United Nations Educational Scientific and Cultural Organization
and
International Atomic Energy Agency
THE ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

**CHARACTERISTIC CLASSES AND TRANSFER RELATIONS
IN COBORDISM**

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Abstract

Decompositions of products of the Ray elements and low dimensional free generators of the symplectic cobordism ring are obtained. In particular it is stated that most of the $4n$ -dimensional generators, for n small, after multiplication by the Ray elements ϕ_i , $i \geq 0$ land in the ideal generated by Ray elements of low dimension.

MIRAMARE – TRIESTE

May 2001

1 Introduction

Most of the relations between Ray elements ϕ_i and free generators in the torsion part of the symplectic cobordism ring up to dimension 32 [14, 15] can be conventionally subdivided into three types: to the first type correspond relations which mainly follow from relations in the integral part coming from $M\mathrm{Sp}_{4n}$. In more detail, relations of the first type have the form $(x + y)\phi_i = 0$, and these relations follow from the fact that the sum of free generators $x + y$ is divisible by 2, whereas the Ray elements have order 2. Relations of the second type have the form $z\phi_i = 0$ where z is again a $4n$ -dimensional generator from the free part. The aim of the paper is to elucidate origins of the relations of third type as in the Abstract.

Let ζ be the universal $\mathrm{Sp}(1)$ -bundle, then $\zeta_1 \otimes_{\mathbb{C}} \zeta_2 \otimes_{\mathbb{C}} \zeta_3$ is a symplectic bundle over $B\mathrm{Sp}(1) \times B\mathrm{Sp}(1) \times B\mathrm{Sp}(1)$; also $\zeta_1 \otimes_{\mathbb{C}} \zeta_2^2$ and $\zeta_1 \otimes_{\mathbb{R}} \zeta_2$ are symplectic bundles over $B\mathrm{Sp}(1) \times B\mathrm{Sp}(1)$.

Section 2 is devoted to the calculation of transfers [1, 5, 2, 11]. In Section 3 we prove the following main result:

Theorem 1.1. *Let $x_i = p_1(\zeta_i)$, $i=1,2$, be the first Conner-Floyd symplectic Pontryagin class. Let ϕ_j , $j \geq 0$, be the Ray elements, and let n be such that $M\mathrm{Sp}_{4m}$ is torsion free for $m \leq 2n - 1$. Then*

- a) *The element $\phi_j p_1(\zeta_1 \otimes_{\mathbb{C}} \zeta_2^2)$ is divisible by $\phi_0 x_1 + \phi_1 x_1^2 + \dots + \phi_{[n/2]} x_1^{2[n/2]}$;*
- b) *$\phi_j p_1(\zeta_1 \otimes_{\mathbb{R}} \zeta_2) = 0$*

in the ring $M\mathrm{Sp}^(\mathbb{H}P(n)^2) = M\mathrm{Sp}^*[[x_1, x_2]]/(x_i^{n+1})$.*

In Section 4 we shall see that in terms of the coefficients a_{klm} of the first Conner-Floyd symplectic Pontryagin class

$$p_1(\zeta_1 \otimes_{\mathbb{C}} \zeta_2 \otimes_{\mathbb{C}} \zeta_3) = \sum_{k+l+m \geq 1} a_{klm} p_1^k(\zeta_1) p_1^l(\zeta_2) p_1^m(\zeta_3)$$

the structure of $M\mathrm{Sp}_{4k}$, $k \leq 4$ can be interpreted as follows:

k	$M\mathrm{Sp}_{4k}$	generators
1	$\mathbb{Z}$	a_{011}
2	$\mathbb{Z} + \mathbb{Z}$	a_{012}, a_{111}
3	$\mathbb{Z} + \mathbb{Z} + \mathbb{Z}$	$a_{022}, a_{011}a_{111}, a_{211}$
4	$\mathbb{Z} + \mathbb{Z} + \mathbb{Z} + \mathbb{Z} + \mathbb{Z}$	$a_{014}, a_{011}a_{211}, a_{122}, a_{111}^2, 2y_4$.

Then Theorem 1.1 implies

Corollary 1.1. *For $i \geq 0$ one has*

- a) $\phi_i a_{001} = \phi_i a_{012} = \phi_i a_{022} = \phi_i a_{014} = 0$;
- b) $\phi_i a_{111}$ and $\phi_i a_{122}$ belong to the ideal $\phi_0 MSp^*$;
- c) $\phi_i a_{211}$ belong to the ideal $\phi_0 MSp^* + \phi_1 MSp^*$.

Relations of Corollary 1.1 imply that multiplication by the elements ϕ_i , $i \geq 0$, carries most of the low dimensional generators from the free part of MSp_{4n} to the ideal generated by the elements ϕ_0 and ϕ_1 .

2 Preliminaries and calculations with transfer

Let ξ and Λ be, respectively, the universal $U(1)$ -bundle and the universal $Spin(3)$ -bundle. Thus the sphere bundle of Λ is $\pi : BU(1) \rightarrow BSp(1)$, and one has

$$\pi^*(\zeta) = \xi + \bar{\xi}; \quad (2.1)$$

$$\pi^*(\Lambda) = \xi^2 + \mathbb{R}; \quad (2.2)$$

$$\zeta \otimes_{\mathbb{H}} \zeta = \Lambda + \mathbb{R}; \quad (2.3)$$

where ζ is the universal $Sp(1)$ -bundle as above. Let N be the normalizer of the torus $U(1)$ in $Sp(1)$. The classifying space BN coincides with the orbit space of the complex projective space $\mathbb{C}P(\infty)$ under the free involution I , which acts via

$$I : [z_0, z_1, \dots] \mapsto [-\bar{z}_1, \bar{z}_0, \dots]$$

in homogeneous coordinates.

The bundle $p : BN \rightarrow BSp(1)$ coincides with the projective bundle of Λ and we have the canonical splitting

$$p^*(\Lambda) = \mu + \nu, \quad (2.4)$$

defined by projectivisation p , where μ and ν are a plane and linear real bundles. Of course for the double covering $q : BU(1) \rightarrow BN$ we have $q^*(\mu) = \xi^2$ and $q^*(\nu) = \mathbb{R}$.

Let τ_π and τ_p be the transfer maps of the bundles π and p [2, 5, 11]. The following lemma follows from [6].

Lemma 2.1. $\pi^* \tau_\pi^* = 1 + I^*$ and $\pi^* \tau_p^* = q^*$

The following lemma follows from definitions

Lemma 2.2. $(\xi_1 \xi_2^2 + \bar{\xi}_1 \bar{\xi}_2^2)! = (\xi_1 + \bar{\xi}_1) \otimes_{\mathbb{R}} \mu$, where ‘!’ is the Atiyah transfer of the double covering $1_{BU(1)} \times q$.

Let f be the map $f : BN \rightarrow B\mathbb{Z}/2$ induced by the projection of N onto the Weyl group $\mathbb{Z}/2$ and let $\tau_{1 \times q}^*$ be the transfer homomorphism for the above double covering $1_{BU(1)} \times q$.

Lemma 2.3. *For some elements $\alpha_i \in \widetilde{M\mathrm{Sp}}^*(B\mathbb{Z}/2)$ the following formula holds*

$$\tau_{1 \times q}^*(p_1(\xi_1 \xi_2^2 + \bar{\xi}_1 \bar{\xi}_2^2)) = p_1((\xi_1 + \bar{\xi}_1) \otimes_{\mathbb{R}} \mu) + \sum_{i \geq 0} f^*(\alpha_i) p_2^i((\xi_1 + \bar{\xi}_1) \otimes_{\mathbb{R}} \mu).$$

Proof. Taking into account Lemma 2.2 the proof follows from the following formula [13]:

Let q be the double covering $q : X \rightarrow B$, let $\eta \rightarrow X$ be the symplectic line bundle, $\eta_l \rightarrow B$ the Atiyah transfer bundle, τ_q the transfer map of the covering q and $f : X \rightarrow B\mathbb{Z}/2$ the classifying map of the real line bundle associated with q . Then for some elements α_i from $\widetilde{M\mathrm{Sp}}^*(B\mathbb{Z}/2)$ the following formula holds

$$\tau_q^*(p_1(\eta)) = p_1(\eta_l) + \sum_{i \geq 0} f^*(\alpha_i) p_2^i(\eta_l).$$

Lemma 2.4. *Let τ be the transfer of the sphere bundle of a $\mathrm{Spin}(3)$ -bundle, then $\phi_j \mathrm{Im} \tau^* = 0$, $j \geq 0$.*

Proof. Of course it suffices to prove this for the universal $\mathrm{Spin}(3)$ -bundle π , that is, $\phi_j \tau_\pi^*(a) = 0$ for all $a \in M\mathrm{Sp}^*(BU(1))$.

Let δ_π be the Boardman map [3]. Then as it is known from [2],

$$\tau_\pi^*(a) = \delta_\pi(ae(\xi_2^2)).$$

Here $e(\xi_2^2)$ is the Euler class of the bundle ξ_2^2 which is the bundle of tangents along the fibers. Then from [10, 7] $\phi_j e(\xi_2^2) = 0$. This proves Lemma 2.4.

Recall from [10, 7, 8] that the bundle Λ is $M\mathrm{Sp}$ -orientable and the corresponding Euler class has the form

$$e(\Lambda) = \phi_0 p_1(\zeta) + \sum_{j \geq 1} \phi_j p_1(\zeta)^j. \quad (2.5)$$

The restrictions of π and p to the symplectic projective space $\mathbb{H}P(n)$ will be denoted by the same symbols. Total spaces of these bundles coincide, respectively, with the complex projective space $\mathbb{C}P(2n+1)$ and with the orbit space $\mathbb{C}P(2n+1)/I$ under the free involution I which acts via

$$[z_0, z_1, \dots, z_{2n}, z_{2n+1}] \mapsto [-\bar{z}_1, \bar{z}_0, \dots, -\bar{z}_{2n+1}, \bar{z}_{2n}]$$

in homogeneous coordinates.

Proposition 2.1. $\phi_j \tau_{\pi \times 1}^*(p_i(r\xi \otimes_{\mathbb{R}} \zeta)) = 0$ for $\pi \times 1 = \pi \times 1_{B\mathrm{Sp}(1)} : BU(1) \times B\mathrm{Sp}(1) \rightarrow B\mathrm{Sp}(1)^2$, $j \geq 0$ and $i = 1, 2$.

Proof. We have $p_i(r\xi \otimes_{\mathbb{R}} \zeta) = \sum_{k \geq 0} \omega_k^{(i)} p_1^k(\zeta)$ in $M\mathrm{Sp}^*(BU(1) \times B\mathrm{Sp}(1)) = M\mathrm{Sp}^*(BU(1))[[p_1(\zeta)]]$. Then it follows from Lemma 2.4, that

$$\phi_j \tau^*\left(\sum_{k \geq 0} \omega_k^{(i)} p_1^k(\zeta)\right) = \sum_{k \geq 0} \phi_j \tau^*(\omega_k^{(i)} p_1^k(\zeta)) = 0.$$

3 Proof of Theorem 1.1

The bundle $\pi \times 1 : \mathbb{C}P(2n+1) \times \mathbb{H}P(n) \rightarrow \mathbb{H}P(n) \times \mathbb{H}P(n)$ coincides with the sphere bundle of the pullback of $\Lambda \rightarrow \mathbb{H}P(n)$ along the projection on the first factor $\mathbb{H}P(n) \times \mathbb{H}P(n) \rightarrow \mathbb{H}P(n)$. So taking into account the formula (2.5) we have to prove that

$$(\pi \times 1_{\mathbb{H}P(n)})^*(\phi_j p_1(\zeta_1 \otimes_{\mathbb{C}} \zeta_2^2)) = 0$$

in $M\mathrm{Sp}^*(\mathbb{C}P(2n+1) \times \mathbb{H}P(n))$. The transfer $\tau^* = \tau_{1 \times \pi}^*$ of the bundle $1_{\mathbb{C}P(2n+1)} \times \pi$ is a composite of two transfers, namely

$$\tau^* = \tau_1^*(\tau_2^*)$$

where τ_1 is the transfer of the bundle $1_{\mathbb{C}P(2n+1)} \times q$ and τ_2 is the transfer of $1_{\mathbb{C}P(2n+1)} \times p$, where the bundles p , π and q are the bundles defined above, that is

$$1 \times q : \mathbb{C}P(2n+1) \times \mathbb{C}P(2n+1) \rightarrow \mathbb{C}P(2n+1) \times \mathbb{C}P(2n+1)/I;$$

$$1 \times p : \mathbb{C}P(2n+1) \times \mathbb{C}P(2n+1)/I \rightarrow \mathbb{C}P(2n+1) \times \mathbb{H}P(n).$$

Using the formula (2.2) we have $(\xi_1 + \bar{\xi}_1) \otimes_{\mathbb{C}} \zeta_2^2 = (\xi_1 + \bar{\xi}_1) \otimes_{\mathbb{R}} (\Lambda + \mathbb{R})$, hence we have

$$p_1((\xi_1 + \bar{\xi}_1) \otimes_{\mathbb{C}} \zeta_2^2) = p_1((\xi_1 + \bar{\xi}_1) \otimes_{\mathbb{R}} \Lambda) + p_1(\xi_1 + \bar{\xi}_1). \quad (3.1)$$

Applying Lemma 2.2 and Lemma 2.3 we have

$$\tau_1^*(p_1(\xi_1 \xi_2^2 + \bar{\xi}_1 \bar{\xi}_2^2)) = p_1((\xi_1 + \bar{\xi}_1) \otimes_{\mathbb{R}} \mu) + \sum_{i \geq 0} f^*(\alpha_i) p_2^i((\xi_1 + \bar{\xi}_1) \otimes_{\mathbb{R}} \mu),$$

Then by the formula (2.4)

$$p_1((\xi_1 + \bar{\xi}_1) \otimes_{\mathbb{R}} \mu) = (1 \times p)^* p_1((\xi_1 + \bar{\xi}_1) \otimes_{\mathbb{R}} \Lambda) - p_1((\xi_1 + \bar{\xi}_1) \otimes_{\mathbb{R}} \nu),$$

hence

$$\begin{aligned} \tau^*(p_1(\xi_1 \xi_2^2 + \bar{\xi}_1 \bar{\xi}_2^2)) &= \\ \tau_2^*((p_1(\xi_1 + \bar{\xi}_1) \otimes_{\mathbb{R}} \mu)) + \tau_2^*(\sum_{i \geq 0} f^*(\alpha_i) p_2^i((\xi_1 + \bar{\xi}_1) \otimes_{\mathbb{R}} \mu)) &= \\ \tau_2^*(1 \times p)^*(p_1((\xi_1 + \bar{\xi}_1) \otimes_{\mathbb{R}} \Lambda)) - \tau_2^*(p_1(\xi_1 + \bar{\xi}_1) \otimes_{\mathbb{R}} \nu) + \\ \tau_2^*(\sum_{i \geq 0} f^*(\alpha_i) p_2^i(\xi_1 + \bar{\xi}_1) \otimes_{\mathbb{R}} \mu) &= \\ p_1((\xi_1 + \bar{\xi}_1) \otimes_{\mathbb{R}} \Lambda) \tau_2^*(1) - \tau_2^*(p_1(\xi_1 + \bar{\xi}_1) \otimes_{\mathbb{R}} \nu) + \end{aligned}$$

$$\tau_2^* \left(\sum_{i \geq 0} f^*(\alpha_i) p_2^i(\xi_1 + \bar{\xi}_1) \otimes_{\mathbb{R}} \mu \right). \quad (3.2)$$

Now we have to prove that $\tau_2^*(1) = 1$, the second summand in (3.2) coincides with $x_1 = p_1(\zeta_1)$ and the third summand is zero.

Note that the bundle $(\xi_1 + \bar{\xi}_1) \otimes_{\mathbb{R}} \nu$ is pullback of the bundle $\zeta \otimes \eta \rightarrow B\mathrm{Sp}(1) \times B\mathbb{Z}/2$ along the map (π, f) . Then $p_1(\zeta \otimes \eta)$ is an element from $M\mathrm{Sp}^*(B\mathbb{Z}/2)[[p_1(\zeta)]]$, hence we have

$$p_1(\zeta \otimes \eta) = p_1(\zeta) + \sum_{i \geq 0} \beta_i p_1^i(\zeta)$$

for some elements $\beta_i \in \widetilde{M\mathrm{Sp}}^*(B\mathbb{Z}/2)$. This implies

$$p_1((\xi_1 + \bar{\xi}_1) \otimes_{\mathbb{R}} \nu) = p_1(\xi_1 + \bar{\xi}_1) + \sum_{i \geq 0} f^*(\beta_i) p_1^i(\xi_1 + \bar{\xi}_1)$$

Similarly the bundle $(\xi_1 + \bar{\xi}_1) \otimes_{\mathbb{R}} \mu$ is pullback of the bundle $\zeta \otimes \eta(2) \rightarrow B\mathrm{Sp}(1) \times BO(2)$, where $\eta(2) \rightarrow BO(2)$ is the universal $O(2)$ -bundle. Hence

$$p_2((\xi_1 + \bar{\xi}_1) \otimes_{\mathbb{R}} \mu) \in M\mathrm{Sp}^*(BN)[[p_1(\xi_1 + \bar{\xi}_1)]]$$

and we have for the third summand of the formula (3.2)

$$\sum_{i \geq 0} f^*(\alpha_i) p_2^i((\xi_1 + \bar{\xi}_1) \otimes_{\mathbb{R}} \mu) = \sum_{i \geq 0} \gamma_i p_1^i(\xi_1 + \bar{\xi}_1)$$

for some $\gamma_i \in \widetilde{M\mathrm{Sp}}^*(BN)$.

So using (3.1) we have

$$\begin{aligned} \tau^*(p_1(\xi_1 \xi_2^2 + \bar{\xi}_1 \bar{\xi}_2^2)) = \\ p_1((\xi_1 + \bar{\xi}_1) \otimes_{\mathbb{R}} \zeta^2) \tau_2^*(1) - x_1(\tau_2^*(1) + 1) + \sum_{i=0}^n \tau_2^*(\delta_i) x_1^i, \end{aligned} \quad (3.3)$$

for some $\delta_i \in M\mathrm{Sp}^*(\mathbb{C}P(2n+1)/I)$.

It is known from [14, 15] that up to dimension 32, $M\mathrm{Sp}_{4n}$ is torsion free. Motivated by this fact let us assume that $M\mathrm{Sp}_{4m}$ is torsion free when $m \leq 2n-1$. Then it follows that $M\mathrm{Sp}^{4k}(\mathbb{H}P(n))$ is torsion free, when $k \geq 1-n$. Then since the minimal dimension of the elements γ_i from the formula (3.3) is $4-4n$, it follows from Lemma 2.1 that the third summand of (3.3) restricts to zero in $M\mathrm{Sp}^*(\mathbb{H}P(n))$. Also $\tau^*(p)(1) = 1$ and $\tau^*(\pi)(1) = 2$.

Thus we obtain from (3.3) and then Lemma 2.4

$$\phi_j p_1(\zeta \otimes_{\mathbb{C}} \zeta_2^2) = \phi_j \tau^*(p_1(\xi_1 \xi_2 + \bar{\xi}_1 \bar{\xi}_2)) = 0.$$

This proves Theorem 1.1 a).

For the proof of b) note that it follows from Lemma 2.1 that for the bundle $\pi \times 1 = \pi \times 1_{\mathbb{H}P(n)}$ we have

$$\begin{aligned} (\pi \times 1)^* \tau_{\pi \times 1}^* (p_1(r\xi_1 \otimes_{\mathbb{R}} \zeta_2)) &= (1 + I)^* (p_1((\xi_1 + \bar{\xi}_1) \otimes_{\mathbb{C}} \zeta_2)) = \\ &= 2p_1((\xi_1 + \bar{\xi}_1) \otimes_{\mathbb{C}} \zeta_2) = (\pi \times 1)^* p_1((\zeta_1 \otimes_{\mathbb{R}} \zeta_2)). \end{aligned}$$

Then by (2.5) any element from $\ker(1 \times \pi)^*$ is divisible by $e(\Lambda)$. On the other hand by hypothesis $M\mathrm{Sp}^{4k}(\mathbb{H}P(n))$ is torsion free for $k \geq 1 - n$. Hence we conclude that restriction of the homomorphism $(\pi \times 1)^*$ to $M\mathrm{Sp}(\mathbb{H}P(n)^2)$ is a monomorphism, thus in $M\mathrm{Sp}^4(\mathbb{H}P(n)^2)$ one has

$$p_1(\zeta_1 \otimes_{\mathbb{R}} \zeta_2) = \tau_{\pi \times 1}^* (p_1(r\xi_1 \otimes_{\mathbb{R}} \zeta_2))$$

Now since Proposition 2.1 says that the right-hand side is zero after multiplication by ϕ_j , this completes the proof of Theorem 1.1.

4 Proof of Corollary 1.1

Let $h : \pi_*(M\mathrm{Sp}) \rightarrow H_*(M\mathrm{Sp}) = \mathbb{Z}[q_1, q_2, \dots]$ be the Hurevicz homomorphism. Since $\pi_{4n}(M\mathrm{Sp})$ is torsion free for small n (see [12, 14, 15]), the Hurevicz homomorphism is a monomorphism in these dimensions. So in low dimensions $4n$ the Hurevicz homomorphism determines all relations. Our aim here is to express the coefficients a_{klm} from the introduction through the generators x -es.

Values of the Hurevicz homomorphism on these a_{klm} are calculated in [9]. In low dimensions we have

$$\begin{aligned} h(a_{100}) &= h(a_{010}) = h(a_{001}) = 4; \\ h(a_{200}) &= h(a_{020}) = h(a_{002}) = 0; \\ h(a_{110}) &= h(a_{101}) = h(a_{011}) = 24q_1; \\ h(a_{111}) &= 360q_2 \\ h(a_{210}) &= \dots = h(a_{012}) = 60q_2 - 24q_1^2; \\ h(a_{300}) &= \dots = h(a_{003}) = 0; \\ h(a_{220}) &= \dots = h(a_{022}) = 280q_3 - 120q_1q_2 + 24q_1^3; \\ h(a_{310}) &= \dots = h(a_{013}) = 112q_3 - 96q_1q_2 + 48q_1^3; \\ h(a_{211}) &= \dots = h(a_{112}) = 1680q_3 - 360q_1q_2; \\ h(a_{122}) &= \dots = h(a_{221}) = 75600q_4 - 3360q_1q_3 + 360q_1^2q_2; \\ h(a_{410}) &= \dots = h(a_{014}) = 180q_4 - 360q_1q_3 + 420q_1^2q_2 - 120q_2^2 - 120q_1^4; \end{aligned}$$

Further the Hurevicz images of generators of $M\mathrm{Sp}$ are calculated in [12]. Namely $h(M\mathrm{Sp}_{4k}) \subset H_{4k}(M\mathrm{Sp})$ has the following generators

$$\begin{aligned} k = 1: & \quad 24q_1; \\ k = 2: & \quad 20q_2 - 8q_1^2; 144q_1^2; \\ k = 3: & \quad 56q_3 - 72q_1q_2 + 24q_1^3; 120q_1q_2 - 48q_1^3; 3456q_1^3; \\ k = 4: & \quad 12q_4 - 24q_1q_3 - 8q_2^2 + 28q_1^2q_2 - 8q_1^4; \\ & \quad 50q_2^2 + 168q_1q_3 - 256q_1^2q_2 + 80q_1^4; 100q_2^2 - 80q_1^2q_2 + 16q_1^4; \\ & \quad 2880q_1^2q_2 - 1152q_1^4; 20736q_1^4. \end{aligned}$$

So we have that the elements $a_{011}, a_{111}, a_{022}, a_{122}, a_{112}, a_{120}, a_{140}$ are generators as in the introduction.

Remark 4.1. In terms of $2x_i$, the generators of $M\mathrm{Sp}_{4n}$ from [12], we have modulo $2M\mathrm{Sp}_*$: $2x_1 = a_{011}$, $2x_2 = a_{012}$, $2x_3 = a_{022}$, $x_4 = a_{014}$, $x_1^2 = a_{111}$, etc.

Remark 4.2. Alternatively images of the elements a_{ijk} in complex cobordism $M\mathrm{U}_*$ can be calculated in terms of two-valued formal groups:

$$\mu^*(p_1(\zeta_1 \otimes_{\mathbb{C}} \zeta_2 \otimes_{\mathbb{C}} \zeta_3)) = \Theta_1(x_1, Y^+) + \Theta_1(x_1, Y^-),$$

where $Y^+ + Y^- = \Theta_1(x_2, x_3)$, $Y^+Y^- = \Theta_2(x_2, x_3)$; Θ_1 and Θ_2 are the coefficient of the two-valued formal group [4] and μ is the obvious map from the symplectic cobordism theory to the complex cobordism theory.

Let us consider now Corollary 1.1. From Theorem 1.1 a) we have in $M\mathrm{Sp}^*(\mathbb{H}P(4) \times \mathbb{H}P(4))$ the relation of the form

$$\begin{aligned} \theta_j(a_{011}x_2^2 + a_{111}x_1x_2^2 + (a_{022})x_2^4 + (a_{211})x_1^2x_2^2 + (a_{122})x_1x_2^4 + \dots) \\ = (\phi_0x_1 + \sum_{1 \leq i \leq n} \phi_i x_1^i)b(x_1, x_2) \end{aligned}$$

for some element $b(x_1, x_2) \in M\mathrm{Sp}^*(\mathbb{H}P(n)^2)$. Then since $\theta_{2i+1} = 0$ [13] by the equality of the coefficients at the monomials $x_1x_2^2$, $x_1^2x_2^2$ and $x_1x_2^4$ we obtain the assertions b) and c) of Corollary 1.1.

Similarly from Proposition 2.1 we have

$$\phi_j(a_{110}x_2^2 + a_{120}x_1x_2^2 + a_{220}x_1^2x_2^2 + a_{140}x_1x_2^4 + \dots) = 0,$$

and hence the assertion a) of Corollary 1.1 is valid.

Proposition 4.1. *In dimension 32 there is an element y_4^2 such that $\phi_2y_4^2$ does not belong to the ideal generated by ϕ_0 and ϕ_1 . Moreover $\phi_{2i}y_4^2$ does not belong to the ideal generated by ϕ_0 , ϕ_1 , ..., ϕ_{2i-1} , $i \geq 1$.*

Proof. It follows from the calculations of the symplectic cobordism ring made in [14, 15].

Acknowledgments: The first author was supported by the CRDF (Grant GM1-2083) and by the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy, where this work was completed.

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