$$u_{\varepsilon}(x) = \begin{cases} u(x) - \varepsilon & \text{if } x \in \Pi_{\varepsilon}, \\ 0 & \text{if } x \in \Omega \setminus \Pi_{\varepsilon}. \end{cases}$$

Clearly, $u_{\varepsilon} \rightarrow u_0$ uniformly, as $\varepsilon \rightarrow 0^+$. Since, for each $\varepsilon \in [0, \varepsilon_0]$, Π_{ε} is a nodal domain of $u - \varepsilon$, Lemma 5.2 in [9], which immediately carries over to $W^{1,p}(\Omega)$, guarantees that $u_{\varepsilon} \in W^{1,p}(\Omega)$.

Testing (2.15) against $w = u_0$, we get

$$\frac{\int_{\Omega} |\nabla u_0|^p}{\int_{\Omega} u_0^p} < \lambda_+.$$

Since

$$\frac{\int_{\Omega} |\nabla u_{\varepsilon}|^p}{\int_{\Omega} u_{\varepsilon}^p} = \frac{\int_{\Pi_0} |\nabla u_{\varepsilon}|^p}{\int_{\Pi_0} u_{\varepsilon}^p} \leq \frac{\int_{\Pi_0} |\nabla u_0|^p}{\int_{\Pi_0} u_0^p} \frac{\int_{\Pi_0} u_0^p}{\int_{\Pi_0} u_{\varepsilon}^p}$$

and

$$\frac{\int_{\Pi_0} u_0^p}{\int_{\Pi_0} u_{\varepsilon}^p} \rightarrow 1 \quad \text{as } \varepsilon \rightarrow 0^+,$$

we can pick $\varepsilon > 0$ so small that u_{ε} satisfies

$$\frac{\int_{\Omega} |\nabla u_{\varepsilon}|^p}{\int_{\Omega} u_{\varepsilon}^p} < \lambda_+.$$

Accordingly, we fix such an ε and we set

$$v_1 = u_{\varepsilon} \quad \text{and} \quad \Omega_1 = \Pi_{\varepsilon}.$$

Now, let Ω_0 be a connected component of $\{x \in \Omega \mid u(x) < 0\}$. It is clear that $\text{dist}(\overline{\Omega}_1, \overline{\Omega}_0) > 0$. By the regularity of $\partial\Omega$, we can choose, for each $x_0 \in \partial\Omega \cap \overline{\Omega}_0$, a N -dimensional open cube $R(x_0)$ such that $\overline{R(x_0)} \cap \overline{\Omega}_1 = \emptyset$, and, possibly after an orthogonal change of coordinates,

$$R(x_0) =]x_1^0 - r, x_1^0 + r[\times \cdots \times]x_N^0 - r, x_N^0 + r[,$$

for some $r > 0$,

$$\partial\Omega \cap \overline{R(x_0)} = \{(x_1, \dots, x_N) \in \overline{R(x_0)} \mid x_N = \varphi(x_1, \dots, x_{N-1})\}$$

and

$$\Omega \cap \overline{R(x_0)} = \{(x_1, \dots, x_N) \in \overline{R(x_0)} \mid x_N < \varphi(x_1, \dots, x_{N-1})\},$$

where $(x_1^0, \dots, x_N^0)$ and $(x_1, \dots, x_N)$ are the coordinates of x_0 and x , respectively, and

$$\varphi : [x_1^0 - r, x_1^0 + r] \times \cdots \times [x_{N-1}^0 - r, x_{N-1}^0 + r] \rightarrow]x_N^0 - r, x_N^0 + r[$$

is a $C^{1,1}$ function. Note that these conditions imply the existence of $\rho(x_0) > 0$ such that, for every $x \in \partial\Omega \cap \overline{R(x_0)}$, $\nu_N(x) \geq \rho(x_0)$, where $\nu_N(x)$ is the N -th component of the unit outer normal ν to Ω at the point x .

Since $\partial\Omega \cap \overline{\Omega}_0$ is compact, there exist points $x_1, \dots, x_p \in \partial\Omega \cap \overline{\Omega}_0$ such that $\partial\Omega \cap \overline{\Omega}_0 \subset R(x_1) \cup \dots \cup R(x_p)$. Let us set

$$A = [R(x_1) \cup \dots \cup R(x_p)] \cap \Omega \quad \text{and} \quad \Gamma_0 = \partial\Omega_0 \cap (\Omega \setminus \overline{A}).$$

Of course, if $\partial\Omega \cap \overline{\Omega}_0 = \emptyset$, then $A = \emptyset$. It is clear that $\text{dist}(\overline{\Gamma}_0, \partial\Omega) > 0$ and $\text{dist}(\overline{\Gamma}_0, \overline{\Omega}_1) > 0$. This implies that, for every $y \in \overline{\Gamma}_0$, there exists an N -dimensional open cube $Q(y)$ centered at

y such that $\overline{Q(y)} \subset \Omega$ and $\overline{Q(y)} \cap \overline{\Omega}_1 = \emptyset$. Since $\overline{\Gamma}_0$ is compact, there exist points $y_1, \dots, y_q \in \overline{\Gamma}_0$ such that $\overline{\Gamma}_0 \subset Q(y_1) \cup \dots \cup Q(y_q)$. We set

$$B = Q(y_1) \cup \dots \cup Q(y_q).$$

Clearly, $\text{dist}(\overline{B}, \partial\Omega) > 0$ and $\partial\Omega_0 \subset \overline{A} \cup \overline{B}$. Finally, let us define

$$\Omega_2 = \Omega_0 \cup A \cup B.$$

It can easily be verified that Ω_2 is a connected open set, which satisfies the cone property, according to Definition 4.2 in [1]. This assertion follows observing that there exists an open finite cone C such that if $x \in \partial\Omega_2$, and hence $x \in \overline{A} \cup \overline{B}$, then by the above construction x is the vertex of an open finite cone C_x , congruent to C , which is contained in $A \cup B$ and, therefore, in Ω_2 . Once, the cone property has been verified for all points on $\partial\Omega_2$, it is easy to extend its validity to the whole of Ω_2 , by using [1, Lem. 4.22] applied to the open set $\Omega_2 \setminus (\overline{A} \cup \overline{B}) \subset \subset \Omega_2$.

Let us denote by V the closure in $W^{1,p}(\Omega_2)$ of the linear space

$$\{v \in W^{1,p}(\Omega_2) \mid v(x) = 0 \text{ a.e. in a neighborhood of } \partial\Omega_2 \cap \Omega\}.$$

V is a reflexive Banach space satisfying $W_0^{1,p}(\Omega_2) \subset V \subset W^{1,p}(\Omega_2)$. Define a functional $J : V \rightarrow \mathbb{R}$ by setting $J(v) = \int_{\Omega_2} |\nabla v|^p$. Clearly, J is C^1 and weakly sequentially lower semicontinuous. Let us set

$$\lambda = \inf \left\{ J(v) \mid v \in V, \int_{\Omega_2} |v|^p = 1 \right\}.$$

and let $(v_n)_n$ be a minimizing sequence. Since $(J(v_n))_n$ is bounded, it follows that $(v_n)_n$ is bounded in V and hence, possibly passing to a subsequence, $(v_n)_n$ is weakly convergent to some $v_0 \in V$. Since Ω_2 has the cone property, Theorem 6.2 in [1] guarantees that $W^{1,p}(\Omega_2)$ is compactly imbedded into $L^p(\Omega_2)$. Hence, $v_n \rightarrow v_0$ in $L^p(\Omega_2)$ and therefore $\int_{\Omega_2} |v_0|^p = 1$. Moreover, one has

$$J(v_0) \leq \liminf_{n \rightarrow +\infty} J(v_n) \leq \lambda.$$

This implies that $J(v_0) = \lambda$, i.e. v_0 is a constrained minimizer. Then, Lagrange multipliers rule yields

$$\int_{\Omega_2} |\nabla v_0|^{p-2} \nabla v_0 \nabla v = \lambda \int_{\Omega_2} |v_0|^{p-2} v_0 v,$$

for every $v \in V$ and, in particular, for every $v \in W^{1,p}(\Omega_2)$ with $\text{supp}(v) \subset \Omega_2$. This means [8, p. 828] that v_0 is a local weak solution of the equation

$$-\Delta_p v = \lambda |v|^{p-2} v \quad \text{in } \Omega_2. \tag{2.19}$$

Now, define the function

$$z(x) = \begin{cases} \frac{u(x)}{\left(\int_{\Omega_0} |u|^p\right)^{\frac{1}{p}}} & \text{if } x \in \Omega_0, \\ 0 & \text{if } x \in \Omega_2 \setminus \Omega_0. \end{cases}$$

Obviously, $z \in V \cap L^\infty(\Omega_2)$ and $\int_{\Omega_2} |z|^p = 1$. Testing (2.15) against the function $w \in W^{1,p}(\Omega)$ defined by

$$w(x) = \begin{cases} u(x) & \text{if } x \in \Omega_0, \\ 0 & \text{if } x \in \Omega \setminus \Omega_0, \end{cases}$$

we get

$$\int_{\Omega_0} |\nabla u|^p = \int_{\Omega_0} q_0 |u|^p \leq \lambda_- \int_{\Omega_0} |u|^p$$

and therefore

$$\int_{\Omega_0} |\nabla z|^p \leq \lambda_-.$$

We are in position to prove that

$$\lambda < \lambda_-. \quad (2.20)$$

Indeed, assuming by contradiction that $\lambda \geq \lambda_-$ and therefore, necessarily, $\lambda = \lambda_-$, we conclude that z is a constrained minimizer for J and hence is a bounded local weak solution of (2.19), with $z \in C^1(\Omega_2)$ (see [8, Cor.]). The strong maximum principle [17, Th. 5] finally implies that $z(x) > 0$ everywhere in Ω_2 , which contradicts the definition of z , since $\Omega_2 \setminus \Omega_0$ has nonempty interior. Accordingly, (2.20) holds.

Now, we define the function

$$v_2(x) = \begin{cases} v_0(x) & \text{in } \Omega_2, \\ 0 & \text{in } \Omega \setminus \Omega_2, \end{cases}$$

where we can suppose, possibly replacing v_0 with $|v_0|$, that $v_0(x) \geq 0$ a.e. in Ω_2 . Finally, we set

$$v = v_1 - v_2.$$

Clearly, v satisfies (2.17). Thus, the proof of the Claim is finished. ■

Hence, also the conclusion of Lemma 2.2 follows. ■

Proof of Theorem 1.1. Without loss of generality, we can assume in the course of this proof that

$$\min(\beta - \alpha) < 0, \quad \sup_{s \leq 0} f(s) < +\infty \quad \text{and} \quad \inf_{s \geq 0} f(s) > -\infty. \quad (2.21)$$

Indeed, otherwise we can find a lower solution $\tilde{\alpha}$ and an upper solution $\tilde{\beta}$ satisfying $\min(\tilde{\beta} - \tilde{\alpha}) \geq 0$. Accordingly, standard results (cf. Claim 1 in the proof of Lemma 2.1) yield the solvability of problem (1.1).

Suppose, for instance, that condition (1.6) holds at $+\infty$ and observe that, by condition (1.5) too, there exist a constant $\eta > 0$ and a sequence $(S_n)_n$, with $S_n \rightarrow +\infty$, such that

$$\frac{f(S_n)}{S_n^{p-1}} \leq \lambda_+ - \eta \quad \text{and} \quad \frac{pF(S_n)}{S_n^p} \leq \lambda_+ - \eta \quad \text{for every } n. \quad (2.22)$$

Indeed, two situations may occur :

$$\limsup_{s \rightarrow +\infty} \frac{pF(s)}{s^p} < \lambda_+ \quad \text{or} \quad \limsup_{s \rightarrow +\infty} \frac{pF(s)}{s^p} = \lambda_+.$$

In the former case, by (1.5) and l'Hospital rule, we necessarily have

$$\liminf_{s \rightarrow +\infty} \frac{f(s)}{s^{p-1}} < \lambda_+$$

and therefore (2.22) holds. In the latter case, by (1.6) there exist a constant $\eta > 0$ and a sequence $(S_n)_n$, with $S_n \rightarrow +\infty$, such that, for each n ,

$$\frac{pF(S_n)}{S_n^p} \leq \lambda_+ - \eta$$

and S_n is a local minimum point of $\frac{pF(s)}{s^p}$. Hence, for each n we get $\frac{d}{ds} \left(\frac{pF(s)}{s^p} \right) \Big|_{s=S_n} = 0$, that is

$$\frac{f(S_n)}{S_n^{p-1}} = \frac{pF(S_n)}{S_n^p} \leq \lambda_+ - \eta.$$

Now, for each n we define the truncations

$$f_n(s) = \begin{cases} f(s) & \text{if } s \leq S_n, \\ f(S_n) & \text{if } s > S_n. \end{cases}$$

Let us observe that, for each n and for every $s \geq S_n$,

$$\frac{f_n(s)}{s^{p-1}} = \frac{f(S_n)}{S_n^{p-1}} \left(\frac{S_n}{s} \right)^{p-1} \leq \lambda_+ - \eta$$

and

$$\frac{pF_n(s)}{s^p} \leq \lambda_+ - \eta, \quad (2.23)$$

where $F_n(s) = \int_0^s f_n(\xi) d\xi$. Indeed, if $s \geq S_n$, we have

$$\begin{aligned} \frac{pF_n(s)}{s^p} &= \frac{pF(S_n)}{s^p} + \frac{pf(S_n)}{s^{p-1}} \frac{s - S_n}{s} \\ &\leq (\lambda_+ - \eta) \left(\frac{S_n}{s} \right)^p + p(\lambda_+ - \eta) \left(\frac{S_n}{s} \right)^{p-1} \left(1 - \frac{S_n}{s} \right) \\ &= (\lambda_+ - \eta) \left[(1-p) \left(\frac{S_n}{s} \right)^p + p \left(\frac{S_n}{s} \right)^{p-1} \right] \\ &\leq \lambda_+ - \eta, \end{aligned}$$

since $(1-p)t^p + pt^{p-1} \leq 1$ for every $t \in [0, 1]$. Without loss of generality, we can also assume that

$$S_n > \max \{ \|\alpha\|_\infty, \|\beta\|_\infty \}$$

for every n . Accordingly, α and β are, respectively, a lower and an upper solution of

$$\begin{cases} -\Delta_p u = f_n(u) - h(x) & \text{in } \Omega, \\ \partial u / \partial \nu = 0 & \text{on } \partial \Omega, \end{cases} \quad (2.24)$$

where f_n satisfies the assumptions of Lemma 2.1. Hence, we can conclude that, for each n , problem (2.24) has at least one solution u_n satisfying

$$\min(u_n - \alpha) \leq 0 \quad \text{and} \quad \max(u_n - \beta) \geq 0. \quad (2.25)$$

The solvability of problem (1.1) will be obviously achieved if we succeed in proving that there exists an integer $\tilde{n}$ such that

$$\max u_{\tilde{n}} < S_{\tilde{n}}. \quad (2.26)$$

Assume by contradiction that for every n

$$\max u_n \geq S_n.$$

Hence, in particular, $\|u_n\|_\infty \rightarrow +\infty$. Observe that, for each n , we can write

$$f_n(s) = p_n(s)(s^+)^{p-1} - q_n(s)(s^-)^{p-1} + r_n(s) \quad \text{for every } s,$$

where p_n, q_n and r_n are continuous functions such that

$$0 \leq p_n(s) \leq \lambda_+, \quad 0 \leq q_n(s) \leq \lambda_- \quad \text{for every } s$$

and

$$\lim_{|s| \rightarrow +\infty} \frac{r_n(s)}{|s|^{p-1}} = 0$$

uniformly with respect to n . Let us set for every n

$$v_n = \frac{u_n}{\|u_n\|_\infty}.$$

Clearly, v_n satisfies $\|v_n\|_\infty = 1$ and

$$\begin{aligned} \int_{\Omega} |\nabla v_n|^{p-2} \nabla v_n \nabla w &= \int_{\Omega} p_n(u_n)(v_n^+)^{p-1} w \\ &- \int_{\Omega} q_n(u_n)(v_n^-)^{p-1} w + \int_{\Omega} \frac{r_n(u_n) - h}{\|u_n\|_\infty^{p-1}} w \end{aligned}$$

for every $w \in W^{1,p}(\Omega)$. Hence, by the regularity theory, there is a constant C such that for every n

$$\|v_n\|_{C^{1,\sigma}} \leq C. \quad (2.27)$$

Then, the compact imbedding of $C^{1,\sigma}(\overline{\Omega})$ into $C^1(\overline{\Omega})$ implies that, possibly passing to a subsequence, $(v_n)_n$ converges in $C^1(\overline{\Omega})$ to some function v , with $\|v\|_\infty = 1$, which satisfies

$$\int_{\Omega} |\nabla v|^{p-2} \nabla v \nabla w = \int_{\Omega} p_0(v^+)^{p-1} w - \int_{\Omega} q_0(v^-)^{p-1} w \quad (2.28)$$

for every $w \in W^{1,p}(\Omega)$. Here, $p_0, q_0 \in L^\infty(\Omega)$ are the weak* limits, possibly of subsequences, of $(p_n(u_n))_n$ and $(q_n(u_n))_n$, respectively, and satisfy

$$0 \leq p_0(x) \leq \lambda_+ \quad \text{and} \quad 0 \leq q_0(x) \leq \lambda_- \quad \text{a.e. in } \Omega.$$

It is clear that, by (2.25), v vanishes somewhere in $\overline{\Omega}$. We now distinguish two cases:

- $p_0(x) = 0$ or $q_0(x) = 0$ a.e. in Ω ;

- $p_0(x) > 0$ and $q_0(x) > 0$ on subsets of Ω of positive measure.

In the first case, if e.g. $q_0(x) = 0$ a.e. in Ω , then testing (2.28) against $w = v^-$, we get $\nabla v^- = 0$ in Ω and, since v vanishes somewhere, $v^- = 0$ and therefore $v = v^+$ in Ω . Testing (2.28) against $w = 1$, we obtain $\int_{\Omega} p_0(v^+)^{p-1} = 0$, which implies $v(x) = 0$ where $p_0(x) > 0$. Finally, testing (2.28) against $w = v$, we conclude that $\nabla v = 0$ in Ω and hence $v = 0$ in Ω , because v vanishes somewhere. This yields a contradiction.

In the second case, Lemma 2.2 implies that v changes sign in Ω and, $p_0(x) = \lambda_+$ where $v(x) > 0$ and $q_0(x) = \lambda_-$ where $v(x) < 0$. Therefore, we have

$$\begin{aligned} \int_{\Omega} |\lambda_+ - p_n(u_n)|(v^+)^{p-1} &= \int_{\Omega} (\lambda_+ - p_n(u_n))(v^+)^{p-1} \rightarrow 0, \\ \int_{\Omega} |\lambda_- - q_n(u_n)|(v^-)^{p-1} &= \int_{\Omega} (\lambda_- - q_n(u_n))(v^-)^{p-1} \rightarrow 0 \end{aligned}$$

and therefore

$$\begin{aligned} &\int_{\Omega} \left| \|u_n\|_{\infty}^{1-p} f_n(u_n) - [\lambda_+(v_n^+)^{p-1} - \lambda_-(v_n^-)^{p-1}] \right| \\ &= \int_{\Omega} \left| [p_n(u_n) - \lambda_+](v_n^+)^{p-1} - [q_n(u_n) - \lambda_-](v_n^-)^{p-1} + \|u_n\|_{\infty}^{1-p} r_n(u_n) \right| \\ &\leq \int_{\Omega} |p_n(u_n) - \lambda_+|(v^+)^{p-1} + \int_{\Omega} |q_n(u_n) - \lambda_-|(v^-)^{p-1} \\ &\quad + \int_{\Omega} |p_n(u_n) - \lambda_+| \left| (v_n^+)^{p-1} - (v^+)^{p-1} \right| \\ &\quad + \int_{\Omega} |q_n(u_n) - \lambda_-| \left| (v_n^-)^{p-1} - (v^-)^{p-1} \right| \\ &\quad + \int_{\Omega} \|u_n\|_{\infty}^{1-p} |r_n(u_n)| \rightarrow 0, \end{aligned}$$

as $n \rightarrow +\infty$. Moreover, since v changes sign in Ω , there are an integer $\hat{n}$ and constants η_1, η_2 , with $0 < \eta_1 < 1 < \eta_2$, such that for every $n \geq \hat{n}$,

$$\eta_1 \leq \frac{\max u_n}{-\min u_n} \leq \eta_2. \quad (2.29)$$

In order to prove (2.26), we exploit condition (2.23) and the following technical result.

Claim. For every $x_0, y_0 \in \overline{\Omega}$, there exists a simple piecewise linear path γ , joining x_0 to y_0 and having range contained in Ω , with the possible exception of the end points x_0 and y_0 , such that, possibly for a subsequence,

$$\int_{\gamma} \left| \|u_n\|_{\infty}^{1-p} f_n(u_n) - [\lambda_+ v_n^+ - \lambda_- v_n^-] \right| \rightarrow 0,$$

as $n \rightarrow +\infty$, where $\int_{\gamma}$ denotes a line integral.

The proof of this result, when $p = 2$, is given in [11]. The extension to the present situation is immediate and therefore is omitted.

We know that u_n changes sign in Ω , provided that n is sufficiently large. Let us denote by $y_n \in \overline{\Omega}$ a point where u_n vanishes and let $x_n \in \overline{\Omega}$ be such that $u_n(x_n) = \max u_n (\geq S_n)$. Clearly, we can suppose that $x_n \rightarrow x_0 \in \overline{\Omega}$ and $y_n \rightarrow y_0 \in \overline{\Omega}$, where $v(x_0) = \max v$ and $v(y_0) = 0$. Let γ_n denote the path joining x_n to y_n constructed in the following way: first join x_n to x_0 by a C^1 path σ_n having range in $\overline{\Omega}$, then join x_0 to y_0 by the piecewise linear path γ provided by the above claim, finally join y_0 to y_n by a C^1 path τ_n having range in $\overline{\Omega}$. By the smoothness of $\partial\Omega$, σ_n and τ_n can be taken such that

$$\ell(\sigma_n) \rightarrow 0 \quad \text{and} \quad \ell(\tau_n) \rightarrow 0 \quad \text{as } n \rightarrow +\infty, \quad (2.30)$$

where $\ell(\cdot)$ denotes the length of the corresponding path. Now, using (2.27) and the properties of f_n , we get

$$\begin{aligned} & \frac{\lambda_+}{p}(\max u_n)^p - F_n(\max u_n) \\ &= \left[\frac{\lambda_+}{p}(u_n^+(x_n))^p - \frac{\lambda_-}{p}(u_n^-(x_n))^p - F_n(u_n(x_n)) \right] \\ & \quad - \left[\frac{\lambda_+}{p}(u_n^+(y_n))^p - \frac{\lambda_-}{p}(u_n^-(y_n))^p - F_n(u_n(y_n)) \right] \\ &= \int_0^1 \left[\lambda_+(u_n^+(\sigma_n))^{p-1} - \lambda_-(u_n^-(\sigma_n))^{p-1} - f_n(u_n(\sigma_n)) \right] \langle \nabla u_n(\sigma_n), \sigma_n' \rangle \\ & \quad + \int_0^1 \left[\lambda_+(u_n^+(\gamma))^{p-1} - \lambda_-(u_n^-(\gamma))^{p-1} - f_n(u_n(\gamma)) \right] \langle \nabla u_n(\gamma), \gamma' \rangle \\ & \quad + \int_0^1 \left[\lambda_+(u_n^+(\tau_n))^{p-1} - \lambda_-(u_n^-(\tau_n))^{p-1} - f_n(u_n(\tau_n)) \right] \langle \nabla u_n(\tau_n), \tau_n' \rangle \\ &\leq (\ell(\sigma_n) + \ell(\tau_n)) \left\| f_n(u_n) - \left[\lambda_+(u_n^+)^{p-1} - \lambda_-(u_n^-)^{p-1} \right] \right\|_\infty \|\nabla u_n\|_\infty \\ & \quad + \left(\int_\gamma \left| f_n(u_n) - \left[\lambda_+(u_n^+)^{p-1} - \lambda_-(u_n^-)^{p-1} \right] \right| \right) \|\nabla u_n\|_\infty \\ &\leq \left(\ell(\sigma_n) + \ell(\tau_n) + \|u_n\|_\infty^{1-p} \int_\gamma \left| f_n(u_n) - \left[\lambda_+(u_n^+)^{p-1} - \lambda_-(u_n^-)^{p-1} \right] \right| \right) \\ & \quad \times C \|u_n\|_\infty^p. \end{aligned}$$

Therefore, by the above Claim and (2.30), we deduce

$$\frac{\lambda_+}{p}(\max u_n)^p - F_n(\max u_n) \leq \varepsilon_n \|u_n\|_\infty^p, \quad (2.31)$$

where $\varepsilon_n \rightarrow 0$ as $n \rightarrow +\infty$. On the other hand, using (2.23) and (2.29), we have

$$\frac{\lambda_+}{p}(\max u_n)^p - F_n(\max u_n) \geq \frac{\eta}{p}(\max u_n)^p \geq \frac{\eta \eta_1^p}{p} \|u_n\|_\infty^p. \quad (2.32)$$

Comparing (2.31) and (2.32) yields a contradiction, letting $n \rightarrow +\infty$. This concludes the proof of Theorem 1.1. ■

Proof of Corollary 1.1. The conclusion immediately follows combining Theorem 1.1 with the following lemma.

Lemma 2.3 Set $\bar{h} = \frac{1}{\text{meas } \Omega} \int_{\Omega} h$. Assume that there is a constant $r > 0$ such that

$$f(s) \geq \bar{h} \quad \text{for every } s \geq r \quad (2.33)$$

$$\left(f(s) \geq \bar{h} \quad \text{for every } s \leq -r, \text{ respectively} \right).$$

Then, for any $c \geq 0$ problem (1.1) has a lower solution α , with $\alpha(x) \geq c$ in Ω ($\alpha(x) \leq -c$ in Ω , respectively). Similarly, assume that there is a constant $r > 0$ such that

$$f(s) \leq \bar{h} \quad \text{for every } s \leq -r$$

$$\left(f(s) \leq \bar{h} \quad \text{for every } s \geq r, \text{ respectively} \right).$$

Then, for any $c \geq 0$ problem (1.1) has an upper solution β , with $\beta(x) \leq -c$ in Ω ($\beta(x) \geq c$ in Ω , respectively).

Proof of Lemma 2.3. Assume that condition (2.33) holds, the other cases being treated similarly. In order to build a lower solution α of (1.1), we will prove that the problem

$$\begin{cases} -\Delta_p w = \bar{h} - h(x) & \text{in } \Omega, \\ \partial w / \partial \nu = 0 & \text{on } \partial \Omega, \\ \int_{\Omega} w = 0 \end{cases} \quad (2.34)$$

has a solution $w \in W^{1,p}(\Omega) \cap L^{\infty}(\Omega)$, and therefore $w \in C^{1,\sigma}(\bar{\Omega})$. Then, for any constant $t > \|w\|_{\infty} + r$, we define $\alpha(x) = t + w(x)$ in Ω . It is immediately seen, by condition (2.33), that α is a lower solution.

First, we prove that (2.34) has a solution $w \in W^{1,p}(\Omega)$. Let us set $W = \{w \in W^{1,p}(\Omega) \mid \int_{\Omega} w = 0\}$. W is closed linear subspace of $W^{1,p}(\Omega)$ and therefore is a reflexive Banach space. Define a functional $J : W \rightarrow \mathbb{R}$ by setting $J(w) = \frac{1}{p} \int_{\Omega} |\nabla w|^p - \int_{\Omega} h w$. Clearly, J is strictly convex and C^1 . Moreover, the Poincaré–Wirtinger inequality implies that J is coercive. Hence, J has a unique global minimum point $w \in W$ and, therefore, for every $z \in W$ and every $c \in \mathbb{R}$, we get

$$\int_{\Omega} |\nabla w|^{p-2} \nabla w \nabla (z + c) = \int_{\Omega} (\bar{h} - h) (z + c),$$

i.e. w is a solution of (2.34).

Second, we show that $w \in L^{\infty}(\Omega)$. Of course, only the case where $p \leq N$ requires a proof. Assume that $w \neq 0$ and observe that, as $\int_{\Omega} w = 0$, w changes sign on subsets of Ω of positive measure. Let us set $B = \{x \in \Omega \mid w(x) < 0\}$ and, for each n , $A_n = \{x \in \Omega \mid w(x) > n\}$ and $w_n = (w - n)^+$. It is clear that $w_n(x) = 0$ a.e. on B . In the sequel, the same symbol C will denote constants which may change time by time, but are all independent of n . We claim that there exists $C > 0$ such that

$$\|w_n\|_p \leq C \|\nabla w_n\|_p \quad (2.35)$$

for every n . Indeed, otherwise, possibly passing to a subsequence, we should have $\frac{\|\nabla w_n\|_p}{\|w_n\|_p} \rightarrow 0$. Setting, for each n , $z_n = \frac{w_n}{\|w_n\|_p}$, we have, possibly passing to a further subsequence, that $(z_n)_n$

converges weakly in $W^{1,p}(\Omega)$, strongly in $L^p(\Omega)$ and pointwise a.e. in Ω to some function $z \in W^{1,p}(\Omega)$. Actually, $(z_n)_n$ converges strongly in $W^{1,p}(\Omega)$ to z . Hence, z satisfies $\|z\|_p = 1$ and $\nabla z = 0$. The Poincaré–Wirtinger inequality implies that z is constant in Ω and, since $z(x) = 0$ a.e. in B , we conclude that $z(x) = 0$ a.e. in Ω : a contradiction. To prove the boundedness from above of w , we apply Lemma 5.1 in [14]. Accordingly, we show that there exist constants $C > 0$ and $\varepsilon > 0$ such that, for every n ,

$$\int_{A_n} w_n \leq C (\text{meas}(A_n))^{1+\varepsilon}. \quad (2.36)$$

Assume that $p < N$; a similar argument applies if $p = N$. By Hölder inequality and the Sobolev imbedding theorem, we have, using (2.35) too,

$$\begin{aligned} \int_{A_n} w_n &\leq (\text{meas}(A_n))^{1-\frac{N-p}{Np}} \left(\int_{A_n} (w_n)^{\frac{Np}{N-p}} \right)^{\frac{N-p}{Np}} \\ &= (\text{meas}(A_n))^{1-\frac{N-p}{Np}} \|w_n\|_{\frac{Np}{N-p}} \\ &\leq C (\text{meas}(A_n))^{1-\frac{N-p}{Np}} \|w_n\|_{1,p} \\ &\leq C (\text{meas}(A_n))^{1-\frac{N-p}{Np}} \|\nabla w_n\|_p. \end{aligned}$$

Testing the equation in (2.34) against $w_n \in W^{1,p}(\Omega)$, we get

$$\int_{\Omega} |\nabla w_n|^p \leq \|h - \bar{h}\|_{\infty} \int_{\Omega} w_n$$

and hence

$$\int_{A_n} w_n \leq C (\text{meas}(A_n))^{1-\frac{N-p}{Np}} \left(\int_{A_n} w_n \right)^{\frac{1}{p}},$$

that is

$$\int_{A_n} w_n \leq C (\text{meas}(A_n))^{1+\frac{p}{N(p-1)}}.$$

Similarly, one proves that w is bounded from below. ■

Proof of Proposition 1.1. Assume, for instance, that condition (1.8) holds at $-\infty$. As in the proof of Theorem 1.1, we can also suppose, without loss of generality, that (2.21) holds. Then, for each n , we define the truncations

$$f_n(s) = \begin{cases} f(s) & \text{if } s \leq n, \\ f(n) & \text{if } s > n. \end{cases}$$

Conditions (1.8) and (2.21) imply the existence of constants $\hat{\lambda} \in]0, \bar{\lambda}[$ and $C > 0$ such that

$$|f_n(s) - h(x)| \leq f_n(s) - h(x) + 2\hat{\lambda}(s^-)^{p-1} + 2C, \quad (2.37)$$

for every n , every s and a.e. x . Consider problem (2.24) and observe that, for every $n > \max\{\|\alpha\|_{\infty}, \|\beta\|_{\infty}\}$, α and β are a lower and an upper solution, respectively. Hence, the assumptions of Lemma 2.1 are satisfied and therefore (2.24) has at least one solution u_n with

$$\min(u_n - \alpha) \leq 0 \quad \text{and} \quad \max(u_n - \beta) \geq 0. \quad (2.38)$$

The solvability of problem (1.1) will be obviously achieved if we are able to prove the existence of an integer $\tilde{n}$ such that

$$\max u_{\tilde{n}} \leq \tilde{n}. \quad (2.39)$$

Assume, by contradiction, that $\max u_n > n$ for every n . Hence, $\|u_n\|_\infty \rightarrow +\infty$ and $\|u_n\|_{1,p} \rightarrow +\infty$, as $p > N$. Let us set

$$v_n = \frac{u_n}{\|u_n\|_{1,p}}.$$

Possibly passing to a subsequence, still denoted by $(v_n)_n$, we have that $(v_n)_n$ converges weakly in $W^{1,p}(\Omega)$ and strongly in $C^0(\overline{\Omega})$ to some function v . Moreover, v_n satisfies

$$\int_{\Omega} |\nabla v_n|^{p-2} \nabla v_n \nabla w = \int_{\Omega} \frac{f_n(u_n) - h}{\|u_n\|_{1,p}^{p-1}} w \quad (2.40)$$

for every $w \in W^{1,p}(\Omega)$. Testing (2.40) against $w = 1$ and using (2.37), we get

$$\int_{\Omega} \frac{|f_n(u_n) - h|}{\|u_n\|_{1,p}^{p-1}} \leq 2 \hat{\lambda} \int_{\Omega} (v_n^-)^{p-1} + \frac{2C \text{ meas } \Omega}{\|u_n\|_{1,p}^{p-1}} \quad (2.41)$$

and hence there is a constant still denoted by C such that

$$\frac{\|f_n(u_n) - h\|_1}{\|u_n\|_{1,p}^{p-1}} \leq c_2 \quad (2.42)$$

for every n . Now, testing (2.40) against $w = v_n - v$ and using (2.42), we obtain

$$\int_{\Omega} |\nabla v_n|^{p-2} \nabla v_n \nabla (v_n - v) \leq c_2 \|v_n - v\|_\infty \rightarrow 0,$$

as $n \rightarrow +\infty$. Hence, condition $(S+)$ in [4, p. 18] implies that $v_n \rightarrow v$ in $W^{1,p}(\Omega)$ and therefore, as $\|v_n\|_{1,p} = 1$, also $\|v\|_{1,p} = 1$. Next, we show that $v^- \neq 0$. Indeed, if $v^- = 0$, then $v_n^- \rightarrow 0$ in $L^\infty(\Omega)$ and, by (2.41), $\frac{\|f_n(u_n) - h\|_1}{\|u_n\|_{1,p}^{p-1}} \rightarrow 0$. Hence, testing (2.40) against $w = v$, we get $\int_{\Omega} |\nabla v_n|^{p-2} \nabla v_n \nabla v \rightarrow 0$. Since $|\nabla v_n|^{p-2} \nabla v_n \rightarrow |\nabla v|^{p-2} \nabla v$ in $\left(L^{\frac{p}{p-1}}(\Omega)\right)^N$, we conclude that $\int_{\Omega} |\nabla v|^p = 0$, i.e. $\nabla v = 0$ a.e. in Ω . As v is continuous and vanishes somewhere in $\overline{\Omega}$ by (2.38), we get $v = 0$: a contradiction. Finally, testing (2.40) against $-v_n^-$ and using (2.37) again, we easily have

$$\begin{aligned} \int_{\Omega} |\nabla (v_n^-)|^p &= - \int_{\Omega} \frac{f_n(u_n) - h}{\|u_n\|_{1,p}^{p-1}} v_n^- \leq \int_{\Omega} \frac{(f_n(u_n) - h)^-}{\|u_n\|_{1,p}^{p-1}} v_n^- \\ &\leq \hat{\lambda} \int_{\Omega} (v_n^-)^p + \frac{C}{\|u_n\|_{1,p}^{p-1}} \end{aligned}$$

and hence, passing to the limit,

$$\int_{\Omega} |\nabla (v^-)|^p \leq \hat{\lambda} \int_{\Omega} |v^-|^p.$$

This yields a contradiction, recalling that $\hat{\lambda} < \overline{\lambda}$, $v^- \neq 0$ and the definition of $\overline{\lambda}$. ■

References

- [1] **R.A. Adams**, *Sobolev Spaces*, Academic Press, New York, 1975.
- [2] **D. Arcoya** and **L. Orsina**, *Landesman–Lazer conditions and quasilinear elliptic equations*, Nonlinear Anal., T.M.A., **28** (1997), 1623-1632.
- [3] **M. Arias**, **J. Campos** and **J.P. Gossez**, *On the antimaximum principle and the Fučík spectrum for the Neumann p -Laplacian*, Diff. Int. Equat., **13** (2000), 217-226.
- [4] **F. E. Browder**, *Existence theorem for nonlinear partial differential equations*, Proc. Sympos. Pure Math., Vol. XVI, Berkeley, Calif., 1968, pp. 1-60, Amer. Math. Soc., Providence, R.I. .
- [5] **M. Cuesta**, **D. de Figueiredo** and **J.P. Gossez**, *The beginning of the Fučík spectrum for the p -Laplacian*, J. Differential Equations, **159** (1999), 212-238.
- [6] **M. Cuesta**, **D. de Figueiredo** and **J.P. Gossez**, *A nodal domain property for the p -Laplacian*, C. R. Acad. Sci. Paris, **330** (2000), 669-673.
- [7] **C. De Coster** and **M. Henrard**, *Existence and localization of solutions for second order elliptic BVP in presence of lower and upper solutions without any order*, J. Differential Equations, **145** (1998), 420-452.
- [8] **E. Di Benedetto**, *$C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations*, Nonlinear Anal., T.M.A., **7** (1983), 827-850.
- [9] **D. de Figueiredo** and **J.P. Gossez**, *On the first curve of the Fučík spectrum of an elliptic operator*, Diff. Int. Equat., **7** (1994), 1285-1302.
- [10] **M. García–Huidobro**, **R. Manàsevich** and **F. Zanolin**, *Strongly nonlinear second-order ODEs with unilateral conditions*, Diff. Int. Equat., **6** (1993), 1057-1078.
- [11] **J.P. Gossez** and **P. Omari**, *On a semilinear elliptic Neumann problem with asymmetric nonlinearities*, Trans. Amer. Math. Soc., **347** (1995), 2553-2562.
- [12] **P. Habets** and **P. Omari**, *Existence and localization of solutions of second order elliptic problems using lower and upper solutions in the reversed order*, Top. Meth. Nonlinear Anal., **8** (1996), 25-56.
- [13] **J. Kazdan** and **F. Warner**, *Remarks on some quasilinear elliptic equations*, Comm. Pure Appl. Math., **28** (1975), 567-597.
- [14] **O.A. Ladyzhenskaya** and **N.N. Ural'tseva**, *Linear and Quasilinear Elliptic Equations*, Academic Press, New York, 1968.

- [15] **G.M. Lieberman**, *Boundary regularity for solutions of degenerate elliptic equations*, Non-linear Anal., T.M.A., **12** (1988), 1203-1219.
- [16] **P. Omari** and **F. Zanolin**, *Nonresonance conditions on the potential for a second-order periodic boundary value problem*, Proc. Amer. Math. Soc., **117** (1993), 125-135.
- [17] **J.L. Vazquez**, *A strong maximum principle for some quasilinear elliptic equations*, Appl. Math. Optim., **3** (1984), 191-202.
- [18] **S. Villegas**, *A Neumann problem with asymmetric nonlinearity and a related minimizing problem*, J. Differential Equations, **145** (1998), 145-155.