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ON WEAKLY FINITE CONDUCTOR RINGS

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1. Introduction

All rings considered below are commutative with identity elements and all modules are unital. For a nonnegative integer n , an R -module E is n -presented if there is an exact sequence of R -modules:

$$F_n \rightarrow F_{n-1} \rightarrow \dots \rightarrow F_1 \rightarrow F_0 \rightarrow E \rightarrow 0$$

where each F_i is a finitely generated free R -module. In particular, 0-presented and 1-presented R -modules are, respectively, finitely generated and finitely presented R -modules.

An ideal I of R is called n -generated ideal if I can be generated by n -elements. We say that R is a finite conductor ring if each 2-generated ideal is finitely presented. On the other hand, R is called a weakly finite conductor ring if $Ra \cap Rb$ is a finitely generated ideal of R for each pair $a, b \in R$. In [13, Theorem 1.2], the authors show that R is a finite conductor ring if and only if R is a weakly finite conductor ring and each 1-generated ideal of R is finitely presented. In [13, Corollary 1.3], the authors also show that if R is a domain, then R is a finite conductor if and only if R is a weakly finite conductor and we obtain the well-known finite conductor domain. For instance, any coherent ring is a finite conductor ring and so it is a weakly finite conductor ring.

Let $n \geq 0$ be integer. We say that R is n -coherent if each n -presented cyclic R -module is $(n+1)$ -presented (i.e. each $(n-1)$ -presented ideal of R is n -presented); and that R is strong n -coherent if each n -presented R -module is $(n+1)$ -presented. In particular, any coherent ring is a strong 2-coherent ring and thus it is a 2-coherent ring.

In section 2, we examine the transfer of weakly finite conductor properties to the trivial ring extensions. Using these results, we construct a class of weakly finite conductor rings which are not finite conductor rings (and so not coherent rings). In the third section, we show that the class of weakly finite conductor rings and the class of (strong) 2-coherent rings are not comparable.

2. Transfer of the weakly finite conductor property to the trivial ring extensions.

Let A be a ring, E be an A -module and $R = A \ltimes E$ be the set of pairs (a, e) with pairwise addition and multiplication given by:

$$(a, e)(a', e') = (aa', ae' + a'e).$$

This is called the trivial ring extension of A by E . An ideal J of R has the form: $J = I \ltimes E'$, where I is an ideal of A and E' is an A -submodule of E such that $IE' \subseteq E'$. If J is a finitely

generated ideal, so is I by [11, Theorem 25.1].

We start by investigating the transfer of the weakly finite conductor property to the trivial ring extensions of a local domain (A, M) by an A -module E such that $ME = 0$.

Theorem 2.1. *Let (A, M) be a local domain with maximal ideal M , E be an A -module such that $ME = 0$, and $R := A \rtimes E$ be the trivial ring extension of A by E . Then R is a weakly finite conductor ring if and only if A is a weakly finite conductor ring and M is a finitely generated ideal.*

Before proving Theorem 2.1, we establish the following five Lemmas.

Lemma 2.2. *Let A be a ring, E be an A -module, and $R := A \rtimes E$ be the trivial ring extension of A by E . If R is a weakly finite conductor ring, then so is A .*

Proof. Let $a, b \in A$. Our aim is to show that $Aa \cap Ab$ is a finitely generated ideal of A . Since R is a weakly finite conductor ring, then $R(a, 0) \cap R(b, 0) = \sum_{i=1}^n R(a_i, e_i)$, where $a_i \in A$, $e_i \in E$, and n is a positive integer. Therefore, $Aa \cap Ab = \sum_{i=1}^n Aa_i$ and so A is a weakly finite conductor ring.

Lemma 2.3. *Let (A, M) be a local ring such that its maximal ideal M contains a regular element a , E be an A -module such that $ME = 0$, and $R := A \rtimes E$ be the trivial ring extension of A by E . Then $R(a, e) \cap R(a, f) = aM \rtimes 0$ for each $e \neq f \in E$.*

In particular, if R is a weakly finite conductor ring, then M is a finitely generated ideal.

Proof. Let $e \neq f \in E$. Let $(l, m)(a, e) = (u, v)(a, f) \in R(a, e) \cap R(a, f)$, where $(l, m), (u, v) \in R$. Hence $la = ua$ and $\bar{l}e = \bar{u}f$ since $a \in M$ and $ME = 0$. Therefore $l = u$ since a is a regular element, so $\bar{l}(e - f) = 0$. Hence $l \in M$ since $e - f \neq 0$ and E is an (A/M) -vector space. Therefore $R(a, e) \cap R(a, f) \subseteq aM \rtimes 0$. Conversely, let $(au, 0) \in aM \rtimes 0$, where $u \in M$. Clearly, $(au, 0) = (u, 0)(a, e) = (u, 0)(a, f)$ since $u \in M$ and so $(au, 0) \in R(a, e) \cap R(a, f)$.

If R is a weakly finite conductor ring, then $R(a, e) \cap R(a, f) (= aM \rtimes 0)$ is a finitely generated ideal of R , so aM is a finitely generated ideal of A . Hence, M is a finitely generated ideal of A since a is a regular element of A .

Lemma 2.4. *Under the hypothesis of Theorem 2.1, $R(0, e) \cap R(0, f)$ is a finitely generated ideal of R for each $e, f \in E$.*

Proof. Let $e, f \in E$ which is an (A/M) -vector space. Two cases are then possible:

Case 1: $\{e, f\}$ are linearly independent. Let $(l, m)(0, e) = (u, v)(0, f) \in R(0, e) \cap R(0, f)$, where $(l, m), (u, v) \in R$. But $(l, m)(0, e) = (0, \bar{l}e)$ and $(u, v)(0, f) = (0, \bar{u}f)$, so $\bar{l}e = \bar{u}f$ and then $\bar{l} = \bar{u} = 0$ since $\{e, f\}$ are linearly independent. Therefore, $R(0, e) \cap R(0, f) = 0$ which is a finitely generated ideal of R .

Case 2: $\{e, f\}$ are linearly dependent. Assume that $e = \bar{w}f$, where $w \in A$. Hence $(0, e) = (w, 0)(0, f) \in R(0, f)$ and so $R(0, e) \cap R(0, f) = R(0, e)$ which is a finitely generated ideal of R . Therefore, $R(0, e) \cap R(0, f)$ is a finitely generated ideal of R in all cases.

Lemma 2.5. *Under the hypothesis of Theorem 2.1, assume that M is a finitely generated ideal of A . Then $R(a, e) \cap R(b, f)$ is a finitely generated ideal of R for each $e, f \in E$ and for each $a, b \in M$ such that $a = wb$, where $w \in M$.*

Proof. Let $(l, m)(a, e) = (u, v)(b, f) \in R(a, e) \cap R(b, f)$, where $(l, m), (u, v) \in R$. Hence, $wbl = al = ub$ and $\bar{l}e = \bar{u}f$ since $a, b \in M$. But, $wbl = ub$ which implies that $u = wl \in M$ since A is a domain; so $\bar{l}e = \bar{u}f = 0$. Two cases are then possible: $e = 0$ or $e \neq 0$.

Case 1: $e = 0$. Hence $\bar{l}e = 0$ for each $l \in A$ and so $R(a, 0) \cap R(b, f) \subseteq R(a, 0)$. Conversely, let $(u, v)(a, 0) \in R(a, 0)$. Clearly, $(u, v)(a, 0) = (u, v)(wb, 0) = (uw, 0)(b, f)$ since $w \in M$, hence $(u, v)(a, 0) \in R(a, 0) \cap R(b, f)$. Therefore, $R(a, 0) \cap R(b, f) = R(a, 0)$ which is a finitely generated ideal of R .

Case 2: $e \neq 0$. Hence, $l \in M$ since $\bar{l}e = 0$ and so $R(a, e) \cap R(b, f) \subseteq aM \propto 0$. Conversely, let $(au, 0) \in aM \propto 0$, where $u \in M$. Hence $(au, 0) = (u, 0)(a, e) = (uw, 0)(b, f) \in R(a, e) \cap R(b, f)$. Therefore, $R(a, e) \cap R(b, f) = aM \propto 0 = \sum_{i=1}^n R(aa_i, 0)$ which is a finitely generated ideal of R

since $M := \sum_{i=1}^n Aa_i$, where $a_i \in A$ is a finitely generated ideal of A . Hence $R(a, e) \cap R(b, f)$ is a finitely generated ideal of R in all cases.

Lemma 2.6. *Under the hypothesis of Theorem 2.1, assume that A is a weakly finite conductor ring and M is a finitely generated ideal of A . Then $R(a, e) \cap R(b, f)$ is a finitely generated ideal of R for each non comparable elements $a, b \in M$ and for each $e, f \in E$.*

Proof. Let $(l, m)(a, e) = (u, v)(b, f) \in R(a, e) \cap R(b, f)$, where $(l, m), (u, v) \in R$. Hence, $Aa \cap Ab = \sum_{i=1}^n Aa_i$, where $a_i \in Aa \cap Ab$ since A is a weakly finite conductor ring, so $a_i = c_i a = d_i b$, where $c_i, d_i \in A$. We claim that $c_i, d_i \in M$ for each $i = 1, \dots, n$. Indeed, assume for example that there exists $j = 1, \dots, n$ such that $c_j \notin M$. So, c_j is invertible and then $Aa_j = Ac_j a = Aa$. Hence $Aa = Aa_j \subseteq \sum_{i=1}^n Aa_i = Aa \cap Ab \subseteq Ab$ and then $a \in Ab$, which is a contradiction since a

and b are not comparable.

Hence $c_i, d_i \in M$ for each $i = 1, \dots, n$, so $la = ub \in Aa \cap Ab = \sum_{i=1}^n Aa_i = a \sum_{i=1}^n Ac_i$ since $(l, m)(a, e) = (u, v)(b, f) \in R(a, e) \cap R(b, f)$. Therefore, $l \in \sum_{i=1}^n Ac_i \in M$ and then $(l, m)(a, e) = (la, 0) \in \sum_{i=1}^n R(ac_i, 0)$, so $R(a, e) \cap R(b, f) \subseteq \sum_{i=1}^n R(ac_i, 0) = \sum_{i=1}^n R(a_i, 0)$. Conversely, let $\sum_{i=1}^n (z_i, t_i)(a_i, 0) \in \sum_{i=1}^n R(a_i, 0)$, where $(z_i, t_i) \in R$ for each $i = 1, \dots, n$. Clearly, $\sum_{i=1}^n (z_i, t_i)(a_i, 0) = \sum_{i=1}^n (z_i a_i, 0) = \sum_{i=1}^n (z_i c_i a, 0) = \sum_{i=1}^n (z_i c_i, 0)(a, e) = \sum_{i=1}^n (z_i d_i b, 0) = (\sum_{i=1}^n z_i d_i, 0)(b, f) \in R(a, e) \cap R(b, f)$. Hence $R(a, e) \cap R(b, f) = \sum_{i=1}^n R(a_i, 0)$ which is a finitely generated ideal of R .

Proof of Theorem 2.1. If R is a weakly finite conductor ring, then A is weakly finite conductor by Lemma 2.2 and M is a finitely generated ideal by Lemma 2.3 since A is a domain. Conversely, assume that A is a weakly finite conductor domain and M is a finitely generated ideal. Let $I = R(a, e)$, $J = R(b, f)$ be two principal proper ideals of R , where $a, b \in M$ (since (u, v) is invertible in R if and only if u is invertible in A). Our aim is to show that $I \cap J$ is a finitely generated ideal of R . Three cases are then possible.

Case 1: $a = b = 0$. In this case, $I \cap J$ is a finitely generated ideal of R by Lemma 2.4.

Case 2: a and b are comparable. Assume for example that $a = cb$, where $c \in A$.

If $c \in M$, then $I \cap J$ is a finitely generated ideal of R by Lemma 2.5.

If $c \notin M$, then c is invertible. Clearly, $R(a, e) = R(bc, \bar{c}c^{-1}e) = R(c, 0)(b, c^{-1}e) = R(b, c^{-1}e)$ since $(c, 0)$ is invertible in R (since c is invertible in A). So, we may assume that $a = b$. Therefore, $I \cap J$ is a finitely generated ideal of R by Lemma 2.3.

Case 3: a and b are not comparable. In this case, $I \cap J$ is a finitely generated ideal of R by Lemma 2.6.

Hence $I \cap J$ is a finitely generated ideal of R in all cases. So R is a weakly finite conductor ring.

Now, we give a first example of a weakly finite conductor ring which is not a finite conductor ring and so not a coherent ring.

Example 2.7. Let (A, M) be a local coherent domain with finitely generated maximal ideal M , $E = (A/M)^\infty$, and $R := A \ltimes E$ be the trivial ring extension of A by E . Then:

1) R is a weakly finite conductor ring by Theorem 2.1.

2) R is not a finite conductor ring by [13, Theorem 2.1] since E is an (A/M) -vector space with infinite rank. In particular, R is not a coherent ring.

Secondly, we study the transfer of the weakly finite conductor property to the trivial ring extensions of any domain by his quotient field.

Theorem 2.8. *Let A be a domain, $Q = qf(A)$, and $R := A \times E$ be the trivial ring extension of A by E . Then R is a weakly finite conductor ring if and only if A is a weakly finite conductor ring.*

Before proving Theorem 2.8, we establish the following Lemma.

Lemma 2.9. *Under the hypothesis of Theorem 2.8, let $I = R(a, e)$ be a principal ideal of R , where $a \in A - \{0\}$ and $e \in Q$. Then, $I = Aa \times Q = R(a, 0)$.*

Proof. Clearly, $I = R(a, e) = \{(b, f)(a, e)/b \in A, f \in Q\} = \{(ba, fa + be)/b \in A, f \in Q\}$. But $\{af/f \in Q\} = Q$, hence $I = Aa \times Q = R(a, 0)$.

Proof of Theorem 2.8. If R is a weakly finite conductor ring, then so is A by Lemma 2.2. Conversely, assume that A is a weakly finite conductor domain. Let $I = R(a, e)$ and $J = R(b, f)$ be two principal ideals, where $a, b \in A$ and $e, f \in Q$. Three cases are then possible.

Case 1: $a = b = 0$. Hence $I = R(0, e) = 0 \times Ae$ and $J = R(0, f) = 0 \times Af$. Let $e = a_1/d_1$ and $f = a_2/d_2$, where $a_1, a_2 \in A$ and $d_1, d_2 \in A - \{0\}$. Hence, $Ae \cap Af = [(Aa_1d_2) \cap (Aa_2d_1)]/(d_1d_2) = (\sum_{i=1}^n Aa_i)/(d_1d_2)$, where $a_i \in A$ for each $i = 1, \dots, n$ since A is a

weakly finite conductor domain. Therefore, $I \cap J = 0 \times (Ae \cap Af) = 0 \times [(\sum_{i=1}^n Aa_i)/(d_1d_2)] =$

$\sum_{i=1}^n R(0, a_i/(d_1d_2))$ is a finitely generated ideal of R .

Case 2: $a \neq 0$ and $b = 0$, or $a = 0$ and $b \neq 0$.

Assume that $a \neq 0$ and $b = 0$. Hence, $I = R(a, e) = Aa \times Q$ by Lemma 2.9 and $J = R(0, f) = 0 \times Af$, so $J \subseteq I$ and $I \cap J = J$ which is a finitely generated ideal of R .

If $a = 0$ and $b \neq 0$, the same argument shows that $I \cap J = I$ which is a finitely generated ideal of R .

Case 3: $a \neq 0$ and $b \neq 0$. Hence, $I = R(a, e) = Aa \times Q$ and $J = R(b, f) = Ab \times Q$. Let $Aa \cap Ab = \sum_{i=1}^n Aa_i$, where $a_i \in A - \{0\}$ since A is a weakly finite conductor ring. Hence,

$I \cap J = (Aa \times Q) \cap (Ab \times Q) = (Aa \cap Ab) \times Q = (\sum_{i=1}^n Aa_i) \times Q = \sum_{i=1}^n R(a_i, 0)$ by Lemma 2.9, so $I \cap J$ is a finitely generated ideal of R .

In all cases, $I \cap J$ is a finitely generated ideal of R . Hence R is a weakly finite conductor ring.

Now, we give a second example of a weakly finite conductor ring which is not a finite conductor ring and so not a coherent ring.

Example 2.10. Let A be a coherent domain, $Q = qf(A)$, and $R := A \propto E$ be the trivial ring extension of A by E . Then:

- 1) R is a weakly finite conductor ring by Theorem 2.8.
- 2) R is not a finite conductor ring by [13, Corollary 2.4]. In particular, R is not a coherent ring.

3. Weakly Finite Conductor and 2-Coherent Properties.

We know that any coherent ring is a weakly finite conductor ring and a (strong) 2-coherent ring. Thus a natural question arises: Is there any relationship (inclusion) between a class of weakly finite conductor rings and a class of (strong) 2-coherent rings?

The goal of this section is to give a negative answer to this question. Hence, we construct an example of a strong 2-coherent ring (and so a 2-coherent ring) which is not a weakly finite conductor ring (Example 3.1). We also give an example of a weakly finite conductor ring which is not a 2-coherent ring (and so not a strong 2-coherent ring) (Example 3.2). We construct an example of a weakly finite conductor strongly 2-coherent ring which is not coherent (Example 3.3), so the intersection of weakly finite conductor class and (strong) 2-coherent class is not only a class of coherent rings. Finally, we give an example of a ring which is neither a weakly finite conductor nor (strong) 2-coherent (Example 3.4).

Example 3.1. Let k be a field, K be a field containing k as a subfield such that $[K : k] = \infty$, $T = K[[X]] = K + M$, where X is an indeterminate over K and $M = XT$ the maximal ideal of T . Set $R = k + M$, then:

- 1) R is a strong 2-coherent domain (and so is a 2-coherent domain) which is not a coherent domain by [5, Corollary 5.2].
- 2) R is not a finite conductor domain by [13, Theorem 3.1] and [5, Corollary 5.2], so R is not a weakly finite conductor domain by [13, Corollary 1.3].

Example 3.2. Let $A = K[[X]]$ be a formal power series, where K is a field of characteristic zero. Let E be the injective hull of the residue field $K := A/(X)$, and let $R := A \propto E^\infty$ be the trivial extension ring of A by E^∞ . Note that, by [4, Exercise 32(b), p.173], we may take $E = K[Y]$ where A operates by $XP = \partial P / \partial Y$ for $P \in K[Y]$. Then:

1) We claim that R is not a 2-coherent ring (and so is not a strong 2-coherent ring). Indeed, let $f \in K^\infty - \{0\}$ and let $J = R(0, f)$ be an ideal of R . It suffices to show that J is finitely presented and is not 2-presented. Consider the exact sequence of R -modules:

$$0 \rightarrow \text{Ker}(u) \rightarrow R \xrightarrow{u} J \rightarrow 0$$

where $u(a, P) = (a, P)(0, f) = (0, af)$. Hence $\text{Ker}(u) = XA \propto E^\infty$ since $f \in K^\infty - \{0\}$, so $\text{Ker}(u) = R(X, 0)$ since $aE = E$ for each $a \in E - \{0\}$. Consider the exact sequence of R -modules:

$$0 \rightarrow \text{Ker}(v) \rightarrow R \xrightarrow{v} \text{Ker}(u) (= R(X, 0)) \rightarrow 0$$

where $v(a, P) = (a, P)(X, 0) = (aX, XP) = (aX, \partial P / \partial Y)$. Hence, $\text{Ker}(v) = 0 \propto K^\infty$ which is not finitely generated. Therefore, $\text{Ker}(u)$ is finitely generated and is not finitely presented, so J is finitely presented which is not 2-presented. Thus R is not 2-coherent and so not strong 2-coherent.

2) We claim that R is a weakly finite conductor ring. Indeed, let $I = R(a, P)$ and $J = R(b, Q)$ be two principal proper ideals of R , where $a, b \in XA$ and $P, Q \in E$ (since (c, f) is invertible in R if and only if c is invertible in A). Our aim is to show that $I \cap J$ is a finitely generated ideal of R . Three cases are then possible.

Case 1: $a \neq 0$ and $b \neq 0$. In this case, $I = R(a, P) = aA \propto E^\infty$ and $J = R(b, Q) = bA \propto E^\infty$ since $dE = E$ for each $d \in E - \{0\}$. Let $c \in XA - \{0\}$ such that $cA = aA \cap bA$ since A is a Noetherian valuation domain. Therefore, $I \cap J = (aA \propto E^\infty) \cap (bA \propto E^\infty) = (aA \cap bA) \propto E^\infty = cA \propto E^\infty = R(c, 0)$ is finitely generated since $cE = E$ (since $c \neq 0$).

Case 2: $a \neq 0$ and $b = 0$, or $a = 0$ and $b \neq 0$.

Assume that $a \neq 0$ and $b = 0$. Hence, $I = R(a, P) = Aa \propto E^\infty$ since $aE = E$ (since $a \neq 0$). On the other hand, $J = R(0, Q) = 0 \propto AQ$. Therefore, $J \subseteq I$ and then $I \cap J = J$ which is finitely generated.

If $a = 0$ and $b \neq 0$, the same argument shows that $I \subseteq J$ and so $I \cap J = I$ which is finitely generated.

Case 3: $a = b = 0$. In this case, $I = R(0, P) = 0 \propto AP = 0 \propto L$ and $J = R(0, Q) = 0 \propto AQ = 0 \propto H$, where $L = AP$ and $H = AQ$. Let $P = (P_i)_i \in E^\infty$ and $Q = (Q_i)_i \in E^\infty$, where $P_i, Q_i \in E (= K[Y])$ and let $n \geq 1$ be an integer such that the polynomials P_i and Q_i have degree at most n for each i . Hence, L and H are K -vector spaces of finite rank since the degree of cP_i (resp. cQ_i) is at most equal to the degree of P_i (resp. Q_i) for each $c \in A$ and for each i , so $L \cap H$ is a K -vector space of finite rank. Let $L \cap H = \sum_{i=1}^m K f_i$, where m is a positive integer

and $f_i \in E^\infty$ for each i . We claim that $I \cap J = 0 \propto (L \cap H) = \sum_{i=1}^m R(0, f_i)$.

Indeed, we have $(0, f_i) \in I \cap J$ for each $i = 1, \dots, m$ since $f_i \in L \cap H$ for each $i = 1, \dots, m$. Hence,

$\sum_{i=1}^m R(0, f_i) \subseteq I \cap J$ since $I \cap J$ is an ideal of R . Conversely, let $(0, h) \in I \cap J (= 0 \propto (L \cap H))$. Hence, $h \in L \cap H = \sum_{i=1}^m K f_i$, so $h = \sum_{i=1}^m a_i f_i$, where $a_i \in K$ for each $i = 1, \dots, m$. Therefore, $(0, h) = (0, \sum_{i=1}^m a_i f_i) = \sum_{i=1}^m (a_i, 0)(0, f_i) \in \sum_{i=1}^m R(0, f_i)$ and so $I \cap J = \sum_{i=1}^m R(0, f_i)$. Hence, $I \cap J = \sum_{i=1}^m R(0, f_i)$ is finitely generated.

Therefore, $I \cap J$ is a finitely generated ideal of R in all cases, so R is a weakly finite conductor ring.

Example 3.3. Let $R = K \propto E$ be the trivial extension ring of K by E , where K is a field and E is a K -vector space of infinite rank. Then:

1) R is a strong 2-coherent ring (and so is a 2-coherent ring) which is not a coherent domain by [15, Theorem 3.4].

2) We claim that R is a weakly finite conductor ring. Indeed, let $I = R(0, f) = 0 \propto K f$ and $J = R(0, g) = 0 \propto K g$ be two principal proper ideals of R , where $f, g \in E - \{0\}$. Hence, $I \cap J = 0 \propto (K f \cap K g)$. But, $K f \cap K g$ is a K -vector space of rank at most 1, so $K f \cap K g = K h$, where $h \in E$. Therefore, $I \cap J = 0 \propto (K f \cap K g) = 0 \propto K h = R(0, h)$ is a finitely generated ideal of R and so R is a weakly finite conductor ring.

Example 3.4. Let (A, M) be a local domain such that its maximal ideal M is not finitely generated, and let $R = A \propto A/M$ be the trivial ring extension of A by (A/M) . Then:

1) R is not a weakly finite conductor ring by Theorem 2.1 since M is not a finitely generated ideal of A .

2) R is not a 2-coherent ring (and thus it is not a strong 2-coherent ring) by [12, proof of Theorem 1.1].

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