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CODIMENSION TWO KÄHLER SUBMANIFOLDS OF SPACE FORMS

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Abstract

In this article we study isometric immersions from Kähler manifolds whose $(1, 1)$ part of the second fundamental form is parallel, the *ppmc* isometric immersions. When the domain is a Riemann surface these immersions are precisely those with parallel mean curvature. P. J. Ryan has classified the Kähler manifolds that admit isometric immersions, as real hypersurfaces, in space forms. We classify the codimension two *ppmc* isometric immersions into space forms.

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1 Introduction and Statement of Results

Surfaces with parallel mean curvature play an interesting part in the study of the geometry of submanifolds. The complex structure of such a surface is a crucial ingredient in the building of this theory. One may ask: which Kähler candidates generalize the parallel mean curvature surfaces?

To study the extrinsic geometry of an isometric immersion $\varphi : M \rightarrow N$ between two Riemannian manifolds, one of the main tools is its second fundamental form $\alpha = \nabla d\varphi$. In such study, when M is Kähler, its complex structure J fulfil an important role.

The complexified α splits in a natural way according to types giving rise to

$$\alpha = \alpha^{(1,1)} + \alpha^{(2,0)} + \alpha^{(0,2)}$$

When $\alpha^{(1,1)} \equiv 0$, φ is called pluriminimal or $(1,1)$ -geodesic. Equivalently φ is pluriminimal if and only if $\varphi|_C$ is minimal for any holomorphic curve $C \subset M$. Pluriminimal isometric immersions have been studied by several authors (see for instance [3], [4], [5]).

When N has non positive curvature operator, every minimal isometric immersion is pluriminimal [4]. Clearly, if $\dim M = 2$, the concepts of minimality and pluriminimality coincide.

Surfaces with parallel mean curvature generalize in a natural way the minimal ones. They can be characterized by the parallelism of the $(1,1)$ - part of their second fundamental form. So, in a higher dimensional setting the isometric immersions with parallel $\alpha^{(1,1)}$ generalize the parallel mean curvature surfaces. One of the fundamental questions is to know which Kähler manifolds admit isometric immersions with $\alpha^{(1,1)}$ parallel.

We will say that an isometric immersion from a Kähler manifold is a parallel pluri-mean curvature immersion (*ppmc*) if $\alpha^{(1,1)}$ is parallel. It is easily seen that φ is *ppmc* if and only if $\varphi|_C$ has parallel mean curvature for any totally geodesic holomorphic curve $C \subset M$.

PPMC isometric immersions into space-forms display, indeed, some special features of the parallel mean curvature surfaces, namely the existence of a 1-parameter deformation through a smooth family of *ppmc* isometric immersions which, up to a parallel isomorphism, have the same normal bundle [7].

The aim of the present work is to classify the, non minimal, codimension two *ppmc* isometric immersions into space-forms. It turns out that if these immersions do not factorize, they are either surfaces with parallel mean curvature or minimal submanifolds of horospheres in an hyperbolic space. This is not true for codimension three as it shows the emdedding of CP^2 into the unit sphere S^7 in the Lie algebra of $SU(3)$.

Kähler hypersurfaces into space-forms have been studied by Ryan, P. J. [12], who presented the following results:

Theorem 1 [12] *Let M be a Kähler hypersurface of a space-form with sectional curvature $c \neq 0$ and $\dim 2m + 1$. Then, when $m > 1$, M is an open set of one of the following manifolds:*

- ($c > 0$) $S^2(c_1) \times S^2(c_2) \subset S^5(c)$, where $S^k(r)$ denotes the k -dimensional Euclidean sphere with constant sectional curvature r and $\frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{c}$.
- ($c < 0$) a horosphere in $H^{2m+1}(c)$ or $H^2(c_1) \times S^2(c_2) \subset H^5(c)$, where $H^k(r)$ denotes the k -dimensional hyperbolic space with constant sectional curvature r and $\frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{c}$.

We remark that all the submanifolds given in the previous theorem provide examples of *ppmc* isometric immersions.

Theorem 2 [12] *Let M be a Kähler hypersurface of R^{2m+1} . Then M is either an open subset of a Riemann surface S or an open set of the Riemannian product $S \times R^{2m-1}$.*

Corollary 3 *Let M be a Kähler manifold with complex dimension m and $\varphi : M \rightarrow R^{2m+1}$ a *ppmc* isometric immersion. Then, locally, one of the following conditions occur:*

- $m = 1$ and $\varphi : M \rightarrow R^3$ has constant mean curvature.
- $M = M_1 \times M_2$ and $\varphi = \varphi_1 \times \varphi_2 : M_1 \times M_2 \rightarrow R^3 \times R^{2m-2}$, where M_1 is a Riemann surface, φ_1 has parallel mean curvature and φ_2 is totally geodesic.

Regarding codimension two isometric immersions from a Kähler manifold with complex dimension m , we have the following results:

Theorem 4 *Assume that $\varphi : M \rightarrow S^{2m+2}(c)$ is a, non minimal, *ppmc* isometric immersion. Therefore one of the following conditions hold:*

- $m = 1$ and $\varphi : M \rightarrow S^4(c)$ has parallel mean curvature.
- $m = 2$, M is an open subset of $S^2(c_1) \times M_2$ and $\varphi = \varphi_1 \times \varphi_2 : M \rightarrow S^2(c_1) \times S^3(c_2)$, where $\varphi_1 : S^2(c_1) \rightarrow R^3$ is the standard embedding, $\varphi_2 : M_2 \rightarrow S^3(c_2)$ is minimal and $\frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{c}$.
- $m = 3$, M is an open subset of $S^2(c_1) \times S^2(c_2) \times S^2(c_3)$, $\frac{1}{c_1} + \frac{1}{c_2} + \frac{1}{c_3} = \frac{1}{c}$ and $\varphi = h|_M$, where $h : S^2(c_1) \times S^2(c_2) \times S^2(c_3) \rightarrow S^8(c)$ is the standard emdedding.

Theorem 5 *Assume that $\varphi : M \rightarrow H^{2m+2}(c)$ is a, non minimal, *ppmc* isometric immersion. Then one of the following conditions hold:*

- $m = 1$ and $\varphi : M \rightarrow H^4(c)$ has parallel mean curvature.
- $m > 1$, $\varphi(M) \subset E^{2m+1} \subset H^{2m+2}(c)$ and $\varphi : M \rightarrow E^{2m+1}(c)$ is minimal.
- M is locally a Riemannian product, $M_1 \times M_2$, $\varphi(M) \subset E^{2m+1}$ and $\varphi = \varphi_1 \times \varphi_2 : M \rightarrow E^{2m+1}$ is a product immersion, where M_1 is a Riemann surface and φ_2 is totally geodesic.

- M is locally a Riemannian product $M_1 \times M_2$, where M_2 is an open subset of $H^2(c_2)$; moreover $\varphi = \varphi_1 \times \varphi_2 : M_1 \times M_2 \rightarrow S^3(c_1) \times H^2(c_2) \subset H^6(c)$, where $\varphi_1 : M_1 \rightarrow S^3(c_1)$ has constant mean curvature and $\frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{c}$.
- M is locally a Riemannian product $M_1 \times M_2$, where M_2 is an open subset of $S^2(c_2)$; moreover $\varphi = \varphi_1 \times \varphi_2 : M_1 \times M_2 \rightarrow H^3(c_1) \times S^2(c_2) \subset H^6(c)$ where $\varphi_1 : M_1 \rightarrow H^3(c_1)$ has constant mean curvature and $\frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{c}$.
- M is an open subset of $H^2(c_1) \times S^2(c_2) \times S^2(c_3)$ and $\varphi = h|_M$ where $h : H^2(c_1) \times S^2(c_2) \times S^2(c_3) \rightarrow H^8(c)$ is the standard embedding and $\frac{1}{c_1} + \frac{1}{c_2} + \frac{1}{c_3} = \frac{1}{c}$.
- M is an open subset of $E^{2r} \times S^2(c_2)$ and $\varphi = h|_M$, where $r = m - 1$, $\frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{c}$, E^{2r} is a horosphere in $H^{2r+1}(c_1)$ and $h : E^{2r} \times S^2(c_2) \rightarrow H^{2m+2}(c)$ is the standard embedding.

Theorem 6 *Let $\varphi : M \rightarrow R^{2m+2}$ be a, non minimal, ppmc isometric immersion. Then one of the following conditions hold:*

- $m = 1$ and $\varphi : M \rightarrow R^4$ has parallel mean curvature.
- $m = 2$, $M = M_1 \times M_2$ and $\varphi = \varphi_1 \times \varphi_2 : M_1 \times M_2 \rightarrow R^6$, where $\dim M_1 = \dim M_2 = 2$ and, for each $1 \leq i \leq 2$, $\varphi_i : M_i \rightarrow R^3$ has constant mean curvature.
- $M = M_1 \times M_2$ and $\varphi = \varphi_1 \times \varphi_2 : M_1 \times M_2 \rightarrow R^{2m+2}$ where φ_1 satisfies one of the two previous conditions and φ_2 is totally geodesic.

Isometric immersions with parallel second fundamental form are trivially ppmc. We shall remark that minimal isometric immersions from a Kähler manifold into a horosphere $E^{2m+1} \subset H^{2m+2}$ provide examples of ppmc isometric immersions which do not have parallel second fundamental form.

2 Preliminaries

Throughout this work all geometric objects are assumed to be smooth and M will denote a connected Kähler manifold with complex dimension m and complex structure J .

Let $\varphi : M \rightarrow N$ be an isometric immersion into a Riemannian manifold. $\overset{\perp}{T}M$ will denote the normal bundle of φ . We will use the same symbol ∇ to represent the Levi-Civita connection of TM or the induced connection on $\varphi^{-1}TN$ and $T^*M \otimes \varphi^{-1}TN$. The symbol $\overset{\perp}{\nabla}$ will be used to represent either the induced connection on $\overset{\perp}{T}M$ or on $T^*M \otimes \overset{\perp}{T}M$.

The complex structure J gives rise to the usual splitting

$$T^C M = T' M \oplus T'' M$$

of the complexified tangent bundle $T^C M$ into its holomorphic and anti-holomorphic parts. We will denote respectively by $\pi' : T^C M \rightarrow T' M$ and $\pi'' : T^C M \rightarrow T'' M$ the orthogonal projections with respect to the hermitian structure of M .

We have therefore

$$\alpha^{(1,1)}(X, Y) = \alpha(X', Y'') + \alpha(X'', Y')$$

where $Z' = \pi'(Z)$ and $Z'' = \pi''(Z)$ for every $Z \in C(T^C M)$.

Similarly

$$\alpha^{(2,0)}(X, Y) = \alpha(X', Y')$$

we can also write

$$\alpha^{(1,1)}(X, Y) = \frac{1}{2}C(X, Y) + \frac{i}{2}C(X, JY) ,$$

where

$$C(X, Y) = \frac{1}{2} \{ \alpha(X, Y) + \alpha(JX, JY) \}$$

and

$$\alpha^{(2,0)}(X, Y) = \frac{1}{2}D(X, Y) - \frac{i}{2}D(X, JY) ,$$

where

$$D(X, Y) = \frac{1}{2} \{ \alpha(X, Y) - \alpha(JX, JY) \}$$

The operators C and D characterize the second fundamental form and play an interesting role in the study of immersions from Kähler manifolds.

As we have seen before, the vanishing of C tells the minimality of the restriction of φ to any holomorphic curve of M . In a similar way, the vanishing of D means that the restriction of φ to any holomorphic curve of M is totally umbilic. These properties are rather special when N is a space form. Under this assumption we have:

Proposition 7 [4] *Let $\varphi : M \rightarrow Q(c)$ be an isometric immersion. Then*

- *if $c = 0$, φ is minimal if, and only if, φ is pluriminimal*
- *if $c \neq 0$ and φ is pluriminimal, $m = 1$.*

We say that C is totally umbilic if $C = \langle \cdot, \cdot \rangle H$, where H denotes the mean curvature of φ and $\langle \cdot, \cdot \rangle$ is the Riemannian structure of M .

Proposition 8 [9] *Let $\varphi : M \rightarrow Q(c)$ be an isometric immersion with C totally umbilic. Then*

- *if $c = 0$, either φ is minimal or $m = 1$.*
- *if $c > 0$, $m = 1$.*
- *if $c < 0$, either $m = 1$ or φ has constant mean curvature $\|H\| = \sqrt{-c}$.*

Proposition 9 [10] *Let $\varphi : M \rightarrow Q(c)$ be an isometric immersion. Then $D = 0$ if and only if φ has parallel second fundamental form.*

3 *ppmc* Isometric Immersions

ppmc isometric immersions were introduced in [7] and [8].

When $m = 1$, *ppmc* isometric immersions are precisely those immersions with parallel mean curvature. It is well known that a parallel mean curvature surface in a three dimensional space-form allows a one parameter family of isometric deformations - the associate family - with the same mean curvature. *ppmc* isometric immersions into a space-form $Q(C)$ also enjoy this property. Indeed, associated to J there is a one parameter family

$$J_\theta = \cos \theta I + \sin \theta J, \theta \in [0, \pi]$$

of tensors, where I denotes the identity map of TM .

For each θ we define $\alpha_\theta \in C \left(\overset{\perp}{\odot} T^*M \otimes \overset{\perp}{TM} \right)$ by

$$\alpha_\theta(X, Y) = \alpha(J_{\frac{\theta}{2}}(X), J_{\frac{\theta}{2}}(Y))$$

α_θ is a symmetric tensor field which satisfy Gauss, Codazzi and Ricci equations. Hence, from the fundamental theorem for submanifolds, we get an isometric immersion $\varphi_\theta : M \rightarrow Q(c)$ whose second fundamental form is α_θ .

ppmc isometric immersions are exactly characterized by the existence of an associated family.

Proposition 10 [8] *Let $\varphi : M \rightarrow Q(c)$ be an isometric immersion. Then φ is *ppmc* if, and only if, φ has an associated family.*

Remark 11 *The hypothesis $(\overset{\perp}{R})^{(2,0)}$ considered in [8] is not necessary, as pointed out in [2]. It follows from the parallelism of $\alpha^{(1,1)}$.*

Assume now that $M = M_1 \times M_2$, where M_1 and M_2 are Kähler manifolds and $\varphi : M_1 \times M_2 \rightarrow Q(c)$ is a codimension two *ppmc* isometric immersion. As a consequence of the previous proposition we can state:

Proposition 12 *Under the above conditions, for each $a \in M_1$ (resp. $b \in M_2$), the immersion $\varphi^a : M_2 \rightarrow Q(c)$ (resp. $\varphi^b : M_1 \rightarrow Q(c)$) defined by $\varphi^a(y) = \varphi(a, y)$ (resp. $\varphi^b(x) = \varphi(x, b)$) is *ppmc* and has flat normal bundle.*

Proof. If φ_θ is an associated family of φ , $\varphi_\theta^a(y) = \varphi_\theta(a, y)$ is a smooth deformation of φ^a . Since the second fundamental form β of φ^a is obtained by the restriction of α to TM_2 , it is easily seen that $\beta_\theta(X, Y) = \beta(J_{\frac{\theta}{2}}^2(X), J_{\frac{\theta}{2}}^2(Y))$ is the second fundamental form of φ_θ^a , where J^2 represents the complex structure of M_2 . Thus φ is *ppmc*.

For $\xi \in C(\overset{\perp}{TM}_2)$ we let A_ξ denote the Weigarten operator for φ_a , corresponding to ξ . Clearly, if $v \in T_a M_1 \subset \overset{\perp}{TM}_2$, $A_v = 0$. Therefore, using Ricci equation for φ_a , we easily conclude that $\overset{\perp}{R}(., .)v = 0$, where $\overset{\perp}{R}$ represents the normal curvature of φ_a .

On the other hand, the fact that the mean curvature H of φ_a is a parallel section of $T^\perp M_2$ which sits on $T^\perp M$ allows us to conclude that, for every $\xi \in C(T^\perp M)$, $\bar{R}(\cdot, \cdot)\xi = 0$, so that $T^\perp M_2$ is flat. ■

4 Proof of the Main Results

Assume that $\varphi : M \rightarrow Q(c)$ is a *ppmc* isometric immersion.

For each $\xi \in C(T^\perp M)$ we let $C_\xi \in C(T^*M \otimes TM)$ be the operator defined by

$$C_\xi(X, Y) = \langle C(X, Y), \xi \rangle ,$$

where C was defined in section two. Equivalently

$$2C_\xi = A_\xi - JA_\xi J$$

C_ξ is a self-adjoint operator which commutes with J . It is clearly parallel whenever ξ is a parallel normal section.

To prove theorems 4, 5 and 6 we use the techniques for decomposing manifolds into products of integral manifolds corresponding to eigenspaces of convenient C_ξ .

For each $x \in M$, let $\lambda_1(x), \dots, \lambda_k(x)$ be the eigenvalues of $C_H(x)$ and let $T_x^1, \dots, T_x^k$ be the corresponding eigenspaces.

$$T_x M = T_x^1 \oplus \dots \oplus T_x^k$$

Since C is parallel for *ppmc* immersions, we obtain in this way parallel distributions $T^1, \dots, T^k$ which are invariant under the action of the holonomy group $Hol(g)$, where g denotes the Riemannian structure of M . Moreover, each T^i , $1 \leq i \leq k$, is J -invariant. From the de Rham theorem we know that M is, locally, a product $(M_1, g_1) \times \dots \times (M_k, g_k)$ and $Hol(g) = Hol(g_1) \times \dots \times Hol(g_k)$ where, for each $1 \leq i \leq k$, $Hol(g_i)$ is a subgroup of $O(2m)$ acting trivially on T^j if $j \neq i$. We get then a foliation whose leaves are Kähler and totally geodesic manifolds.

Using N to represent a unit normal vector field orthogonal to H , we consider now, for each $1 \leq i \leq k$, the operator $\tilde{C}_N = p_{T^i} \circ C_N : T^i \rightarrow T^i$ which commutes also with J , is parallel and self-adjoint. Each M_i decomposes again, locally, as a product $M_i^1 \times \dots \times M_i^{r_i}$ of Kähler manifolds, so that, after these two factorization steps, we end up with

$$M = M_1 \times \dots \times M_r ,$$

where each M_i is Kähler. However, at this stage is not clear that the T^i may be eigenspaces for C_N . This turns out to be true.

Since φ , by assumption, is not minimal we can select $i \in \{1, \dots, r\}$ such that $trace \alpha|_{T^i \times T^i} \neq 0$. To simplify we assume $i = 1$ and consider $M = M_1 \times \tilde{M}$, where $\tilde{M} = M_2 \times \dots \times M_r$.

We consider $a = (a_1, a_2) \in M_1 \times \tilde{M}$ and claim that $\alpha(v, w) = 0$ whenever $v \in T_{a_1} M_1$ and $w \in T_{a_2} \tilde{M}$.

To prove the claim we fix $a_2 \in \widetilde{M}$ and consider $\varphi_{a_2} : M_1 \rightarrow Q(c)$. According to proposition 12 φ_{a_2} is *ppmc* and has flat normal bundle. It follows that its first normal bundle is parallel and we can reduce the codimension (cf.[6]). Therefrom $\varphi_{a_2}(M_1)$ is contained in a totally geodesic submanifold $\widetilde{Q}(c)$ of $Q(c)$, where $\dim \widetilde{Q}(c) \leq \dim M_1 + 2$. The claim is now a consequence of the fact that the first normal bundle of φ_{a_2} at x coincides with the first normal bundle of φ at (x, a_2) .

We proceed now analysing separately each of the three situations.

c = 0 We know from proposition 8 that M_1 must be a Riemann surface. It follows from Lemma 1 [11] that $\varphi = \varphi_1 \times \varphi_2$ is a product of immersions and two alternative possibilities come out:

1. $\varphi_1 : M_1 \rightarrow R^4$ has parallel mean curvature and φ_2 is totally geodesic.
2. $\varphi_1 : M_1 \rightarrow R^3$ and $\varphi_2 : M_2 \rightarrow R^{2m-1}$ are two isometric immersions with constant mean curvature.; from corollary 3 we infer that either $\dim M_2 = 2$ or $M_2 = \widetilde{M}_2 \times \widetilde{M}$ and $\varphi = \rho \times \sigma$, where $\widetilde{M}_2$ is a Riemann surface, ρ has parallel mean curvature and σ is totally geodesic.

c > 0 As above, we know from proposition 8 that M_1 is a surface. Applying Lemma 1 [11] to $\varphi : M_1 \times \widetilde{M} \rightarrow R^{2m+3}$ we conclude that $\varphi = \varphi_1 \times \varphi_2$ is a product of *ppmc* immersions and three distinct situations may be considered:

1. $m = 1$ and $\varphi : M \rightarrow S^4$ has parallel mean curvature.
2. $\varphi_1(M_1) \subset R^4$ and φ_2 has codimension 1. Then $\varphi_1(M_1) \subset S^3(c_1)$, and $\varphi_1 : M_1 \rightarrow S^3(c_1)$ has constant mean curvature. On the other hand, since $\varphi(M) \subset S^{2m+2}$, it follows from proposition 8 that $\dim \widetilde{M} = 2$ and $\varphi_2(\widetilde{M}) \subset S^2(c_2)$, $c_2 > 0$.
3. $\varphi_1(M_1) \subset R^3$ and φ_2 has codimension 2. Therefore $\varphi_1(M_1) \subset S^2(c_1)$, φ_2 is *ppmc* and $\varphi_2(M_2) \subset S^{2m-1}(c_2)$ with $\frac{1}{c} = \frac{1}{c_1} + \frac{1}{c_2}$. Thus from theorem 1, either $\dim M_2 = 2$ and $\varphi_2 : M_2 \rightarrow S^3(c_2)$ has constant mean curvature, or $\varphi_2(M_2)$ is an open subset of $S^2(a) \times S^2(b) \subset S^5(c_2)$, where $\frac{1}{a} + \frac{1}{b} = \frac{1}{c_2}$.

c < 0 As a model for the hyperbolic space we use the hyperbolic space model.

Now a new situation appears, namely when C is totally umbilic and $m > 2$. In this case we claim that H is an umbilical direction of α . To see this we consider a local orthonormal frame field $v_1, Jv_1, \dots, v_m, Jv_m$ of TM and, for each $1 \leq k \leq m$, we consider $V_k = \frac{1}{\sqrt{2}}(v_k - iJv_k)$. Clearly $C_N(V_k) = 0$ and $C_H(V_k) = \lambda V_k$ for a certain real λ .

It follows from Gauss equation that

$$C_{\alpha(X_1, X_k)} X_j = C_{\alpha(X_j, X_k)} X_i \quad , \quad \forall k, \forall i \neq j$$

Therefrom $C_{\alpha(X_1, X_k)} X_j = 0$ and $C_{\alpha(X_j, X_k)} X_i = 0$. We also know that $\alpha(X_i, X_k) = \alpha^H(X_i, X_k)H + \alpha^N(X_i, X_k)N$, where for $\xi \in C(T\bar{M})$, $\alpha^\xi = \langle \alpha(\cdot, \cdot), \xi \rangle$.

The claim follows now from the fact that, $\forall i, k$, $\alpha^H(X_i, X_k) = 0$, since $C_{\alpha(X_1, X_k)} X_j = \alpha^H(X_i, X_k)C_H X_j = 0$.

We may now conclude that $\varphi(M)$ is contained in a totally umbilical submanifold W of $H^{2m+2}(c)$ and that $\varphi : M \rightarrow W$ is minimal. Therefore we obtain one of the following situations:

$W = H^{2m+1}(d)$; then $m = 1$ and $\varphi : M \rightarrow H^{2m+1}(d)$ is minimal.

$W = S^{2m+1}(d)$; then $m = 1$ and M is minimal in $S^{2m+1}(d)$.

$W = E^{2m+1}$, a horosphere, so that $\varphi : M \rightarrow E^{2m+1}$ is pluriminimal.

Assume now that $M = M_1 \times M_2$, where $\text{trace } \alpha|_{T^1 \times T^1} \neq 0$.

Using an adapted version of Lemma 1 [11] we know that φ is a product immersion $\varphi_1 \times \varphi_2$, that is, there exist two non-degenerate orthogonal affine subspaces E_1 and E_2 of the Minkowski space R_1^{2m+3} such that $\varphi_1(M_1) \subset E_1$ and $\varphi_2(M_2) \subset E_2$. Now two cases must be considered: $\dim E_1 = 2r_1 + 1$ or $\dim E_2 = 2r_1 + 2$. We let $r_2 = m - r_1$.

Assume first that $\dim E_1 = 2r_1 + 2$. Then $\varphi_1(M_1)$ is contained either, in a hyperbolic space $H^{2r_1}(c_1)$, or in an euclidean sphere $S^{2r_1}(c_1)$, or in a horosphere E^{2r_1} . Since M_1 is Kähler we end up with one of the following situations:

- $\varphi_1(M_1)$ is an open subset of $H^2(c_1)$ so that $\varphi(M_2) \subset S^{2m-1}(c_2)$, $\frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{c}$, and $\varphi_2 : M_2 \rightarrow S^{2m-1}(c_2)$ is *ppmc*. Therefore it follows from theorem 1 that:

1. $\dim M_2 = 2$ and $\varphi_2 : M_2 \rightarrow S^3$ is *ppmc*.

2. $\varphi_2(M_2)$ is an open subset of $S^2(a) \times S^2(b) \subset S^5(c_2)$, where $\frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c_2} = \frac{1}{c}$, and we get $\varphi_1(M_1) \times \varphi_2(M_2) \subset H^2(c_1) \times S^2(a) \times S^2(b) \subset H^8(c)$.

- $\varphi_1(M_1)$ is an open subset of $S^2(c_1)$. We know then that $\varphi(M_2) \subset H^{2m-1}(c_2)$, $\frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{c}$, and $\varphi_2 : M_2 \rightarrow H^{2m-1}(c_2)$ is *ppmc*. From theorem 1 we infer that:

1. $\dim M_2 = 2$ and $\varphi = \varphi_1 \times \varphi_2 : M_1 \times M_2 \rightarrow S^2(c_1) \times H^3(c_2) \subset H^6(c)$.

2. $\varphi_2(M_2)$ is an open subset of $H^2(a) \times S^2(b) \subset H^5(c_2)$, $\frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c_2} = \frac{1}{c}$, and $\varphi_1(M_1) \times \varphi_2(M_2) \subset S^2(c_1) \times H^2(a) \times S^2(b) \subset H^8(c)$.

3. $\varphi_2(M_2)$ is an open subset of a horosphere E^{2m-2} of $H^{2m-1}(c_2)$. Then $\varphi_1(M_1) \times \varphi_2(M_2) \subset S^2(c_1) \times E^{2m-2} \subset S^2(c_1) \times H^{2m-1}(c_2) \subset H^{2m+2}(c)$.

- $\varphi_1(M_1)$ is an open subset of a horosphere E^{2m+1} of $H^{2r_1+1}(c)$. Therefore $\varphi(M_1 \times M_2) \subset E^{2m+1}$ and, from corollary 3, we conclude that $\dim M_1 = 2$, φ_1 is *ppmc* with image contained in a three-dimensional space and φ_2 is totally geodesic.

- $\varphi_1(M_1)$ is an open subset of a horosphere E^{2r_1} of $H^{2r_1+1}(c_1)$. In this case $\varphi_2(M_2) \subset S^{2(m-r_1)}(c_2)$ and, since M_2 is Kähler, $\varphi_2(M_2)$ is an open subset of $S^2(c_2)$. Then $\varphi_1(M_1) \times \varphi_2(M_2) \subset E^{2m-2} \times S^2(c_2) \subset H^{2m-1}(c_1) \times S^2(c_2)$.

Now if $\dim E_1 = 2r_1 + 2$, $\varphi_1(M_1)$ is contained either, in a hyperbolic space $H^{2r_1+1}(c_1)$, or in an euclidean sphere $S^{2r_1+1}(c_1)$, or in a horosphere. So one of the following situations come out:

- $\varphi_1 : M_1 \rightarrow H^{2r_1+1}(c_1)$ is *ppmc* and $\varphi_2(M_2) \subset S^{2(m-r_1)}(c_2)$, so that $r_1 = m - 1$ and $\varphi_2(M_2)$ is an open subset of $S^2(c_2)$. Then, from theorem 1, we get one of the following situations:
 1. $\dim M_1 = 2$ and $\varphi = \varphi_1 \times \varphi_2 : M_1 \times M_2 \rightarrow H^3(c_1) \times S^2(c_2) \subset H^5(c)$, $\frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{c}$.
 2. $\varphi_1(M_1)$ is an open subset of $H^2(a) \times S^2(b) \subset H^5(c_1)$, $\frac{1}{a} + \frac{1}{b} + \frac{1}{c_1} = \frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{c}$, and $\varphi_1(M_1) \times \varphi_2(M_2) \subset H^2(a) \times S^2(b) \times S^2(c_2) \subset H^8(c)$.
 3. $\varphi_1(M_1)$ is an open subset of a horosphere E^{2m-2} of $H^{2m-1}(c_1)$ and $\varphi_1(M_1) \times \varphi_2(M_2) \subset E^{2m-2} \times S^2(c_2) \subset H^{2m+2}(c)$.
- $\varphi_1 : M_1 \rightarrow S^{2r_1+1}(c_1)$ is *ppmc* and $\varphi_2(M_2)$ is an open subset of $H^2(c_2)$, $\frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{c}$. In this case we know from theorem 1 that one of the following situations hold:
 1. $\dim M_1 = 2$ so that $\varphi = \varphi_1 \times \varphi_2 : M_1 \times M_2 \rightarrow S^3(c_1) \times H^2(c_2) \subset H^6(c)$.
 2. $\dim M_1 = 4$ and $\varphi_2(M_2)$ is an open subset of $S^2(a) \times S^2(b) \subset S^5(c_1)$, so that $\varphi_1(M_1) \times \varphi_2(M_2) \subset S^2(a) \times S^2(b) \times H^2(c_2) \subset H^8(c)$.
- $\varphi_1(M_1)$ is an open subset of a horosphere $E^{2r_1+1} \subset H^{2r_1+2}(c)$. In this case $\varphi(M_1 \times M_2) \subset E^{2m+1}(c)$ and, from corollary 3, we get $\dim M_1 = 2$ and $\varphi = \varphi_1 \times \varphi_2$, where φ_1 is *ppmc*, $\varphi_1(M_1)$ is contained in a three-dimensional subspace of $E^{2m+1}(c)$ and φ_2 is totally geodesic.

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