A model of the Universe including Dark Energy accounted for by both a Quintessence Field and a (negative) Cosmological Constant

Rolando Cardenas*, Tame Gonzalez†, Osmel Martin‡ and Israel Quiros§

Departamento de Fisica. Universidad Central de Las Villas. Santa Clara. CP: 54830 Villa Clara.

Cuba

(June 19, 2002)

Abstract

In this work we present a model of the universe in which dark energy is modelled explicitly with both a dynamical quintessence field and a cosmological constant. Our results confirm the possibility of a collapsing universe (for a given region of the parameter space), which is necessary for an adequate definition of string theory. We have also reproduced the measurements of modulus distance from supernovae with good accuracy.

I. INTRODUCTION

From 1998 to date several important discoveries in the astrophysical sciences have being made, which have given rise to the so called New Cosmology [1,2]. Amongst its more important facts we may cite: the universe expands in an accelerated way [3,4]; the first Doppler peak in the cosmic microwave background is strongly consistent with a flat universe
whose density is the critical one [5], while several independent observations indicate that
matter energy density is about one third of the aforementioned critical density [6,7]. The
last two facts implied that some unknown component of the Universe "was missing", it was
called dark energy, and represents near two thirds of the energy density of the universe.
The leading candidates to be identified with dark energy involve fundamental physics and
include a cosmological constant (vacuum energy), a rolling scalar field (quintessence), and
a network of light, frustrated topological defects [8].

On the other hand, an eternally accelerating universe seems to be at odds with string
theory, because of the impossibility of formulating the S-matrix. In a deSitter space the
presence of an event horizon, signifying causally disconnected regions of space, implies the
absence of asymptotic particle states which are needed to define transition amplitudes [9,10].

Due to the above there is a renewed interest in exponential quintessence, because in
several scenarios exponential potentials can reproduce the present acceleration and pre-
dict future deceleration, so again string theory has well defined asymptotic states [10,11].
Worthwhile to notice that exponential quintessence had been so far overlooked on fine tun-
ing arguments, but several authors have recently pointed out that the degree of fine tuning
needed in these scenarios is no more than in others usually accepted [10–12].

The cosmological constant can be incorporated into the quintessence potential as a con-
stant which shifts the potential value, especially, the value of the minimum of the potential,
where the quintessence field rolls towards. Conversely, the height of the minimum of the
potential can also be regarded as a part of the cosmological constant. Usually, for separating
them, the possible nonzero height of the minimum of the potential is incorporated into the
 cosmological constant and then set to be zero. The cosmological constant can be provided
by various kinds of matter, such as the vacuum energy of quantum fields and the potential
energy of classical fields and may also be originated in the intrinsic geometry. So far there
is no sufficient reason to set the cosmological constant (or the height of the minimum of
the quintessence potential) to be zero [13]. In particular, some mechanisms to generate a
negative cosmological constant have been pointed out [14,15].
The goal of this paper is to present a model of the universe in which the dark energy component is accounted for by both a quintessence field and a negative cosmological constant. The quintessence field accounts for the present stage of accelerated expansion of the universe. Meanwhile, the inclusion of a negative cosmological constant warrants that the present stage of accelerated expansion will be, eventually, followed by a period of collapse into a final cosmological singularity (AdS universe).

Our scenario is a generalization of that of Rubano and Scudellaro [12]. We consider a model consisting of a three-component cosmological fluid: matter, scalar field (quintessence with an exponential potential) and cosmological constant. "Matter" means barionic + cold dark matter, with no pressure, and the scalar field is minimally coupled and noninteracting with matter. This model cannot be used from the very beginning of the universe, but only since decoupling of radiation and dust. Thus, we don’t take into account inflation, creation of matter, nucleosynthesis, etc. Also, we use the experimental fact of a spatially flat universe [16]. We apply the same technique of adimensional variables we used in [17] to determine the integration constants without additional assumptions.

The paper has been organised in the following way. In Sec. II we present the details of the model (action, etc.) and apply it to the case when a single exponential is considered. The case when a potential with two exponentials combined is considered, is studied in Sec. III. The relevant results are analysed in Sec. IV and brief conclusions are given in the last section.

II. AN EXPONENTIAL POTENTIAL

The action of the model under consideration is given by

\[
S = \int d^4x \sqrt{-g}\left\{ \frac{c^2}{16\pi G} (R - 2\Lambda) + \mathcal{L}_\phi + \mathcal{L}_m \right\},
\]

where \(\Lambda\) is the cosmological constant, \(\mathcal{L}_m\) is the Lagrangian for the matter degrees of freedom and the Lagrangian for the quintessence field is given by
\[ \mathcal{L}_\phi = -\frac{1}{2} \phi_{,n} \phi^{,n} - V(\phi). \] (2.2)

We use the dimensionless time variable \( \tau = H_0 t \), where \( t \) is the cosmological time and \( H_0 \) is the present value of the Hubble parameter. In this case \( a(\tau) = \frac{a(t)}{a(0)} \) is the dimensionless scale factor. Then we have that, at present \( (\tau = 0) \)

\[
\begin{align*}
a(0) &= 1, \\
\dot{a}(0) &= 1, \\
H(0) &= 1,
\end{align*}
\] (2.3)

Considering a spatially flat, homogeneous and isotropic universe, the field equations derivable from (2.1) are

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{2}{9} \sigma^2 \left\{ \frac{\bar{D}}{a^3} + \frac{1}{2} \dot{\phi}^2 + \bar{W}(\phi) \right\},
\] (2.4)

\[
2 \frac{\ddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 = -\frac{2}{3} \sigma^2 \left\{ \frac{1}{2} \phi^2 - \bar{W}(\phi) \right\},
\] (2.5)

and

\[
\ddot{\phi} + 3 \frac{\dot{a}}{a} \dot{\phi} + \bar{W}'(\phi) = 0,
\] (2.6)

where the dot means derivative in respect to \( \tau \) and,

\[
\begin{align*}
\bar{W}(\phi) &= \bar{V}(\phi) + \frac{3}{2} \bar{\Lambda}, \\
\bar{V}(\phi) &= B^2 e^{-\sigma \phi},
\end{align*}
\] (2.7)

\[
\begin{align*}
\bar{B}^2 &= \frac{B^2}{H_0^2}, \\
\bar{\Lambda} &= \frac{\Lambda}{H_0^2}, \\
\bar{D} &= \frac{D}{a_0^3 H_0^2} = \frac{\rho_{mo}}{H_0^2},
\end{align*}
\] (2.8)

with \( \rho_{mo} \) - the present density of matter, \( \sigma^2 = \frac{12 \pi G}{c^2} \) and \( B^2 \) - a generic constant.
Applying the Noether Symmetry Approach [18–21], it can be shown that the new variables we should introduce to simplify the field equations are the same used in [12]:

\[ a^3 = uv, \]  

(2.9)

and

\[ \phi = -\frac{1}{\sigma} \ln \left( \frac{u}{v} \right). \]  

(2.10)

In these variables the field Eqs. (2.4-2.6) may be written as the following pair of equations

\[ \frac{\ddot{u}}{u} + \frac{\ddot{v}}{v} = \bar{B}^2 \sigma^2 \frac{u}{v} - \sigma^2 \bar{V}_0, \]  

(2.11)

and

\[ \frac{\ddot{u}}{u} - \frac{\ddot{v}}{v} = -\sigma^2 \bar{B}^2 \frac{u}{v}, \]  

(2.12)

respectively. In Eq (2.11) we have introduced explicitly a negative cosmological constant:

\[ \bar{\Lambda} = -\frac{2}{3} \sigma^2 \bar{V}_0. \]  

(2.13)

Combining of Eqs. (2.11) and (2.12) yields

\[ \ddot{u} = -\frac{\sigma^2 \bar{V}_0}{2} u, \]  

(2.14)

and

\[ \ddot{v} = -\frac{\sigma^2 \bar{V}_0}{2} v + \sigma^2 \bar{B}^2 u. \]  

(2.15)

The solutions of the equations (2.14) and (2.15) are found to be

\[ u(\tau) = u_1 \sin \left( \sigma \sqrt{\frac{\bar{V}_0}{2}} \tau \right) + u_2 \cos \left( \sigma \sqrt{\frac{\bar{V}_0}{2}} \tau \right), \]  

(2.16)

and

\[ v(\tau) = \left\{ v_2 + \frac{\bar{B}^2}{2 \bar{V}_0} u_2 - \frac{\sigma \bar{B}^2}{\sqrt{2 \bar{V}_0}} u_1 \tau \right\} \cos \left( \sigma \sqrt{\frac{\bar{V}_0}{2}} \tau \right) + \right. \]  

\[ \left. + \left\{ v_1 + \frac{\bar{B}^2}{2 \bar{V}_0} u_1 + \frac{\sigma \bar{B}^2}{\sqrt{2 \bar{V}_0}} u_2 \tau \right\} \sin \left( \sigma \sqrt{\frac{\bar{V}_0}{2}} \tau \right) \right\}, \]  

(2.17)
where $u_1, u_2, v_1$ and $v_2$ are the integration constants. These can be related with the initial conditions for $u$ and $v$ (and their $\tau$-derivatives) throughout the following equations,

$$u(0) = u_2,$$
$$\dot{u}(0) = \sigma \sqrt{\frac{V_0}{2}} u_1;$$

(2.18)

and

$$v(0) = v_2 + \frac{B^2}{2V_0} u_2,$$
$$\dot{v}(0) = \sigma \sqrt{\frac{V_0}{2}} (v_1 - \frac{B^2}{2V_0} u_1).$$

(2.19)

In finding the integration constants we use the following equations. First we evaluate Eq. (2.9) for $\tau = 0$ and then we consider the relationships (2.3),

$$1 = u(0)v(0).$$

(2.20)

Another expression is provided by the Hubble parameter:

$$H(\tau) = \frac{1}{3} \left\{ \frac{\dot{u}(\tau)}{u(\tau)} + \frac{\dot{v}(\tau)}{v(\tau)} \right\},$$

(2.21)

evaluated at $\tau = 0$,

$$3 = \frac{\dot{u}(0)}{u(0)} + \frac{\dot{v}(0)}{v(0)}.$$

(2.22)

The other two relationships needed for our purposes are given by the field equations (2.4) and (2.5) rewritten as

$$9H(\tau)^2 = \frac{2\sigma^2 \tilde{D}}{a(\tau)^3} + \left\{ \frac{\dot{u}(\tau)}{u(\tau)} - \frac{\dot{v}(\tau)}{v(\tau)} \right\}^2 + 2\sigma^2 B^2 u(\tau) \frac{u(\tau)}{v(\tau)} - 2\sigma^2 V_0,$$

(2.23)

and

$$3H(\tau)^2 \{1 - 2q(\tau)\} = -\left\{ \frac{\dot{u}(\tau)}{u(\tau)} - \frac{\dot{v}(\tau)}{v(\tau)} \right\}^2 + 2\sigma^2 B^2 u(\tau) \frac{u(\tau)}{v(\tau)} - 2\sigma^2 V_0,$$

(2.24)

respectively. In the last equation we have introduced the deceleration parameter
\[ q(\tau) = -\left\{1 + \frac{H(\tau)}{H(\tau)^2}\right\} \]  

(2.25)

If we evaluate Eqs. (2.23) and (2.24) at \( \tau = 0 \) (recalling that \( H(0) = a(0) = 1 \)) and, we consider the equations (2.20) and (2.21), then we have the following system of four equations for determining the four integration constants:

\[
1 = u_2(v_2 + \frac{\bar{B}^2}{2V_0}u_2),
\]

\[
\frac{3\sqrt{2}}{\sigma\sqrt{V_0}} = v_2u_1 + u_2v_1,
\]

\[
9 - 2\sigma^2D + 2\sigma^2V_0 = \frac{\sigma^2V_0}{2}((v_2 + \frac{\bar{B}^2}{2V_0}u_2)u_1 - u_2v_1)^2 + 2\sigma^2\bar{B}^2u_2^2,
\]

\[
3(1 - 2q_0) + 2\sigma^2V_0 = -\frac{\sigma^2V_0}{2}((v_2 + \frac{\bar{B}^2}{2V_0}u_2)u_1 - u_2v_1)^2 + 2\sigma^2\bar{B}^2u_2^2,
\]

(2.26)

where \( q_0 \) is the present value of the deceleration parameter. By solving the system of equations (2.26) we find 4 sets of solutions, or, strictly speaking, 4 branches of a same solution.

\[
u_2(\pm) = \pm \sqrt{\frac{3(2 - q_0) - \sigma^2D + 2\sigma^2V_0}{2\sigma^2\bar{B}^2}},
\]

(2.27)

\[
v_2(\pm) = \frac{1 - \frac{\bar{B}^2}{2V_0}u_2^2}{u_2(\pm)},
\]

(2.28)

\[
u_1(\pm) = \frac{3 - [\pm]\sqrt{3(1 + q_0) - \sigma^2D}}{\sqrt{2\sigma^2V_0}}u_2(\pm),
\]

(2.29)

and

\[
u_1(\pm) = \frac{6 - \sqrt{2\sigma^2V_0}v_2(\pm)u_1(\pm)}{\sqrt{2\sigma^2V_0 u_2(\pm)}}.
\]

(2.30)

If we introduce new parameters:

\[
\Omega_{m_0} = \frac{2}{9}\sigma^2\bar{D},
\]

(2.31)

\[
\Omega_{Q_0} = \frac{2}{9}\sigma^2\left\{\frac{1}{2}\dot{\phi}(0)^2 + \bar{V}(\phi(0))\right\},
\]

(2.32)
and
\[ \Omega_\Lambda = -\frac{2}{9} \sigma^2 \bar{V}_0, \]
(2.33)
where, for flat FRW spacetimes, \( \Omega_m + \Omega_Q + \Omega_\Lambda = 1 \), and we propose the following relationship between the parameters \( \bar{B}^2 \) and \( \bar{V}_0 \),
\[ \bar{B}^2 = n \bar{V}_0, \]
(2.34)
where \( n \) is a positive real number, then the above integration constants can be written in the following way:
\[ u_2^{(\pm)} = \pm \sqrt{\frac{2 - q_0 - 1.5 \Omega_{m_0} - 3 \Omega_\Lambda}{-3n \Omega_\Lambda}}, \]
(2.35)
\[ v_2^{(\pm)} = \frac{1 - \frac{u_2^{(\pm)}}{u_2^{(\pm)}}}{u_2^{(\pm)}}, \]
(2.36)
\[ u_1^{(\pm)} = \frac{\{\sqrt{3} - \sqrt{1 + q_0 - 1.5 \Omega_{m_0}}\} u_2^{(\pm)}}{\sqrt{-3 \Omega_\Lambda}}, \]
(2.37)
and
\[ v_1^{(\pm)} = \frac{2 - \sqrt{-\Omega_\Lambda v_2^{(\pm)} u_1^{(\pm)}}}{\sqrt{-\Omega_\Lambda u_2^{(\pm)}}}, \]
(2.38)
respectively.

Since \( \sqrt{1 + q_0 - 1.5 \Omega_{m_0}} \) should be real (see equation (2.37)) then, the following constrain on the present value of the deceleration parameter follows
\[ q_0 \geq -1 + 1.5 \Omega_{m_0}. \]
(2.39)

It can be noticed that the constants (and, consequently, the solutions) depend on 4 physical parameters: \( \Omega_{m_0} \), \( \Omega_\Lambda \), \( q_0 \) and on the positive real number \( n \). In general, the experimental values for \( \Omega_{m_0} \) and \( q_0 \) are model-dependent, because this magnitudes are not directly measured (though Turner and Riess have developed a model-independent test for past deceleration [22]).
Concerning the parameter $n$, it should be noticed that most of the relevant cosmological parameters (such as the scale factor and the deceleration parameter) are quite insensitive to it’s value. However, the state parameter is dependent on it. Though we made calculations for several values of $\Omega_\Lambda$ in the range -0.01 a -0.30, for simplicity we present results for -0.15, having in mind that they change little for other values.

III. TWO EXPONENTS COMBINED

Now we apply the former procedure, step by step, to the field equations (2.4)-(2.7) with the "bar" potential,

$$\bar{V}(\phi) = \bar{B}^2 e^{-\sigma \phi} + \bar{A}^2 e^{\sigma \phi},$$  \hspace{1cm} (3.1)

where, as before,

$$\bar{B}^2 = \frac{B^2}{H_0^2},$$

$$\bar{A}^2 = \frac{A^2}{H_0^2},$$

$$\bar{\Lambda} = \frac{\Lambda}{H_0^2},$$

$$\bar{D} = \frac{D}{a_0^3 H_0^2} = \frac{\rho_{m0}}{H_0^2}. \hspace{1cm} (3.2)$$

Applying the Noether Symmetry Approach [18–21], it can be shown that the new variables we should introduce to simplify the field equations are again the same of those used in [12] for this potential:

$$a^3 = \frac{u^2 - v^2}{4}, \hspace{1cm} (3.3)$$

and

$$\phi = \frac{1}{\sigma} \ln\left[\frac{B(u + v)}{A(u - v)}\right]. \hspace{1cm} (3.4)$$

Now the field equations take the form:
\[ \ddot{u} = (\bar{A} \bar{B} \sigma^2 - \sigma^2 \bar{V}_0), \]  

(3.5)

and

\[ \ddot{v} = -(\bar{A} \bar{B} \sigma^2 + \sigma^2 \bar{V}_0)v. \]

(3.6)

In what follows, for definiteness, we assume the following relation between the constants:

\[ \bar{A}^2 = n \bar{B}^2 = m \bar{V}_0, \]  

(3.7)

where \( n \) and \( m \) are real parameters.

The solutions are

\[ u(\tau) = b \exp[-\sqrt{-\frac{9}{2} \Omega_\Lambda (m \sqrt{\frac{1}{n}} - 1)t}] + c \exp[-\sqrt{-\frac{9}{2} \Omega_\Lambda (m \sqrt{\frac{1}{n}} - 1)t}], \]  

(3.8)

\[ v(\tau) = d \sin[\sqrt{-\frac{9}{2} \Omega_\Lambda (m \sqrt{\frac{1}{n}} + 1)t}] + e \cos[\sqrt{-\frac{9}{2} \Omega_\Lambda (m \sqrt{\frac{1}{n}} + 1)t}], \]  

(3.9)

where, the integration constants are given through

\[ e = -\sqrt{\frac{1 - \Omega_{m_0} - 4 \Omega_\Lambda + \frac{1 - 2 q_0}{3}}{-2 \Omega_\Lambda m \sqrt{\frac{1}{n}}}} - 2, \]  

(3.10)

\[ c = \frac{e \sqrt{2(9 - 9 \Omega_{m_0} - 3(1 - 2 q_0)) - 6 e^2 + 4}}{8 \sqrt{-\frac{9}{2} (m \sqrt{\frac{1}{n}} - 1) \Omega_\Lambda}} + \frac{\sqrt{e^2 + 4}}{2}, \]  

(3.11)

\[ b = \sqrt{e^2 + 4} - c, \]  

(3.12)

\[ d = \frac{-6 + (b + c)(b - c) \sqrt{-\frac{9}{2} (m \sqrt{\frac{1}{n}} - 1) \Omega_\Lambda}}{e \sqrt{-\frac{9}{2} (m \sqrt{\frac{1}{n}} + 1) \Omega_\Lambda}}. \]  

(3.13)

Our constants (and, consequently, the solution) depend now on five physical parameters: \( \Omega_{m_0}, \Omega_\Lambda, q_0 \) and on the positive real numbers \( m \) and \( n \).
IV. ANALYSIS OF RESULTS

After making a detailed study, it was determined that the only relevant cosmological magnitude that has a sensible dependence on parameters $m$ and $n$ is the state parameter $\omega$. We fixed $\Omega_m = 0.3$, in accordance with experimental evidence.

Figure 1 shows the evolution of the scale factor for $\Omega_\Lambda = -0.15$. It was shown both algebraically and graphically that, in the case of the single exponential potential, the evolution of the universe is independent of $n$, but not written for the sake of simplicity. For the double exponential potential, it can be shown that, in order to have a collapsing universe, the following condition should be fulfilled:

$$m < \frac{1}{\sqrt{n}} \quad (4.1)$$

We also saw that with the decrease (modular increase) of $\Omega_\Lambda$, the time of collapse diminishes.

We got the dependence of the state parameter and of the deceleration parameter with the redshift and then we selected the values of $m$, $n$ and $q_0$. For this purpose we considered the results of Turner and Riess [22].

Figure 2 shows the behaviour of the deceleration parameter as function of the redshift $z$ for the same values of the parameters. This figure shows an early stage of deceleration and a current epoch of acceleration. A transition from an accelerated phase to a decelerated one is seen approximately for $z =0.5$. We appreciate an increase of the deceleration parameter upon increasing the value of $z$. This points at a past epoch in the evolution when gravity of the dark energy was attractive. As follows from figure 1, acceleration is not eternal: in the future $q > 0$ again, which gives rise to the collapse.

Figure 3 shows the evolution of the state parameter of the effective quintessence field $\omega_\phi$. It’s noticeable that the effective quintessence field has state parameter $\omega_\phi$ near $-1$ today, which means that its behaviour is similar to the ”pure” cosmological constant, as a vacuum fluid. In the case of the double exponential potential, one can see that when $z$ tends to zero (current time) the state parameter tends to - 0.8. This is in agreement with the results of
when only the first Doppler peak is considered. If we are to explain the very desirable for today’s cosmology recent and future deceleration obtained in our model, it’s important to look at the dynamical quintessence field. We see that in the recent past $\omega_\phi > 0$, which implies that quintessence field behaved (or simply was) like ordinary attractive matter, giving rise to the logical deceleration. In the future this will happen again ($\omega_\phi > 0$), with the consequent deceleration.

The present values of the physical parameters ($\Omega_{m_0} = 0.3$, $\Omega_\Lambda = -0.15$, $q_0 = -0.44$ and $q_0 = -0.34$, for single and double exponential potential, respectively) were chosen after a detailed analysis of the behaviour of these parameters shown in figs. 2 and 3.

Now we proceed to analyze how our solution reproduces experimental results. With this purpose, in Fig. 4 we plot the distance modulus $\delta(z)$ vs redshift $z$, calculated by us and the one obtained with the usual model with a constant $\Lambda$ term. The relative deviations are of about 0.5% and 0.3%, for single and double exponential potential, respectively.

V. CONCLUSIONS

In a recent paper [13] it is pointed out that the ultimate fate of our universe is much more sensitive to the presence of the cosmological constant than any other matter content. In particular, the universe with a negative cosmological constant will always collapse eventually, even though the cosmological constant may be nearly zero and undetectable at all at the present time. Our results support the very general assertions of [13], we have shown that for a determined region of the parameter space, the universe collapses. This also favours the formulation of string theory, as explained in the introduction. The experimental measurements of modulus distance from the supernovae are adequately reproduced within an accuracy of 0.5%. So far, we have investigated one of the several possible branches of the solution, leaving for the future the investigation of the others. We have also reserved for future work the careful examination of this universe near its beginning (i.e., just after the decoupling of matter and radiation).
We acknowledge Claudio Rubano, Mauro Seleno and Paolo Scudellaro, from Universita di Napoli ”Federico II”, Italy, for useful comments and discussions and Andro Gonzales for help in the computations.
REFERENCES


Single Exponential Potential
Double Exponential Potential
Single Exponential Potential

Double Exponential Potential