Fractal dust model of the Universe based on Mandelbrot’s Conditional Cosmological Principle and General Theory of Relativity

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Abstract

We present a fractal dust model of the Universe based on Mandelbrot’s proposal to replace the standard Cosmological Principle by his Conditional Cosmological Principle within the framework of General Theory of Relativity. This model turns out to be free from the de-Vaucouleurs paradox and is consistent with the SNeIa observations. The expected galaxy count as a function of red-shift is obtained for this model. An interesting variation is a steady state version, which can account for an accelerating scale factor without any cosmological constant in the model.

Running Title: Fractal dust model of the Universe

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1. INTRODUCTION

Fractality is ubiquitous in nature. Should Cosmology be an exception? Mandelbrot’s vision, followed by elaborate demonstration of fractality in galaxy distributions by Pietronero, have perhaps been singularly responsible for ensuring that any modern text on cosmology is rather incomplete without at least a chapter on Fractals. Unfortunately, it stops here as there is no definite ansatz that could be used to match cosmological predictions. This article explores a way out of this impasse.

Standard cosmology is based on the assumption of homogeneity and isotropy of the Universe, the so-called Cosmological Principle, on scales greater than \( \approx 10^8 \) light years. The Friedmann metric, and consequently the Hubble Law, follow from this assumption. Although the metric is expected to be valid only on scales larger than the scale of homogenization, the Hubble law is found to hold on smaller scales. This seeming contradiction goes by the name of the ‘Hubble-deVaucouleurs’ [H-deV] paradox\(^1\). On the other hand, during the last decade, the assumption of homogeneity on scales greater than \( 10^8 \) light years itself has come to be challenged\(^1,2\). It is now believed that the scale of homogenization, if any, is definitely greater than 100 Mpc. The number of galaxies \( N(r) \) within a sphere of radius \( r \) centred on any galaxy, is not proportional to \( r^3 \) as would be expected of a homogeneous distribution. Instead \( N(r) \) is found to be proportional to \( r^D \), where \( D \) is approximately equal to 2. Without assigning a special central position to an observer, such a scaling can be explained only by assuming that galaxies are distributed on points belonging to a fractal set of dimension \( D \). It has further been argued that available evidence indicates that fractal distribution of visible matter extends well up to the present observational limits without any evidence of cross-over to homogeneity\(^1,3\). This suggests that the entire Universe could be a fractal. At present this question is being hotly debated\(^1−6\).

No generally acceptable structure formation scenario, that could explain the observed inhomogeneities, has yet emerged, either within the framework of the standard big-bang cosmologies \(^7,8\) or alternative cosmologies such as the quasi-steady state cosmology\(^9\).

Although, a fair amount of evidence has been collected in support of in-
homogenous distribution of visible matter, there is no evidence to contradict isotropy in a statistical sense from any galaxy. In view of the observed fractality and isotropy, Mandelbrot\textsuperscript{10} proposed the replacement of the standard Cosmological Principle by the Conditional Cosmological Principle. According to this principle the Universe appears to be the same statistically from every galaxy (point of the fractal) and in every direction.

In Sec 2, we present a model fractal Universe that is based on the Mandelbrot’s Conditional Cosmological Principle and the General Theory of Relativity. We show how the Conditional Cosmological Principle leads to the Friedmann metric and how the Einstein equation can be satisfied in the fractal context.

In Sec 3, we obtain the time dependence of the scale function for two cases. If we assume that the number of galaxies is conserved, we obtain the FRW metric with a non-zero effective density, whereas the average density for the fractal Universe is zero. Thus the redshift-distance relation in the fractal model turns out to be the same everywhere as that over scales greater than the homogenization scale in the standard model. Hence this fractal model is free from the Hubble-deVaucouleurs paradox. On the other hand, if we assume that the large scale fractal distribution of galaxies is in a steady state, we obtain an accelerating Universe whose acceleration is related to the fractal density and the Hubble’s constant.

It has previously been argued \textsuperscript{11} that fractal scaling can be obtained along the past light cone in a perturbed Einstein - de Sitter Cosmology so any observed fractality is not necessarily inconsistent with the Cosmological Principle and there is no Hubble-deVaucouleurs paradox. In Sec 4, we obtain galaxy counts along the past light cone on the basis of our model, assuming that the ‘effective density’/ ‘fractal density’ can be neglected. Fractal scaling on a constant time hypersurface implies a fractal scaling along the past light cone for small red-shifts.

It is pertinent to recall that the association of the FRW metric obtained from smoothed out homogeneous Universe and the actual Universe is not clearly established to date. This goes to the root of “the averaging problem” in General Theory of Relativity \textsuperscript{12–14}. Once the manner in which the FRW metric holds is established, it may have an essential bearing on the inferences drawn from the fractal model in this paper.

In Sec 5, we discuss the previous attempts at reconciling the observed fractal structure with a relativistic description of the Universe and how the Conditional Cosmological Principle, as used in our model, leads to a more
satisfactory picture.

Conclusion is presented in Sec 6.

2. THE FRACTAL MODEL

The Cosmological Principle provides the symmetry necessary to derive the FRW metric. Mandelbrot’s Conditional Cosmological Principle weakens the Cosmological Principle as it demands that the Universe appears statistically the same to all observers situated on a galaxy (point of a fractal) but not in a region of void. More specifically, in a reference frame with origin P, the distribution of matter is independent of P under the sole condition that P must be a material point. If P is not a material point and R is fixed, a sphere of radius R centred on P is empty with probability equal to one.

Just as the standard model follows naturally from the Cosmological Principle when General Theory of Relativity is applied, an ansatz for a fractal model follows naturally from Mandelbrot’s Conditional Cosmological Principle, once the necessary change in perspective required to deal with fractal distributions is made. This ansatz is based on the observation that in a fractal universe, density is not defined at any point. Hence Einstein’s equations do not mean anything at a point. However, by replacing the concept of density at a point by that of a conditional ‘mass measure’ defined over sets, it is possible to formally satisfy the Einstein’s equations integrated over sets. Conditional Cosmological Principle then means that the conditional mass measure will be the same for all observers situated at points belonging to the fractal.

We define a “hypersurface of homogeneous fractality of dimension $D$” as the hypersurface in which the mass measure over a sphere of radius $R$ centred on the observer is proportional to $R^D$. We say that the Universe is a fractal universe of dimension $D$, if through each galaxy in the Universe, there passes a spacelike “hypersurface of homogeneous fractality of dimension $D$”.

Isotropy of the Universe means that, at any event, an observer who is at rest in this hypersurface cannot statistically distinguish any space direction from another.

It is widely believed that isotropy from all points of observation implies homogeneity. Thus an inhomogeneous Universe like a fractal could not be isotropic. It was shown from the observed isotropy of the Universe that the fractal dimension of the Universe could not differ appreciably from 3: \( |D - 3| < 0.001 \). However, Mandelbrot has demonstrated how to construct
fractals of any given dimension whose lacunarity could be tuned at will to make the distribution as close to isotropy (from any occupied point of the fractal) as desired. Thus in a fractal scenario isotropy from all galaxies does not rule out surfaces of homogeneous fractality.

Isotropy of a fractal universe implies that the world lines of the cosmological fluid are orthogonal to each hypersurface of homogeneous fractality. This allows the slicing of spacetime into hypersurfaces of constant time as for the standard model.

For the standard homogeneous model the Cosmological Principle leads to the Friedmann metric $g_{\mu\nu}^{FRW}$ specified by the line element:

$$ds^2 = g_{\mu\nu}^{FRW} dx^\mu dx^\nu = -dt^2 + a^2(t)\{d\chi^2 + \Sigma^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2)\}$$

where

$$\Sigma(\chi) = \begin{cases} 
\sin(\chi) & \text{for positive spatial curvature, } k = 1 \\
\chi & \text{for zero spatial curvature } k = 0 \\
\sinh(\chi) & \text{for negative spatial curvature, } k = -1 
\end{cases}$$

This metric yields the component $G^{00}$ of the Einstein tensor,

$$G_{FRW}^{00} = 3\{\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\}$$

As demanded by the Cosmological Principle, this has the same value everywhere on a hypersurface of constant time.

For a fractal Universe, the Conditional Cosmological Principle demands that on any given hypersurface $S$ of constant time, $G^{00}$ has the same value at the points of the fractal and zero elsewhere. This is consistent with the Einstein Equation

$$G^{00} = 8\pi \rho$$

because for a fractal

$$\rho(P) = \Sigma_i m\delta(P, P_i).$$

Here $P_i$ denote the points of the fractal each having mass $m$. $G^{00}$ denotes the moment of rotation density, which is the sum of intrinsic and extrinsic curvature at the point $P$. With every point of mass $m$ is associated a moment of rotation $8\pi m$. 

5
If $S^3_P(R)$ denotes a hypersphere of radius $R$ centred at a point $P$ belonging to a fractal of dimension $D$, then
\[
\int_{S^3_P(R)} \rho dV = \int_{S^3_P(R)} d\mu = M_P(R) = C(t)R^D
\] (6)
The discrete mass measure $d\mu = \rho dV$ may be replaced by a smoothed out conditional measure \( \hat{\rho} dV = (D/4\pi)C(t)r^{D-3}dV \).

As is clear from the above, for a fractal distribution of matter, the concept of density is undefined and has to be replaced by the notion of a measure on sets. This implies that $G^{00}$, the moment of rotation density, is not defined at any point. However, over any set in constant-time surface, we must have
\[
\int G^{00} dV = \int d\mu_M = 8\pi \int d\mu
\] (7)
The exact moment of rotation measure $d\mu_M$ may be replaced by the smoothed out conditional measure $\hat{G}^{00}_{\text{fractal}} dV$, for an observer at point $P$, where,
\[
\hat{G}^{00}_{\text{fractal}}(t, \chi, \theta, \phi) = \begin{cases} 
\hat{f}(\chi)G^{00}_{\text{FRW}}(t) & \text{if } P \in \text{the fractal} \\
0 & \text{otherwise}
\end{cases}
\] (8)
Then,
\[
4\pi G^{00}_{\text{FRW}}(t)a^3(t) \int_0^\chi \hat{f}(\chi)\Sigma^2(\chi)d\chi = 8\pi C(t)a^D(t)\chi^D
\] (9)
This is satisfied by
\[
G^{00}_{\text{FRW}}(t) = 2\nu C(t)a^{D-3}(t)
\] (10)
\[
\hat{f}(\chi) = \frac{D\chi^{D-1}}{\nu \Sigma^2(\chi)}
\] (11)
The value of $\nu$ is determined by demanding that $\hat{f}(\chi) = 1$ for $k = 0$ and $D = 3$. This gives $\nu = 3$. $\hat{G}^{00}_{\text{fractal}}$ satisfies the integrated Einstein equation over a sphere of radius $R$. Here $\hat{G}^{00}_{\text{fractal}}$ is not a function. It is an ansatz for defining a smoothed out moment of rotation measure on sets containing the point $P$ just as the mass measure is expressed by using the smoothed out measure $\hat{\rho}$ proportional to $r^{D-3}$. Here $\hat{\rho}$ does not mean the density at a point, but merely an ansatz to compute the mass measure. In this way Einstein’s equations for a fractal distribution of mass are expressed by a relation connecting $\hat{G}^{00}_{\text{fractal}}$ to $\hat{\rho}$, remembering clearly that these are not functions but ansatz to compute
conditional measures. Thus the dependence of $\hat{G}_{\text{fractal}}^{00}$ on $\chi$ and of $\hat{\rho}$ on $r$ should not be seen as an indication of inhomogeneity but rather as a means of concrete realization of the Conditional Cosmological Principle.

The averaging procedure over a constant time hypersurface used here, is in fact tacitly assumed in the standard model while making the fluid approximation. Both for the homogeneous model and the fractal model, the dust mass distribution is the sum of delta functions. In one case, these delta functions are distributed homogeneously whereas in the other they are distributed on a fractal set. In both cases Einstein’s equations can be satisfied only when integrated over sets as they are otherwise ill defined for point mass distributions. This integration essentially sums over discrete masses for the ‘matter’ side of the Einstein’s equations and over discrete moments of rotation for the ‘geometry’ side of the Einstein’s equations. In both cases, a clumpy matter distribution and a clumpy geometry are smoothed out in a manner that integrations over finite sets give the same result as for the clumpy case. Cosmological principle in the case of homogenous distributions and Conditional Cosmological Principle in the case of fractal distribution allow simplified descriptions of the Universe.

The above argument can be made more explicit by assuming that the Universe is made up of homogeneous galaxies of mass $M_g$ and radius $R_g$. First, let us consider the case of a homogeneous distribution of these galaxies. It is clear from the Einstein eqn $G^{00} = 8\pi \rho$ that

$$G^{00}(P) = \begin{cases} \frac{3 \times 8\pi M_g}{4\pi R_g^3}, & \text{if } P \in \text{some galaxy} \\ 0, & \text{otherwise} \end{cases} \quad (12)$$

Let us call this function $G_{\text{exact}}^{00}$. Suppose there are $N_{\text{hom}}$ galaxies in a sphere of radius $R$. Then,

$$\int_{S^3(R)} G_{\text{exact}}^{00} dV = 8\pi N_{\text{hom}} M_g \quad (13)$$

In the fluid approximation, the discrete mass distribution is replaced by a smoothed density distribution. For this distribution,

$$\int_{S^3(R)} G_{\text{smooth}}^{00} dV = 8\pi N_{\text{hom}} M_g \quad (14)$$

It is clear that $G_{\text{exact}}^{00}$ is not equal to $G_{\text{smooth}}^{00}$. The metric coefficients $g_{\mu\nu}^{\text{exact}}$ that would give rise to $G_{\text{exact}}^{00}$ will be different from the metric coefficients $g_{\mu\nu}^{\text{smooth}}$ which give rise to $G_{\text{smooth}}^{00}$. The FRW metric gives the $g_{\mu\nu}^{\text{smooth}}$ and
inferences about red-shift etc are drawn from it. It is assumed that these inferences hold for the exact distribution.

For our model fractal Universe, suppose there are \( N_{\text{frac}} \) galaxies in a sphere of radius \( R \). Then

\[
\int_{S^3(R)} G^0_0 \hat{G}_{\text{frac-exact}} dV = 8\pi N_{\text{frac}} M_g = \int_{S^3(R)} \rho dV
\]  

(15)

We see that \( G^0_0 \hat{G}_{\text{frac-smooth}} = f(\chi) G^0_0_{FRW} \) satisfies

\[
\int_{S^3(R)} G^0_0 \hat{G}_{\text{frac-smooth}} dV = 8\pi N_{\text{frac}} M_g = \int_{S^3(R)} \rho dV
\]  

(16)

To deal with fractal distributions the Einstein equation may be generalized to

\[
\int_{S^3(R)} d\mu_{MR} = 8\pi \int_{S^3(R)} d\mu
\]

(17)

where \( d\mu_{MR} \) denotes a measure for moment of rotation. For a homogeneous distribution \( d\mu_{MR} = G^0_0_{FRW} dV \) and \( d\mu = \rho dV \). For a fractal distribution \( d\mu_{MR} = \hat{f}(\chi) G^0_0_{FRW} dV \) and \( d\mu = \hat{\rho} dV \).

3. TIME DEPENDENCE OF THE SCALE FACTOR

We obtain the time dependence of the scale factor in two cases:

3.1 Case 1: Conserved galaxy number:

In this case we assume that the number of galaxies remains unchanged as the scale factor changes with time. Then \( C(t) a^D(t) \chi^D = C(t_0) a^D(t_0) \chi^D \) so that

\[
C(t) = \frac{a^D(t_0)}{a^D(t)} C(t_0)  
\]

(18)

In this way, we get,

\[
3\left(\frac{\dot{a}}{a} - \frac{k}{a^2}\right) + \frac{k}{a^2} = 6 \frac{a^D_0}{a^3} C_{a_0} 
\]

(19)

where \( a_0 \) is the scale factor at time \( t_0 \).

The dynamics of the scale factor due to a fractal distribution of matter, which satisfies the conditional cosmological principle is the same as in
standard cosmology for homogeneously distributed matter with an effective density

$$\rho_{\text{eff}} = \frac{3}{4\pi} \frac{a_0^D C_{a_0}}{a^3} \propto \frac{1}{a^3}$$  \hspace{1cm} (20)$$

It should be noted that although the time dependence of the scale factor is the same as for a homogeneous Universe, the effective density for the fractal Universe is different from the average density which is zero.

The value of the ‘fractal density’ $C_{a_0}$ at the present epoch $t_0$, may be obtained from the observed number of galaxies in a sphere of radius $R$. Then the scale factor for the present epoch may be obtained from

$$H_0^2 + \frac{k}{a_0^2} = 2a_0^{D-3}C_{a_0}$$  \hspace{1cm} (21)$$

From the values $C_{a_0}$ and $a_0$ the proportionality constant in eqn(20) can be determined. For $D = 2$,

$$a_0 = \frac{C_{a_0} + \sqrt{C_{a_0}^2 - kH_0^2}}{H_0^2}$$  \hspace{1cm} (22)$$

From the galaxy number count data of Labini et al$^1$, the average conditional number density $\Gamma^*$ of galaxies over a radius of 100 Mpc is $\approx 10^{-2} (Mpc)^{-3}$. The total number of galaxies in a sphere of radius $R$ is given by:

$$N(R) = nR^2$$  \hspace{1cm} (23)$$

where $n$ is the “fractal number density”. This gives the average conditional number density

$$\Gamma^* = \frac{3N}{4\pi R^3} = \frac{3n}{4\pi R}$$  \hspace{1cm} (24)$$

One therefore gets $n \approx 4 (Mpc)^{-2}$. Taking a typical galaxy mass as $\approx 1.8 \times 10^{11} M_\odot$, gives $C_{a_0} \approx 10^{-4} gms cm^{-2}$. In gravitational units this amounts to $C_{a_0} \approx 2 \times 10^{-24} sec^{-1}$. This is small in comparison to the observed Hubble parameter $H_0 \approx 2 \times 10^{-18} sec^{-1}$. Such a universe would be curvature dominated even for redshifts as high as $10^5$ and its coasting would be indistinguishable from a linear coasting Milne model: $a(t) = t$ at lower redshifts.

It is straightforward to put this scaling to classical cosmological tests, viz.: (1) The galaxy number count as a function of redshift; (2) The angular
diameter of “standard” objects (galaxies) as a function of redshift; and finally
(3) The apparent luminosity of a “standard candle” as a function of redshift.
The first two tests are marred by evolutionary effects and for this reason
have fallen into disfavour as reliable indicators of a viable model. However,
the discovery of Supernovae type Ia [SNe Ia] as reliable standard candles has
raised hopes of elevating the status of the third test to that of a precision
measurement that could determine the viability of a cosmological model.
The main reason for regarding these objects as reliable standard candles are
their large luminosity, small dispersion in their peak luminosity and a fairly
accurate modeling of their evolutionary features.

For a linearly coasting model, the apparent magnitude of an object is
related to its redshift $z$ by:

$$m = 25 + M + 5\log[a_0 \sinh(\chi)(1 + z)]$$ (25)

It is straightforward to reduce it to

$$m(z) = 5\log\left(\frac{z^2}{2} + z\right) + M$$ (26)

with $M \equiv M - 5\log(H_0) + 25$

Figure ‘1’ sums up the Supernova Cosmology project data for super-
novae with redshifts between 0.18 and 0.83 together with the low redshift set
at redshifts below 0.1. Also plotted is the latest SNe Ia at redshift 1.7 [see eg.
Wright]. Clearly, the Fractal model described here is as good a fit as
the constrained Standard Cosmology model with $(\Omega_\Lambda, \Omega_M) = (0.72, 0.28)$.
The goodness of concordance can be judged by the fact that the $\chi^2$ per
degree of freedom is roughly unity for the fit. As a matter of fact a linear
coasting is accommodated even in the 68% confidence region. This finds a
passing mention in the analysis of Perlmutter who noted that the curve for
$\Omega_\Lambda = \Omega_M = 0$ (for which the scale factor would have a linear evolution), is
“practically identical to the best fit plot for an unconstrained cosmology”.

It is interesting to note that, unlike the standard model, for any observed
set of values $H_0$ for the Hubble constant and $C_{a_0}$ for the fractal density, eqns
(19) and (21) also admit a $k = 0$ solution for appropriate choice of $a_0$.

3.2: Case 2: Steady State Fractality:

Another interesting model may be obtained if we assume that the number
of galaxies in a sphere of radius $R$ is the same at all epochs so that $C(t)$ is
a constant. This gives a fractal version of the steady state model, although it lacks the maximal symmetry in space-time of the “Perfect Cosmological Principle”.

In this case we get,

\[ 3\left\{ \left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} \right\} = 6C a^{D-3} \]  

(27)

For \( D = 2 \), we obtain,

\[ \left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} = \frac{2C}{a} \]  

(28)

The solution of this equation is

\[ a(t) = a_0 + H_0 a_0 (t - t_0) + \frac{C}{2} (t - t_0)^2 \]  

(29)

where \( a_0 \) is the scale factor at the present epoch \( t_0 \) and satisfies the equation,

\[ H_0^2 a_0^2 + k = 2Ca_0 \]  

(30)

For \( k = 0 \) the deceleration parameter is given by

\[ q = -\frac{\ddot{a}a_0}{\dot{a}^2} = -\frac{C}{a_0 H_0^2} = -\frac{1}{2} \]  

(31)

For \( k = -1 \),

\[ q = -\frac{C}{C + \sqrt{C^2 + H_0^2}} \]  

(32)

If \( C << H_0 \), \( q \) is approximately equal to \(-C/H_0\). This is a rather low value and its concordance again coincides with that of the empty model.

However, it should be noted that the Conditional Cosmological Principle could be the consequence of an underlying fractal structure of space-time, in which case any dark matter would also have the same fractal structure as visible matter. This steady-state model therefore offers the possibility of accommodating the acceleration of the scale factor without invoking any cosmological constant.

4. RED-SHIFT DEPENDENCE OF GALAXY COUNTS

As astronomical observations take place on the past light cone, the number count of galaxies inside a hypersphere of radius \( R = a(t) \chi \) on a constant
time hypersurface is unobservable. The red-shift dependence of galaxy counts can be derived as follows:

We assume that $C_{a0} << H_0$ so that,

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{1}{a^2} = 0$$  \hspace{1cm} (33)

For the present epoch this gives $a_0 = H_0^{-1}$ so that the time dependence of the scale factor is given by

$$a(t) = (t - t_0) + \frac{1}{H_0}$$  \hspace{1cm} (34)

If a light ray is emitted from a source at event $(t_e, \chi_e, \theta, \phi)$ and received by an observer at event $(t_0, 0, \theta, \phi)$, then along the light cone, we must have

$$\frac{dt}{d\chi} = -a(t) = -(t - t_0 + \frac{1}{H_0})$$  \hspace{1cm} (35)

so that

$$\chi = -ln\{H_0(t - t_0) + 1\}$$  \hspace{1cm} (36)

The red-shift is given by

$$1 + z = \frac{a(t_0)}{a(t_e)} = e^{\chi_e}$$  \hspace{1cm} (37)

Number of galaxies having $\chi < \chi_e$ is given by $C_0 a_0 D \chi_e D$. Therefore the number of galaxies having red-shift less than $z$ is given by:

$$N(< z) = \frac{C_0}{H_0 D} [ln(1 + z)]^D$$  \hspace{1cm} (38)

For small $z$, fractal scaling is seen along the past-light cone. For larger $z$ the deviations from pure power law behaviour may be compared with observational data to test the model.

5. DISCUSSION

There have been several attempts to incorporate large-scale inhomogeneities of the Universe in the framework of General Theory of Relativity. However,
they have not exploited the Conditional Cosmological Principle proposed by Mandelbrot as a replacement of the Standard Cosmological Principle.

In case future observations unambiguously demonstrate that the Cosmological Principle is not valid, entire standard cosmology scenario will break down. In such an eventuality our model based on Mandelbrot’s Conditional Principle provides the simplest alternative around which modified cosmologies may be built. For the present, this model can help resolve many of the vexing questions of relevance to the ‘fractal debate’.

Abdella et al\cite{11} have suggested that fractal scaling observed by Pietronero and coworkers\cite{1,2} is simply an apparent scaling due to the fact that observational quantities such as density lie along the past light cone and depend more significantly on the red shift than had hitherto been assumed. The average density along the past light cone becomes inhomogeneous, even in the spatially homogeneous spacetime of standard cosmology. However, it does not have the observed fractal scaling. By introducing perturbations, they could obtain an approximate scaling. In this way they tried to reconcile the observed fractal scaling with the standard Cosmological Principle.

Compared to this procedure, our model based on the Mandelbrot’s Conditional Cosmological principle derives the galaxy count scaling law along the past light cone in a simple straightforward manner. This scaling agrees with the observed fractal scaling for low red-shifts. If Conditional Cosmological Principle is to hold along the same lines as the standard Cosmological Principle, the fractal scaling has to hold along the constant time hypersurface and not along the past light cone. For low red-shifts there is negligible difference between the two. Deviations at higher red-shifts may be used to test our model.

From the apparent fractal conjecture perspective, the Hubble-deVaucouleurs paradox is resolved by attributing apparent fractality to observations of a perturbed FRW universe on the past light cone. It is claimed that there is no inconsistency between apparent fractal scaling and the observed linear Hubble Law on scales smaller than the homogenization scale.

In the approach of this paper, the Conditional Cosmological Principle forces the points of the fractal to follow the Hubble flow, that is, to remain at rest in the comoving coordinates. Thus the expected increase in peculiar velocities with greater inhomogeneities observed on larger scales is not to be found. Hitherto all attempts to treat the fractal structure in a relativistic context have explicitly or implicitly regarded the fractal structure as inhomogeneities, with a background homogeneity providing the relativistic
framework in the form of Friedmann metric. In our approach, the homogeneous background is replaced by homogeneous fractality. In the fractal picture, there is no average density and therefore no inhomogeneity. All the points of the fractal are on equal footing, each at rest in the comoving coordinates. The observed linear Hubble Law is consistent with the observed fractal scaling thus resolving the Hubble de-Vaucouleurs paradox.

The apparent fractal conjecture is based upon an inhomogeneous spherically symmetric metric. This does not put all the points on equal footing and the derived scaling would hold only from one point, the centre of spherical symmetry. Contrary to this, the observed fractal scaling of galaxy distribution, is a power law scaling from every galaxy. Only non-analytic fractal sets can give rise to this kind of scaling. It is not sufficient to obtain power law scaling from one point to claim that the observed fractal scaling has been explained without giving up the standard Cosmological Principle.

Being based on the Mandelbrot’s Conditional Cosmological Principle, the model presented here puts all the points of the fractal on equal footing. It deals with non-analytic distribution of matter in the General Theory of Relativity framework with the help of conditional measures. Non-analytic distribution of matter will necessarily be associated with a non-analytic space-time geometry. Till a totally satisfactory mathematical framework for dealing with fractals emerges, one has to try to deal with them using smoothing methods. However, in the case of fractals the smoothing has to be identical from every point of the fractal. The use of smoothed out conditional measures is our suggestion to achieve this.

Further, the basic assumption of the apparent fractal conjecture is that the fractal scaling law deduced by Pietronero and coworkers\textsuperscript{1,2} is based on taking the Euclidean space approximation, so that no distinction has been made between the observable past light cone and the unobservable constant time hypersurface. This does not appear to be correct. Labini et al\textsuperscript{1} have clearly stated that comoving distances have been computed by using the Mattig formula for $q = 1/2$. It has also been stated that the use of different values of $q$ does not have significant effect on the results for small red-shifts. The justification of this procedure, in the absence of a relativistic framework for fractal cosmology, is another matter. For the Conditional Cosmological Principle and our model to apply, it is necessary that fractal scaling on a constant time hypersurface exists up to very large scales, so that the fractal can be treated as infinite. Comparison of the red-shift dependence of galaxy counts derived for our model with the observed data for moderate and large
red-shifts, would provide another test for the model.

From the above arguments, we feel that the model of this paper, which may be called the *homogeneous fractal expanding model*, is better suited than the apparent fractal conjecture to provide a starting point for developing a theoretical framework that can replace the standard framework.

6. CONCLUSION

The Standard Cosmological Principle is not merely an esthetically pleasing and philosophically satisfying principle; it plays a crucial role in developing the framework of Standard Cosmology. However, Standard Cosmology has not been able to satisfactorily explain the observed large scale distribution of galaxies, which seems to satisfy a fractal scaling law up to the largest scales investigated.

On the other hand the fractal scenario till now had no satisfactory explanation even for observed red-shift of galaxies. It has generally been believed that if the Universe would be hierarchical then there is no known analysis of redshift data that is self-consistent and if the Cosmological Principle could be shown to be false, then cosmology would not be the coherent body of knowledge that many theorists believe that it is\(^{19}\). However, the use of Mandelbrot’s Conditional Cosmological Principle in the framework of General Theory of Relativity, as described in the model presented here, provides the means to explain the observed red-shifts of galaxies. If the Cosmological Principle is eventually shown to be false, the Conditional Cosmological Principle may provide cosmology with a theoretical underpinning necessary for the analysis and interpretation of observational data. Mc Caulley\(^{19}\) has claimed that visible matter provides no evidence to support either the standard cosmological principle or that the Universe is a fractal/multifractal. That may well be true, because neither the Cosmological nor the Conditional Cosmological Principle are required by any other known law of Physics. Nevertheless, the Cosmological Principle has played an important indispensable role in development of cosmology so far. The Conditional Cosmological Principle may play a similar role in the fractal scenario. After all, the idealized homogeneous Universe is a special case of the idealized homogeneous fractal Universe.

It is hoped that this model will lead to more realistic models that incorporate fluctuations, radiation and nucleosynthesis. All these issues need to be scrutinized afresh from a fractal perspective, looking carefully for hidden
assumptions of homogeneity and continuity in the analysis of observed data, specially because the Standard Model is based on several untested physical theories and parameter fitting.

Acknowledgements:

We thank Inter University Centre of Astronomy and Astrophysics (IUCAA) for hospitality and facilities to carry out this research.

References


Figure 1: Hubble diagram for SNe1a$^{18}$