Geometric Transitions, del Pezzo
Surfaces and Open String Instantons

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We continue the study of a class of geometric transitions proposed by Aganagic and Vafa which exhibit open string instanton corrections to Chern-Simons theory. In this paper we consider an extremal transition for a local del Pezzo model which predicts a highly nontrivial relation between topological open and closed string amplitudes. We show that the open string amplitudes can be computed exactly using a combination of enumerative techniques and Chern-Simons theory proposed by Witten some time ago. This yields a striking conjecture relating all genus topological amplitudes of the local del Pezzo model to a system of coupled Chern-Simons theories.

e-print archive: http://xxx.lanl.gov/hep-th/0206163
1 Introduction

In the original formulation [?], geometric transitions have predicted a remarkable relation between Chern-Simons theory on $S^3$ and closed topological strings on the small resolution of a conifold singularity. This correspondence has been extended to knots and links in [?, ?, ?, ?, ?] and it has been recently proven from a linear sigma model perspective in [?]. A different generalization has been proposed in [?], where the Chern-Simons theory was corrected by open string instanton effects. This new class of dualities yields very interesting predictions relating topological open string amplitudes in various toric backgrounds to certain open string expansions. The new feature of these transitions is a fascinating interplay of open string enumerative geometry and Chern-Simons theory proposed by Witten in [?]. Open string enumerative techniques have been developed in [?]-[?],[?],[?],[?]-[?],[?]-[?],[?]-[?]. Applying some of these results, we have successfully tested this approach for a simple exactly soluble model in [?].

The question we would like to address in this paper is if one can perform similar high precision tests of the duality in more general toric backgrounds. In particular, we consider the local $dP_2$ model, which is a toric noncompact Calabi-Yau threefold fibered over the del Pezzo surface of degree two. This is the simplest local model containing a compact divisor which exhibits extremal transitions. Since there is an abundance of holomorphic curves on the del Pezzo surface, the topological closed string amplitudes are quite complicated. So far, concrete computations have been performed only for genus zero Gromov-Witten invariants [?]. The extremal transition in question is obtained by contracting two $(-1,-1)$ curves on the noncompact threefold, and then smoothing out the conifold singularities. After a somewhat technical analysis, one can show that the resulting open string theory consists of two Chern-Simons theories supported on two disjoint 3-spheres which are coupled by instanton effects. Systems of this kind have been predicted by Witten in [?].

The main result of this paper is that the open string instanton corrections can be summed exactly using the techniques developed in [?],[?],[?]. This yields a fairly simple system of Chern-Simons theories by interpreting the instanton corrections as Wilson loop perturbations of the Chern-Simons theories [?]. Then large $N$ duality predicts that the ’t Hooft expansion of these coupled Chern-Simons theories computes all topological closed string amplitudes of the local $dP_2$ model! We show by direct computations that this conjecture is valid up to degree four in the expansion in terms of Kähler parameters. This is very strong evidence that the conjecture is true to all orders, but we do not have a general proof.

This paper is structured as follows. In section two we study the geometry of the extremal transition and construct the primitive open string instantons after defor-
Section three consists of a review of the topological closed string theory for the local $dP_2$ model following [?]. In section four we present the main results, namely the open string instanton expansion accompanied by Chern-Simons computations. Here we find a precise agreement with the known genus zero Gromov-Witten invariants, and make some higher genus predictions. Sections five and six are devoted to open string enumerative computations based on localization techniques as in [?, ?, ?]. Finally, some technical details and calculations are presented in the two appendixes.

Acknowledgements. During this work we have greatly benefited from interactions with Mina Aganagic, Marcos Mariño and Cumrun Vafa who were working simultaneously on a similar project [?]. We would like to express our special thanks to them for sharing their ideas and insights with us regarding the framing dependence (section 4).

We would also like to thank Ron Donagi and Tony Pantev for collaboration on a related project, and Bobby Acharya, Michael Douglas, John Etnyre, Albrecht Klemm, John McGreevy and Harald Skarke for very stimulating conversations. We owe special thanks (and lots of tiramisù) to Corina Florea for invaluable help with the LaTeX conversion of the original draft. The work of D.-E. D. has been supported by DOE grant DOE-DE-FG02-96ER40959; A.G. is supported in part by the NSF Grant DMS-0074980.

2 Geometric Transitions for Local $dP_2$ Model

The local $dP_2$ model is a toric Calabi-Yau threefold $X$ isomorphic to the total space of the canonical bundle $O(K_{dP_2})$. We have $X = (\mathbb{C}^5 \setminus F) / (\mathbb{C}^*)^3$ defined by the following toric data

$$
\begin{array}{cccccc}
X_0 & X_1 & X_2 & X_3 & X_4 & X_5 \\
1 & -1 & 1 & -1 & 1 & 0 \\
0 & -1 & 0 & 1 & -1 & 1 \\
0 & -1 & 0 & 0 & 1 & -1 \\
\end{array}
$$

with disallowed locus $F = \{X_1 = X_3 = 0\} \cup \{X_2 = X_4 = 0\} \cup \{X_3 = X_5 = 0\}$. This toric quotient can be equivalently described as a symplectic quotient $\mathbb{C}^6 / U(1)^3$ with moment maps

$$
\begin{align*}
|X_1|^2 + |X_3|^2 - |X_2|^2 - |X_0|^2 &= \xi_1 \\
|X_2|^2 + |X_4|^2 - |X_3|^2 - |X_0|^2 &= \xi_2 \\
|X_3|^2 + |X_5|^2 - |X_4|^2 - |X_0|^2 &= \xi_3
\end{align*}
$$

where $\xi_1, \xi_2, \xi_3 > 0$. The toric fan of $X$ is a cone over the two dimensional polytope represented below.
Figure 1: A section in the toric fan of $X$. The resulting polytope describes $\mathbb{P}^2$ blown-up at two points.

There is a single compact divisor $S$ on $X$ which is the zero section of $\pi : X \rightarrow dP_2$ defined by $X_0 = 0$. The Mori cone of $X$ is generated by the curve classes $e_1, h - e_1 - e_2, e_2$ corresponding the cones over $v_0v_2, v_0v_3$ and respectively $v_0v_4$. One can check that these are rigid rational curves with normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. Since $X$ is a toric manifold, it can be represented as a topological $T^2 \times \mathbb{R}$ fibration over $\mathbb{R}^3$ whose discriminant is the two dimensional planar graph represented in fig. 1. Then the curves $e_1, h - e_1 - e_2, e_2$ can be represented as $S^1$ fibrations over certain edges of the graph as shown there.

The moment maps (2) yield the following parameterization of the Kähler cone

$$J = \xi_1(h - e_1) + \xi_2 h + \xi_3(h - e_2)$$

(3)

where $J$ represents the restriction of the Kähler class to $S$. In the following we will use alternative Kähler parameters $(s_1, t, s_2)$ defined by

$$J = -s_1e_1 + th - s_2e_2.$$ 

(4)

We are interested in a extremal transition consisting of a contraction of $e_1, e_2$ on $X$ followed by a smoothing of the two nodal singularities. It turns out that the resulting singular threefold $\hat{X}$ can be described as a nodal hypersurface in a toric variety $Z = (\mathbb{C}^3 \setminus \{0\}) \times \mathbb{C}^2 / \mathbb{C}^*$. The ($\mathbb{C}^*$) action is defined by

$$\begin{align*}
\mathbb{C}^* & \quad Z_1 \quad Z_2 \quad Z_3 \quad U \quad V \\
\mathbb{C}^* & \quad 1 \quad 1 \quad 1 \quad -1 \quad -2.
\end{align*}$$

(5)
Obviously, $Z$ is isomorphic to the total space of $\mathcal{O}(-1) \oplus \mathcal{O}(-2)$ over $\mathbb{P}^2$. The embedding of $i : \hat{X} \hookrightarrow Z$ is given terms of homogeneous coordinates by

$$Z_1 = X_2X_3X_4, \quad Z_2 = X_1X_2, \quad Z_3 = X_4X_5, \quad U = X_0X_1X_5, \quad V = -X_0X_3.$$  

(6)

In order to make sure this is a regular map, one has to check that (6) is compatible with the toric actions (1), (5) and that the disallowed loci agree. We have included the details in appendix A.1. The image of $\hat{X}$ in $Z$ is given by

$$UZ_1 + VZ_2Z_3 = 0.$$  

(7)

One can easily check that this hypersurface has exactly two nodal singularities, as expected. The singular locus is described by

$$U = 0, \quad Z_1 = 0, \quad Z_2Z_3 = 0, \quad VZ_2 = 0, \quad VZ_3 = 0.$$  

(8)

Since $Z_1, Z_2, Z_3$ do not vanish simultaneously, the singular points are $\{U = V = 0, Z_1 = Z_2 = 0\} \cup \{U = V = 0, Z_1 = Z_3 = 0\}$. Therefore we obtain the two expected conifold singularities.

This representation of $\hat{X}$ allows a concrete description of the extremal transition. We can resolve the singularities by blowing up $Z$ along the zero section $U = V = 0$. The proper transform of $\hat{X}$ will then be isomorphic to the local $dP_2$ model $X$ discussed before. Alternatively, we can smooth out (7) by deforming the polynomial equation to

$$UZ_1 + VZ_2Z_3 = \mu.$$  

(9)

We obtain a family of hypersurfaces $Y/\Delta$ parameterized by a complex parameter $\mu \in \Delta$, where $\Delta$ is the unit disc. Without loss of generality we can take $\mu$ to be real and positive, and we denote by $Y \equiv Y_\mu$ the corresponding fiber. The transition can be very conveniently described in terms of the $T^2$ fibration structure represented in fig. 1. The discriminant of the torus fibration undergoes the following sequence of transformations. As a differential manifold, $Y$ is obtained from $X$ by performing surgery along the links of the two nodes, which are both isomorphic to $S^2 \times S^3$ [?]. The third homology is generated by the two vanishing cycles $L_1, L_2$ associated to the nodal singularities, subject to the relation $[L_1] - [L_2] = 0$. Topologically, these cycles are 3-spheres which can be locally described as fixed point sets of local antiholomorphic involutions. We will give some details below, since this is an important point for the rest of the paper. Let us cover $Z$ with three coordinate patches

$$U_1 = \{Z_1 \neq 0\} : \quad x_1 = Z_2Z_1, \quad y_1 = Z_3Z_1, \quad u_1 = UZ_1, \quad v_1 = VZ_2^2$$  

$$U_2 = \{Z_2 \neq 0\} : \quad x_2 = Z_1Z_2, \quad y_2 = Z_3Z_2, \quad u_2 = UZ_2, \quad v_2 = VZ_3^2$$  

$$U_3 = \{Z_3 \neq 0\} : \quad x_3 = Z_1Z_3, \quad y_3 = Z_2Z_3, \quad u_3 = UZ_3, \quad v_3 = VZ_3^2.$$  

(10)
In local coordinates, the hypersurface equation (7) can be written

\[ U_1 : \quad u_1 + v_1 x_1 y_1 = \mu \]
\[ U_2 : \quad u_2 x_2 + v_2 y_2 = \mu \]
\[ U_3 : \quad u_3 x_3 + v_3 y_3 = \mu. \] (11)

One can then see two conifold singularities at \( \mu = 0 \) in the patches \( U_2, U_3 \). The corresponding vanishing cycles are defined by the real sections

\[ U_3 : \]
\[ u_3 = \overline{x}_3, \]
\[ v_3 = \overline{y}_3. \]

We show in appendix A.2. that one can choose a symplectic Kähler form \( \omega \) on \( Y \) so that \( L_1, L_2 \) are lagrangian cycles. More precisely, one can construct \( \omega \) so that it
is locally isomorphic to the standard symplectic form on a deformed conifold near \( L_1, L_2 \). To conclude this section, let us discuss the second homology of \( Y \). An exact sequence argument shows that \( H_2(Y, L_1 \cup L_2, \mathbb{Z}) \cong H_2(Y, \mathbb{Z}) = \mathbb{Z} \). One can construct certain nontrivial relative 2-cycles on \( Y \) as holomorphic discs \( D_1, D_2 \) with boundaries on \( L_1, L_2 \). Let \( \Sigma_{0,1} \) be the disc \( \{|t| \leq \mu^{-1/2}\} = \{\mu^{1/2} \leq |t'|\} \) in a projective line \( \mathbb{P}^1 \) with affine coordinates \( t, t' \). We construct a holomorphic embedding \( f_1 : \Sigma_{0,1} \longrightarrow Y \) given in local coordinates by

\[
(16)
\]

\[
U_2 : \\
x_2(t') = t' , \\
y_2(t') = 0 , \\
u_2(t') = \mu t' , \\
v_2(t') = 0 .
\]

It easy to check that \( f \) is well defined and it maps the boundary \( |t'| = \mu^{1/2} \) of \( \Sigma_{0,1} \) to an unknot \( \Gamma_1 \) in \( L_1 \). We will denote the image of \( \Sigma_{0,1} \) in \( Y \) by \( D_1 \). We can construct similarly a disc ending on \( L_2 \). The embedding map is locally given by

\[
(18)
\]

\[
U_3 : \\
x_3(t') = t' , \\
y_3(t') = 0 , \\
u_3(t') = \mu t' , \\
v_3(t') = 0 .
\]

To complete this discussion, note that one can also embed a holomorphic annulus \( C \) in \( Y \), the boundary components being mapped to \( L_1, L_2 \). For this we have to
use the coordinate patches \( U_2, U_3 \). Let \( \Sigma_{0,2} \) be the cylinder \( \{ \mu^{1/2} \leq |t| \leq \mu^{-1/2} \} = \{ \mu^{-1/2} \geq |t'|^2 \geq \mu^{1/2} \} \) in \( \mathbb{P}^1 \) with affine coordinates \((t, t')\) (recall that \( \mu \) is a positive real number inside the unit disc, hence \( \mu < 1 \)). We define a map \( f : \Sigma_{0,2} \to Y \) by

\[
(20)
\]

\[
U_3 : \\
x_3(t') = 0, \\
y_3(t') = t', \\
u_3(t') = 0, \\
v_3(t') = \mu t'.
\]

Then the boundary component \( |t| = \mu^{1/2} \) is mapped to \( L_1 \), while the boundary \( |t| = \mu^{-1/2} \) is mapped to \( L_2 \). Note that the discs \( D_1, D_2 \) and the cylinder \( C \) can be in principle covered by a single coordinate patch. We have used two coordinate patches for reasons that will be clear in section six. Let us denote the two boundary components of \( C \) by \( \Xi_1, \Xi_2 \), which are again to be regarded as knots in \( L_1, L_2 \). An important point for Chern-Simons computations is that \( \{ \Gamma_1, \Xi_1 \} \) and respectively \( \{ \Gamma_2, \Xi_2 \} \) are algebraic links in \( L_1, L_2 \). This can be seen by noting that locally we can identify for example \( L_1 \) to the sphere

\[
|x_2|^2 + |y_2|^2 = \mu
\]

in \( \mathbb{C}^2 \) with coordinates \((x_2, y_2)\). Then the disc \( D_1 \) and \( C \) can be locally described by the equation \( x_2 y_2 = 0 \) in \( \mathbb{C}^2 \). It is a well known fact that the intersection of this singular curve with the sphere surrounding the origin is an algebraic link with linking number one in the orientations induced by the complex structure. More precisely, if we parameterize the two boundary components as \( x_2 = \mu^{1/2} e^{i\theta_x}, y_2 = \mu^{1/2} e^{i\theta_y} \), the 1-forms \( d\theta_x, d\theta_y \) define orientations of \( \Gamma_1, \Xi_1 \) such that the linking number is 1 \([?]\). The same is true for \( \Gamma_2, \Xi_2 \) in \( L_2 \). In particular this shows that \( D_1, C \) and \( D_2, C \) are disconnected. Note that from now on we will fix the above orientations for \( \Gamma_1, \Xi_1, \Gamma_2, \Xi_2 \). In terms of the \( T^2 \) fibrations, \( D_1, D_2, C \) can be represented as in fig. 3.

Using this picture it is easy to see that there is a continuous family of 2-spheres interpolating between \( D_1 \) and \( D_2 \), hence the relation \([D_1] - [D_2] = 0 \). Similarly,
we have \([C] = [D_1] = [D_2]\), and we will denote this relative homology class by \(\beta\). However note that these 2-spheres are not holomorphically embedded in \(Y\). In fact we will show in section five that there are no holomorphic curves on \(Y\) and that \(\beta\) is a generator of \(H_2(Y, L_1 \cup L_2; \mathbb{Z})\). Therefore \(D_1, D_2, C\) constructed above are primitive open string instantons. Since \(L_1, L_2\) are lagrangian, this shows that \(D_1, D_2, C\) have the same symplectic area

\[
t_{op} = \int_{D_1} \omega = \int_{D_2} \omega = \int_{C} \omega.
\]

(22)

By deforming the discs as topological spheres away from the vanishing cycles, one can show that classically \(t = t_{op}\). This is so because the transition leaves the symplectic form essentially unchanged away from the singular locus. This concludes our discussion of the extremal transitions for the local \(dP_2\) model from a geometric point of view. The physics of the transitions will be explored in the next sections.

3 Closed String Amplitudes and Duality Predictions

In the context of geometric transitions we are interested in a relation between the topological closed string \(A\) model on \(X\) and a topological open string \(A\) model on the deformation space \(Y\). The topological open string theory on \(Y\) is defined by wrapping \(N_1\) and respectively \(N_2\) D-branes on the lagrangian cycles \(L_1, L_2\). The target space action of this theory consists of two Chern-Simons theories with gauge groups \(U(N_1), U(N_2)\) supported on the two cycles [?]. We will see later that these theories are coupled by open string instanton effects.
In order to have the right integrality properties, the topological amplitudes must be written in terms of flat coordinates. On the closed string side we have flat coordinates \((\hat{s}_1, \hat{s}_2, \hat{t})\) corresponding to the classical coordinates \((s_1, s_2, t)\). For simplicity, we will drop the notation \(\hat{\phantom{\hat}}\), keeping in mind that topological amplitudes will always be written in terms of flat coordinates. The open string theory contains a classical geometric parameter defined in (22). Accordingly we have a flat coordinate \(\hat{t}_{\text{op}}\), which will also be denoted by \(t_{\text{op}}\) from now on. As discussed above, classically, one would predict a relation of the form \(t = \hat{t}_{\text{op}}\), but this has to be refined at quantum level, as discussed in [?]. Moreover, we will see later in section four that the open string amplitudes depend in fact on three flat coordinates corresponding to the three primitive instantons constructed in section two.

Without going into details for the moment, note that large \(N\) duality predicts a relation between closed and open string amplitudes of the form

\[
\mathcal{F}_{\text{cl}}(g_s, s_1, s_2, t) = \mathcal{F}_{\text{op}}(g_s, \lambda_1, \lambda_2, t_{\text{op}}). \tag{23}
\]

Here \(\lambda_1 = N_1 g_s, \lambda_2 = N_2 g_s\) are the \(t\) Hooft coupling constants of the two Chern-Simons theories on \(L_1, L_2\) which should be related to the closed string parameters \((s_1, s_2)\). We will discuss the precise relation in section four.

According to [?], the closed string free energy on \(X\) has the following structure\(^1\)

\[
\mathcal{F}_{\text{cl}}(t, s_1, s_2, g_s) = \sum_{r=0}^{\infty} \sum_{n=1}^{\infty} \sum_{C \in H_2(X, \mathbb{Z})} N_C^r n \int (2 \sin n g_s 2)^{2r-2} e^{-n<J,C>}. \tag{24}
\]

In this expression, \(N_C^r\) are the Gopakumar-Vafa invariants of \(X\) which count the number of BPS states of charge \(C\) and spin quantum number \(r\) in M-theory compactified on \(X\); \(n\) counts multicovers. In the following we will refer to them as GV invariants. In terms of the generators \((e_1, h - e_1 - e_2, e_2)\) of the Mori cone we have

\[
C = d_1 e_1 + d(h - e_1 - e_2) + d_2 e_2, \quad d, d_1, d_2 \in \mathbb{Z} \tag{25}
\]

Note that \(C\) is representable by an irreducible reduced curve only if \(0 \leq d_1, d_2 \leq d\) or \(d = 0\) and \(d_1 = d_2 = 1\). Recall that the Kähler class \(J\) is given by (4), \(J = -s_1 e_1 + th - s_2 e_2\). Then we can rewrite (24) as

\[
(26)
\]

\(^1\)Throughout this paper, we will consider truncated expressions for the free energy, that is we will omit the polynomial terms.
Note that the first two terms represent the universal contributions of the two exceptional curves, whose open string interpretation is well understood [?]. In the second series, the GV invariants are not known for the local $dP_2$ model, except for $r = 0$, when they can be computed using mirror symmetry. This calculation has been performed in [?].

4 Open String Amplitudes and The Duality Map

We now consider the open string A model defined by wrapping $N_1$ and respectively $N_2$ branes on the lagrangian cycles $L_1, L_2$ in $Y$. According to [?], the target space physics of this model is captured by two Chern-Simons theories with gauge groups $U(N_1)$ and respectively $U(N_2)$ supported on the cycles $L_1, L_2$. As explained in [?], the Chern-Simons theory is in general corrected by open string instantons which give rise to Wilson loop operators in the target space action. The complete action can then be schematically written in the form

$$S(A_1, A_2) = S_{CS}(A_1) + S_{CS}(A_2) + F_{\text{inst}}(g_s, t_{\text{op}}, A_1, A_2)$$ \quad (27)$$

where $A_1, A_2$ denote the two gauge fields on $L_1, L_2$. The concrete form of the instanton expansion depends on the details of the model. In general $A_1, A_2$ enter $F_{\text{inst}}(g_s, t_{\text{op}}, A_1, A_2)$ via holonomy operators associated to the boundary components of open string instantons interpreted as knots in $L_1, L_2$. For large volume, the instanton corrections can be treated perturbatively from the Chern-Simons point of view. Therefore in the large $N$ limit, the open string free energy can be written as

$$F_{\text{op}}(t_{\text{op}}, \lambda_1, \lambda_2, g_s) = F_{1CS}^{CS}(\lambda_1, g_s) + F_{2CS}^{CS}(\lambda_2, g_s) + \ln \langle e^{F_{\text{inst}}(g_s, t_{\text{op}}, A_1, A_2)} \rangle$$ \quad (28)$$

where $\lambda_1 = N_1 g_s$ and $\lambda_2 = N_2 g_s$ denote the 't Hooft coupling constants of the two Chern-Simons theories. In the last term of (28) we have a double functional integral over both gauge fields. Therefore the computation of the free energy consists of two steps. First we have to find $F_{\text{inst}}(g_s, t_{\text{op}}, A_1, A_2)$ using open string enumerative techniques, and then compute the Wilson line expectation values in Chern-Simons theory. For convenience, we will denote the last term in (28) by $F_{\text{inst}}(g_s, t_{\text{op}}, \lambda_1, \lambda_2)$ so that (28) becomes

$$F_{\text{op}}(t_{\text{op}}, \lambda_1, \lambda_2, g_s) = F_{1CS}^{CS}(\lambda_1, g_s) + F_{2CS}^{CS}(\lambda_2, g_s) + F_{\text{inst}}(g_s, t_{\text{op}}, \lambda_1, \lambda_2).$$ \quad (29)
The above discussion is quite schematic since the interaction between Chern-Simons theory and open string instantons is more subtle. According to [?], the perturbative Chern-Simons expansion should be interpreted as a sum over degenerate open string Riemann surfaces which develop infinitely thin ribbons. The ribbons, which are mapped to the spheres $L_1, L_2$ as geodesic graphs, have been interpreted in [?] as virtual instantons at infinity. So far there is no rigorous mathematical treatment of this type of degenerate behavior at infinity. In particular it is not known how to actually write down the open string amplitudes as finite dimensional integrals on a well defined moduli space. The prescription outlined above following [?] is to first sum over nondegenerate instantons, i.e. Riemann surfaces which have at worst double node singularities, and then sum over degenerate instantons by performing the Chern-Simons path integral.

The sum over non-degenerate instantons should be in principle defined in terms of intersection theory on some moduli space of stable open string maps\(^2\) $\overline{M}_{g,h}(Y, L, d\beta)$, where $L = L_1 \cup L_2$. This theory has not been rigorously developed so far, but at the level of rigor of [?, ?, ?], one can give a computational definition of open string amplitudes. The main idea is to proceed by localization with respect to a torus action induced by a torus action on $Y$ preserving $L_1, L_2$. Although the structure of the moduli space is unknown, one can describe in detail the structure of the fixed point loci. From this data, we can obtain enumerative invariants essentially by adapting on spot the known closed string techniques [?, ?, ?, ?] in order to evaluate the contribution of each fixed point locus. This approach has been successfully implemented for noncompact lagrangian cycles in [?, ?, ?, ?]. In that case one can fix a flat unitary connection on the lagrangian cycles as a background field.

In the present context, since the cycles are compact, the unitary connections become dynamical variables and one should integrate over all (gauge equivalence classes) of such fields [?]. This is achieved by performing the Chern-Simons theory path integral. The coupling between finite area instantons and Chern-Simons theory is quite subtle [?]. As explained there, if one had only isolated open string instantons, their effects would be encoded in a series of holonomy (Wilson loop) operators added to the Chern-Simons action. For each rigid isolated instanton $D \subset Y$, one adds a term of the form $e^{-t \text{Tr} V}$, where $V$ is the holonomy around the boundary of $D$, which is a knot in $L$. It is important to note that in this formula $V$ is not a flat gauge field, therefore this operator is not invariant under deformations of the knot. If $D$ is rigid and isolated, this is not a problem. However, what happens if we have families of such instantons? Then the holonomy $V$ depends on the particular member in the family, and it is not clear how one should write the associated corrections. A general answer to this question is not known at the

\(^2\)This is a schematic discussion. Since $Y$ is noncompact, the compactification of this moduli space is a very subtle issue. Some aspects will be mentioned in section six.
present stage, but we would like to propose an answer for situations in which there is a torus action on $Y$ preserving the lagrangian cycles. In such cases, one can simply use localization arguments to argue that all nontrivial contributions to the instanton sum come from fixed maps under this action. Then, to each component of the fixed point locus we associate a certain series of holonomy operators in Chern-Simons theory, as detailed below. Because of this coupling with Chern-Simons theory, the procedure described above should not be thought as localization of a virtual cycle on a moduli space of maps in the usual sense. It would certainly be desirable to have a more precise mathematical formulation of this construction, but this is not known at the present stage.

The final step is to perform the Chern-Simons functional integral with the instanton corrections included in order to compute the open string free energy. This approach has been successfully tested in a simple geometric situation in [?]. Note that this last step requires the choice of a framing for each boundary $\partial D$ in order to regulate the divergences in Chern-Simons perturbation theory. We will comment more in this point below.

In the present situation, we define an $S^1$ action on $Z$ by

$$Z_1 \ Z_2 \ Z_3 \ U \ \ V$$

$$\mathbb{C}^* \ \lambda_1 \ \lambda_2 \ 0 \ -\lambda_1 \ -\lambda_2.$$  (30)

Obviously, this action preserves $Y$ and $L_1, L_2$. Then, a somewhat technical analysis shows that the only primitive open string instantons left invariant by this action are the discs $D_1, D_2$ and the annulus $C$ constructed in the previous section. The proof follows the lines of [?] (section 5); one has to take a projective completion $\bar{Y}$ of $Y$ and show that the problem reduces to finding invariant curves on $\bar{Y}$ subject to certain homology constraints. We leave the details for the next section.

Note that the two discs $D_1, D_2$ have a common origin, therefore they form a nodal (or pinched) cylinder $D_1 \cup D_2$. As discussed at the end of section two, $D_1 \cup D_2$ and $C$ are disconnected. Therefore the fixed locus of the torus action on $\bar{M}_{g,h}(Y, L, d\beta)$ consists of two disconnected components: multicovers of $D_1 \cup D_2$ and respectively multicovers of $C$. The precise structure of these components will be discussed in detail in section five. For now, let us note that on general grounds, an open string map to the pinched cylinder $D_1 \cup D_2$ is characterized by two degrees $d_1, d_2$ and two sets of winding numbers $m_i, i = 1, \ldots, h_1, n_j, j = 1, \ldots, h_2$, where $h_1, h_2$ are the numbers of boundary components mapped to $\Gamma_1$ and respectively $\Gamma_2$. We have the constraints $\sum_{i=1}^{h_1} m_i = d_1, \sum_{j=1}^{h_2} n_j = d_2$. Similarly, a generic map to the cylinder $C$ is characterized by a degree $d$ and two sets of winding numbers $m_i, n_j, i = 1, \ldots, h_1, j = 1, \ldots, h_2$ with $\sum_{i=1}^{h_1} m_i = \sum_{j=1}^{h_2} n_j = d$, where $h_1, h_2$ are the numbers of boundary components mapped to $\Xi_1, \Xi_2$.

Based on these elements, we can write down the general form of the open string
instanton expansion on $Y$. We noticed earlier that $D_1, D_2$ and $C$ have the same symplectic area, therefore one would be tempted to write down this expansion in terms of a single open string Kähler modulus $t_{op}$. However, things are more subtle here since the instanton expansion should be written in terms of flat coordinates rather than classical moduli. One of the lessons of \[\text{?}\] was that in the presence of dynamical A branes, the open string flat coordinates can receive nonperturbative corrections generated by virtual instantons at infinity. These corrections can be different for different open string instantons, depending on the lagrangian cycle they end on. For this reason, we will refine (28) by writing the instanton expansion in terms of three distinct flat Kähler moduli $t_1, t_2, t_c$ corresponding to $D_1, D_2, C$. We will show later that this is in precise agreement with the duality predictions from the closed string side. We also introduce the following holonomy variables

\[
\begin{align*}
F_{\text{inst}}^{(1)}(g_s, t_c, U_1^m U_2^n) = & \sum_{g=0}^\infty \sum_{h_1, h_2 = 0}^\infty \sum_{d=0}^\infty \sum_{m_i, n_j \geq 0} \epsilon^{h_1 + h_2} g_s^{2g-2+h_1+h_2} \\
& \times C_{g, h_1, h_2}(d; m_i, n_j) e^{-d t_c} \prod_{i=1}^{h_1} \text{Tr} U_1^{m_i} \prod_{j=1}^{h_2} \text{Tr} U_2^{n_j} \\
F_{\text{inst}}^{(2)}(g_s, t_1, t_2, V_1^m V_2^n) = & \sum_{g=0}^\infty \sum_{h_1, h_2 = 0}^\infty \sum_{d_1, d_2 = 0}^\infty \sum_{m_i, n_j \geq 0} \epsilon^{h_1 + h_2} g_s^{2g-2+h_1+h_2} \\
& \times F_{g, h_1, h_2}(d_1, d_2; m_i, n_j) e^{-d_1 t_1 - d_2 t_2} \prod_{i=1}^{h_1} \text{Tr} V_1^{m_i} \prod_{j=1}^{h_2} \text{Tr} V_2^{n_j}.
\end{align*}
\]
In (34) we have a sum over multicovers of the annulus $C$, while (35) represents a sum over multicovers of the two discs $D_1, D_2$, which form a nodal cylinder. Note that the winding numbers are subject to the constraints mentioned above, that is $\sum_{i=1}^{h_1} m_i = \sum_{j=1}^{h_2} n_j = d$ for $C$, and $\sum_{i=1}^{h_1} m_i = d_1$, $\sum_{j=1}^{h_2} n_j = d_2$ for $D_1 \cup D_2$.

As discussed earlier in this section, the coefficients $C_{g,h_1,h_2}(d; m_i, n_j)$ as well as $F_{g,h_1,h_2}(d_1, d_2; m_i, n_j)$ can be computed by evaluating the contribution of the fixed loci in $\overline{M}_{g,h}(Y, L, d\beta)$. Note however, that to each component of the fixed locus we assign a certain holonomy operator in the Chern-Simons theory. Therefore, one does not simply sum over all fixed loci as in standard localization computations. This means that the contribution of each fixed component depends on the weights of the toric action used in the localization process. In order to obtain a physically sensible answer, we have to make a certain choice of weights similar to the choices made in [?, ?, ?]. Moreover, in our case the situation is more complicated since we also have to make a choice of framing in Chern-Simons theory. The two choices are in fact related, as discussed below and in more detail in section 6.3.

Before giving the details, note that given the geometric context, there may be many choices of weights and/or framings that result in distinct open string expansions. At this stage we do not know if there is a preferred choice based on certain intrinsic consistency criteria of the open string theory. This problem is very hard, and it cannot be answered without a better development of the mathematical formalism. In the following we will pursue a more modest goal, namely we will try to find a set of choices which leads to an agreement with the dual closed string expansion. Formulated differently, we will try to find the correct prescriptions for the duality map in this geometric situation.

For a single disc $D \subset \mathcal{C}^3$ with boundary on a noncompact lagrangian cycle $L$, it was shown in [?] that the choice of weights is equivalent to the choice of an equivariant section of the normal bundle $N_{\partial D/L}$. This prescription formalizes the relation between framing and toric action found for the first time in [?]. In particular, the instanton expansion for $D$ depends on an integer ambiguity $a$ which parameterizes isomorphism classes of $S^1$ equivariant sections of the normal bundle $N_{\partial D/L}$. For the discs $D_1, D_2$ embedded in $Y$, one can still choose the boundary data in the form of two sections to $N_{\Gamma_1/L_1}, N_{\Gamma_2/L_2}$ which are labeled by two integers $a, b$. Generalizing the strategy proposed by [?], we assume that these choices have to be compatible with the global $S^1$ action on $Y$. By explicitly writing these conditions in local coordinates, we show in section 6.3. that we are left with only two consistent choices, namely $(a, b) = (0, 0)$ or $(a, b) = (2, 2)$. In the following we will choose $(a, b) = (0, 0)$ since in this case the instanton expansion takes a very simple compact form. The second choice is not logically ruled out, but leads to a very complicated formula for the instanton corrections. We leave it for future work.
Having made this choice, the open string topological theory is still not completely determined, since we also have to specify the framing of the knots \( \Gamma_1, \Gamma_2, \Xi_1 \) and \( \Xi_2 \). In principle, the equivariant sections introduced in the previous paragraph should determine the framings of \( \Gamma_1 = \partial D_1 \) and \( \Gamma_2 = \partial D_2 \). However, there is a subtlety at this point explained in detail in section 6.3. Briefly, the choice of a single section does not determine the framing as an integer number; one also needs a reference section which is typically provided by the geometric context. In our case we have a natural reference section since \( \Gamma_1, \Gamma_2 \) are algebraic knots. Then a short local computation shows that the framings of \( \Gamma_1, \Gamma_2 \) are \((2 - a, 2 - b)\). Therefore for \((a, b) = (0, 0)\) we obtain framings \((2, 2)\).

For the annulus \( C \), the choice of framing is more subtle since the localization computation does not require the choice of special values of toric weights. Therefore one does not have to choose equivariant sections on the two boundaries components \( \Xi_1, \Xi_2 \). In the present context, this framing can be related to the framings of the discs by a deformation argument detailed in section six. The resulting values are \((1 - a^2, 1 - b^2) = (1, 1)\) for the two boundaries of \( C \). Moreover, it is shown in [?] that for an annulus with framings \((p, p)\), the \( p \)-dependence of the amplitudes can be absorbed in a simple shift of the open string Kähler parameters, leaving the amplitudes otherwise unchanged. This allows us to choose canonical framing without loss of generality. We are very grateful to the authors of [?] for explaining this to us.

The open string instanton expansions (34),(35) can be determined by adding the contributions of all fixed points of the \( S^1 \) action. These are computed in section six, equations (6.30), (6.100), (6.101) and (6.102). There is one subtle aspect at this point, namely given the choices made so far, one has to count the contribution of the pinched cylinder, eqn (6.102), twice in order to match the predictions of large \( N \) duality. This factor of two does not follow directly from localization computations, and it cannot be satisfactorily explained using our present knowledge of moduli spaces of open string maps. In fact, since the only criterion for introducing this factor is agreement with the closed string dual, we should think of it as a prescription of the duality map. A more conceptual explanation would require a much deeper mathematical understanding of the open/closed string duality, which is beyond the purpose of this work. We hope to report on this aspect in the future.

To summarize this discussion, we propose the following large \( N \) Chern-Simons dual to the local \( dP_2 \) model

\[
F^{(1)}_{inst}(g_s, t_c, U_1, U_2) = - \sum_{d=1}^{\infty} e^{-dt_c} d \text{Tr} U_1^d \text{Tr} U_2^d
\]  
(36)
\begin{align}
\mathcal{F}_{\text{inst}}(g_s, t_1, t_2, V_1, V_2) &= \sum_{d=1}^{\infty} i e^{-dt_1} 2d \sin d g_s 2 \text{Tr} V_1^d + \sum_{d=1}^{\infty} i e^{-dt_2} 2d \sin d g_s 2 \text{Tr} V_2^d \\
&\quad + 2 \sum_{d=1}^{\infty} e^{-d(t_1+t_2)} d \text{Tr} V_1^d \text{Tr} V_2^d \tag{37}
\end{align}

where the framings of the knots \( \Gamma_1, \Gamma_2, \Xi_1, \Xi_2 \) are \((2, 2, 0, 0)\). In the next subsection, we will present very convincing evidence for this conjecture by computing the open string free energy up to degree four in \( e^{-t_1}, e^{-t_2}, e^{-t_c} \) and finding perfect agreement with the closed string results.

### 4.1 Chern-Simons Computations

As discussed above, we have to evaluate

\[ \mathcal{F}_{\text{op}}(t_1, t_2, t_c, \lambda_1, \lambda_2) = \mathcal{F}_1^{CS}(\lambda_1, g_s) + \mathcal{F}_2^{CS}(\lambda_2, g_s) + \mathcal{F}_{\text{inst}}(g_s, t_1, t_2, t_c, \lambda_1, \lambda_2) \tag{38} \]

where

\[ \mathcal{F}_{\text{inst}}(g_s, t_1, t_2, t_c, \lambda_1, \lambda_2) = \ln \left\langle e^{\mathcal{F}_{\text{inst}}(g_s, t_1, t_2, t_c, U_1, U_2, V_1, V_2)} \right\rangle. \tag{39} \]

Moreover, the instanton expansion is obtained by adding (36) and (37)

\[ F_{\text{inst}}(g_s, t_1, t_2, t_c, U_1, U_2, V_1, V_2) = F_{\text{inst}}^{(1)}(g_s, t_c, U_1, U_2) + F_{\text{inst}}^{(2)}(g_s, t_1, t_2, V_1, V_2). \tag{40} \]

The holonomy variables \((U_1, V_1), (U_2, V_2)\) are associated to the unknots \((\Xi_1, \Gamma_1), (\Xi_2, \Gamma_2)\) in \(L_1\) and respectively \(L_2\). Each pair of knots form a link with linking number one in the present choice of orientations. Moreover, \((\Xi_1, \Xi_2)\) are canonically framed while \((\Gamma_1, \Gamma_2)\) have framings \((2, 2)\). This completely specifies the Chern-Simons system. The first two terms in (38) are well understood. Performing an analytic continuation as in [? , ? , ? , ?], we can write them in the form

\[ \mathcal{F}_1^{CS}(\lambda_1, g_s) + \mathcal{F}_2^{CS}(\lambda_2, g_s) = \sum_{n=1}^{\infty} \ln(2 \sin n g_s)^2 (e^{in\lambda_1} + e^{in\lambda_2}). \tag{41} \]

The third term is more complicated. We will evaluate it perturbatively up to terms of order three in \( e^{-t_1}, e^{-t_2}, e^{-t_c} \).

For a systematic approach, let us write the instanton corrections in the form

\[ F_{\text{inst}}(g_s, t_1, t_2, t_c, U_1, U_2, V_1, V_2) \]

\[ = \sum_{n=1}^{\infty} \left[ ia_n e^{-nt_1} + ib_n e^{-nt_2} - c_n e^{-nt_c} + 2d_n e^{-n(t_1+t_2)} \right] \tag{42} \]