Constraints on discrete symmetries from anomaly cancellation in compactified superstring theories

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Abstract

Compactified string theories give rise to discrete symmetries which are essential if they are to provide a realistic low energy theory. We find that in a class of four dimensional string theories these symmetries are constrained by similar conditions to those discrete anomaly cancellation conditions found in the case the discrete symmetry is a residue of a spontaneously broken gauge symmetry. Such conditions strongly constrain the allowed form of the low energy effective theory.

While four dimensional string theories appear to have all the necessary ingredients needed to give the structure of the Standard Model, progress towards the construction of realistic string theory models has been hampered by the embarrassingly large number of candidate theories. Given this, any generic property of a string theory which may be used to limit the effective low-energy theory without committing the analysis to a specific example is important.

In this letter we explore the possibility that the discrete symmetries that follow from compactified string theories are constrained by conditions similar to that were found [1] to result from anomaly cancellation when a local gauge symmetry is broken leaving a discrete symmetry factor unbroken [2]. The importance of these conditions stems from the fact that they involve only the states left light after the breaking of the gauge symmetry and thus constrain the low energy theory on its own. The power of these constraints is well illustrated by the study of discrete gauge symmetries in supersymmetric versions of the Standard Model where it was shown [3] that only two non-R symmetry discrete groups of low dimension satisfy the discrete anomaly cancellation conditions while suppressing

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nucleon decay giving just two candidate versions for the supersymmetric Standard Model, the minimal version which conserves B and L and a variant violating L but not B.

An advantage of a discrete gauge symmetry is that the constraints following from such a symmetry are not violated by gravitational corrections [4], an extremely important consideration if such constraints are needed to stabilise the nucleon for example. While in what follows we will be only able to show that some of the discrete non-R symmetries following from compactified string theories satisfy discrete anomaly cancellation conditions it is suggestive that, if they do, they too are discrete gauge symmetries and protected from large gravitational corrections. The string theories we consider here are all based on the heterotic string compactification. The main part of the paper is concerned with the 4D-construction of Gepner, although we will briefly comment on the implications for associated Calabi-Yau constructions.

The Gepner models are D-dimensional \((D < 10, \text{even})\) heterotic string models in which compactification is achieved by tensoring minimal \(N = 2\) superconformal models to form an internal sector of the theory in such a way that conformal anomaly cancellation is achieved and a \(N = 1\) space-time supersymmetric, modular invariant theory with correct spin-statistics is generated [9]. These models are characterized by their level \(k_i\) and their type according to the \(ADE\)-classification scheme [5]. They have conformal anomaly

\[
c_i = \frac{3k_i}{k_i + 2} .
\]

The states of the theory are formed from tensor products of the primary fields \(\Phi_{q_i, s_i}^{l, \pi}\) associated with each factor. The conformal dimensions and \(U(1)\)-charges of these fields are given by

\[
h = \frac{l(l + 2) - q^2}{4(k + 2)} + \frac{s^2}{8} \quad (2)
\]

\[
Q = -\frac{q}{k + 2} + \frac{s}{2} .
\]

The heterotic string theory then possesses in general a maximal gauge symmetry \(E_{D/2+4} \otimes E_8' \otimes U(1)^{r-1}\). The full set of discrete symmetries with generators in the left-moving sector may be written as a \(Z_{k_i+2} \otimes Z_2\) symmetry with charges \(\tilde{q}^I_i, \tilde{s}^I_i\) satisfying [7, 8]

\[
\sum_{J=1}^n \tilde{q}^I_i = 0 \mod k_i + 2 \quad (\forall i) \quad (4)
\]

\[
\sum_{J=1}^n \tilde{s}^I_i = 0 \mod 2 \quad (\forall i) .
\]

The right moving part of the string has a \(\tilde{Z}_{k_i+2} \otimes \hat{Z}_2\) symmetry which is generically an R symmetry with charges \(q_i, s_i\) satisfying [7, 8]

\[
\sum_{J=1}^n q^I_i + 2 = 0 \mod k_i + 2 \quad (\forall i) \quad (6)
\]

\[
\sum_{J=1}^n s^I_i + 2\sum_{J=4}^n d^I_i = 0 \mod 2 \quad (\forall i) \quad (7)
\]
and follows from the selection rules for non-vanishing correlation functions involving \( n \) fields which come from the parafermionic nature of the primary fields [6]. Here \( d_i^j \) describes the vertex operators in the \((0)\) picture. We have adopted the convention that the quantum numbers \( q, s \) refer to the fields in the \((-1)\) picture and the \( \bar{q}, \bar{s} \) refer to the appropriate choice of representation leading to a gauge invariant correlation function.

It will be important, in what follows, to identify which discrete symmetries are discrete subgroups of the original gauge symmetries. In the left moving part of the string the \( U(1) \) gauge group of each tensor factor \( i \) contains a discrete subgroup \( Z_{2(k_i+2)} \) if \( k_i + 2 \) is odd or \( Z_{k_i+2} \) if \( k_i + 2 \) is even. The charges, \( q_i^j, \) of the fields entering in a non-vanishing correlation function are constrained by these symmetries to satisfy

\[
\sum_{j=1}^{n} q_i^j \equiv \left\{ \begin{array}{ll}
\sum_{j=1}^{n} [-2q_i^j + (k_i + 2)\bar{s}_i^j] = 0 \mod 2(k_i + 2) & \text{for } k_i + 2 \text{ odd} \\
\sum_{j=1}^{n} [-q_i^j + \frac{k_i+2}{2}\bar{s}_i^j] = 0 \mod (k_i + 2) & \text{for } k_i + 2 \text{ even } (\forall i),
\end{array} \right.
\]

where \( n \) is the number of fields appearing in the correlation function [7, 8] and \( i \) refers to the \( i^{th} \) tensor factor. These symmetries are a subgroup of the symmetries of eqs.(4),(5).

For all modular invariants the conformal theory is related to a Calabi-Yau theory [10]. The diagonal sum \((q_i + \bar{q}_i)/2\) are the Gepner charges defined for the \( 10_{\text{scalar}} \) representation. The associated symmetry is that found in the Calabi-Yau construction [11]. Due to correlations between left and right sectors the discrete group may be somewhat larger than that of eqs.(4)-(7). In fact the diagonal sum and difference of eqs.(4) and (6) is a \( Z_{2(k_i+2)} \otimes Z_{2(k_i+2)} \) symmetry for \( A, D_{\text{odd}}, \) and \( E_6 \) invariants and a \( Z_{k_i+2} \otimes Z_{k_i+2} \) symmetry for \( D_{\text{even}}, E_7 \) and \( E_8 \) invariants while for (5) and (7) it is a \( Z_4 \otimes Z_4 \) symmetry and a \( Z_2 \otimes Z_2 \) symmetry for the two classes of invariants respectively [8]. However, as we will discuss, the larger symmetries are all trivially realised on the spectrum or R symmetries; the non-R symmetries of interest here are all contained in eqs.(4) and (6).

In the following let us start with a study of the \( q_i, \bar{q}_i, s, \bar{s} \) charges separately and then apply the result to linear combinations of the left and right moving charges. We consider the case that the discrete symmetries associated with these charges are not spontaneously broken although the underlying \( U(1) \)s may be. The \( \bar{q}_i \) charges automatically fulfil the discrete anomaly cancellation relations \( i \) to \( iv \) of [1] since the \( Z_{2(k_i+2)} \) or \( Z_{k_i+2} \) symmetry is a subgroup of the \( U(1) \)s. Except for an embedding into \( E_{D/2} \) and \( E'_8 \) simultaneously leading to an \( U(1) \) contained partially in one and the other, all gauge symmetries are anomaly free [12]. In the case of the anomalous \( U(1)_A \) the discrete Green-Schwarz mechanism is at work [13] implying the required cancellation through the axion field.

However (for \( k_i + 2 \) even) the \( \bar{q}, \bar{s} \) charges are associated with a larger discrete symmetry and need not satisfy the discrete anomaly cancellation conditions. Even so, we can deduce constraints on these charges for the light states from the connection of these symmetries with the \( U(1)_r \) gauge symmetries of the Gepner construction. The latter are all anomaly free and so the \((M)\) massless states of the theory satisfy anomaly cancellation conditions:

i) Mixed \( Z_N \)-gravitational conditions.

For the \( U(1) \) factors we have the usual mixed \( U(1) \) gravitational anomaly cancellation conditions
\[
\sum_{j=1}^{M} \alpha_{\bar{q}_j, \bar{z}_j} = 0 \quad \forall i \in 1, \ldots, r \, ,
\]

where
\[
\alpha_{\bar{q}, \bar{z}} = [k(k+2)]^{-\frac{1}{2}}[-\bar{q} + \frac{1}{2}(k+2)\bar{z}] .
\]

Using eq. (10) in eq. (9)
\[
\sum_{j=1}^{M} [-\bar{q}_j + \frac{1}{2}(k_i + 2)\bar{z}_i] = 0 .
\]

we immediately derive the associated mixed \(Z_N\)-gravitational conditions
\[
\sum_{j=1}^{M} \bar{q}_j = p\bar{N} + \eta_\bar{q} \frac{N}{2} \, , \, p, q \in Z ,
\]

where \(\eta_\bar{q} = 1,0\) for \(N = k_i + 2\) even, odd, following from the fact that \(\sum_{j=1}^{M} \bar{q}_j\) has to be an integer. These are the usual discrete gravitational anomaly cancellation conditions, \(i\) of [1].

Using the same arguments it is now straightforward to derive the remaining anomaly cancellation conditions for the discrete symmetries in the same basis as above:

ii) Pure discrete \(Z_N Z_M Z_L\) cancellation conditions
\[
\sum_i q_i p_i a_i = t\bar{N} + s M + r\bar{L} + \eta_\bar{q} u \frac{N}{2} + \eta_p v \frac{M}{2} + \eta_o w \frac{L}{2} \, , \, t, s, r, u, v, w \in Z \, ,
\]

where \(\eta_\bar{q}, \eta_p, \eta_o = 1,0\) for \(N, M, L\) even, odd and \(q_i, p_i, a_i\) are the discrete charges \((= \bar{q}_i)\). Note that this is weaker than the result \(ii\) of [1] in which the last three terms vanish if any one of the discrete groups is odd. We will discuss below the conditions under which the two results coincide.

iii) Mixed discrete and gauge cancellation conditions
\[
\sum_i T_\epsilon q_i = \frac{1}{2} r'' N + s'' \eta_\bar{q} \frac{N}{4} \, , \, r'', s'' \in Z \, ,
\]

where \(T(R)\) is the quadratic Casimir corresponding to each given representation \(R\) (the normalization is such that the Casimir for the \(M\)-plet of \(SU(M)\) is \(\frac{1}{2}\)). Note that this differs from the result \(iii\) of [1] due to the appearance of the last term.

iv) Mixed discrete and \(U(1)\) conditions; \(Z_N U(1)_X U(1)_Y\) and \(Z_N Z_M U(1)_X\)
\[
\sum_i x_i y_i q_i = r'''' N + s'''' \eta_\bar{q} \frac{N}{2} \, , \, r'''', s'''' \in Z \, ;
\]
\[
\sum_i x_i q_i p_i = s'' N + r'' M + \eta_\bar{q} i'''' \frac{N}{2} + \eta_p u'''' \frac{M}{2} \, , \, r'', s'', t'', u'''' \in Z \, .
\]
where \( x_i, y_i \) denote the \( U(1) \) charges. Again the last two terms are in general different from the result \( iv \) in [1].

Of course, starting with only a knowledge of the discrete charges of the light states in an effective low energy theory it is always possible to identify these charges with a larger discrete symmetry in such a way as to satisfy these mixed conditions (i.e. the charges \( x'_i = 2x_i ; q'_i = 2q_i \) will satisfy eqs.\((15)\) and \((16)\) for \( N' = 2N \)). As with the discrete gauge anomalies these mixed conditions are only useful if we restrict the size of the discrete gauge group and assume the heavy states have conventional \( U(1) \) charges.

In certain cases the differences between the conditions derived here and the discrete anomaly cancellation conditions disappear. This happens if the original \( E_6 \) gauge symmetry is not broken by nontrivial embeddings of twists. The fields that can contribute to the discrete anomaly are the matter fields transforming as singlets under \( E_6 \) or as the 27 dimensional representations. For the case of \( E_6 \) singlets or the fields that transform as the \((1 + 10)\) representations under the \( SO(10) \) subgroup of \( E_6 \) (components of the 27 dimensional representation of \( E_6 \) the \( \bar{s} \) are even. For the case of the 16 dimensional representations of \( SO(10) \) the \( \bar{s} \) are odd but the multiplicity is even. Thus in both cases the contribution to the second term of eq.\((11)\) is an integer multiple of \((k_i + 2)\). Hence we derive the discrete anomaly cancellation condition \( i \) without the \( \eta \) term. The same applies for \( ii \) to \( iv \). However in the case that the original \( E_6 \) gauge symmetry is broken by twists nontrivially embedded (see below for details), it is easy to show that there can be contributions with \( \bar{s} \) odd and odd multiplicity. In this case the new conditions derived above apply.

So far we have discussed only the anomaly cancellation conditions associated with the \( Z_{k+2} \) symmetries of the \( \bar{q} \) charges. The \( \bar{s} \) charges also fulfill eqs.\((12)-(16)\), but in this case the symmetry is \( Z_2 \) and the relations are trivial. The anomaly cancellation condition for the \( \bar{Z}_N \) symmetries associated with the right sector charges (cf. eqs.\((6)\) and \((7)\)) follow from these results by using the Gepner construction of massless states as we now discuss.

Consider first the case of untwisted models. One starts with a combination of half integer charge \( Q_{tot} \) and appropriate conformal dimension \( h \) in the right internal sector and obtains the ones in the left sector by adding \((n)\) multiples of \( \beta_0 \), and \((m)\) multiples of \( \beta_i \) [9] in such a way that the resulting states have again half integer \( \bar{Q}_{tot} \) and appropriate \( \bar{h} \). Here \( \beta_0 \) is the generator of supersymmetry and acts by adding 1 to each \( q_i \) and \( s_i \) component. The vector \( \beta_i \) only acts by adding 2 to the \( \bar{s} \) index. Thus

\[
q_i^J = \bar{q}_i^J - n , \quad s_i^J = \bar{s}_i^J - n - 2m_i \quad (\forall i) .
\]

Let us first consider those charges, \( q_i^J \), related by eq.\((17)\) to the charges \( \bar{q}_i^J \). It is convenient to define a commuting basis which projects the non-R symmetries by choosing the linear combination of the \( Z_{k+2} \) symmetries generated by \( \sum_{i=1}^{r} v_i q_i^J \), i.e. labelled by the vector \( (v_1, \cdots , v_r) \), such that

\[
\sum_{i=1}^{r} \frac{k_i}{k_i + 2} v_i^p v_i^q = Z \quad (\forall \; p, q) .
\]

This basis contains the R symmetry \( \beta_0 \). The remaining ones which commute with it are non-R symmetries. It is clear from eq.\((17)\) that under these non-R symmetries the
massless states have $q'$ charges which differ only by multiples of $2(k + 2)$ or $k + 2$, for the two cases of eq.(8), from the $\vec{q}'$. Therefore all these symmetries fulfil the discrete anomaly gauge cancellation conditions $i$ to $iv$ of ref. [1] without the additional terms of eqs.(12)-(16).

Of course, the $q'$ charges in general do not generate the complete set of symmetries in the right sector. The full set of symmetries with $q_i$ charges may be treated in a similar manner although in this case the appropriate basis needed to project the non-R symmetries is given by orthogonal combinations of the vectors $(v_1, \cdots, v_r)$ (again containing the R symmetry $\beta_0$) which satisfy

$$\sum_{i=1}^{r} \frac{q_i^p q_i^q}{k_i + 2} = Z \quad \forall p, q .$$  \hspace{1cm} (19)

Again the non-R symmetries have charges $q$ which differ only by multiples of $2(k + 2)$ or $k + 2$ from the $\vec{q}$. These satisfy the same conditions given in eqs.(12)-(16) as were found for the left sector.

In the case of twisted models the Gepner construction is slightly more complicated. Instead of eq. (17) we have

$$q_i^j = \vec{q}_i^j - n - 2pt_i , \quad s_i^j = \vec{s}_i^j - n - 2m_i ,$$  \hspace{1cm} (20)

where $t_i$ is the twistvector [9]. The twist must commute with supersymmetry (generated by $\beta_0$) and so $t_i$ must satisfy

$$\sum_{i=1}^{r} \frac{t_i}{k_i + 2} \in Z .$$  \hspace{1cm} (21)

The residual discrete symmetries must commute with the twist and hence for them the connection between the $\vec{q}$ and $q$ charges does not involve the new term proportional to $t_i$ in eq.(20). As a result the analysis of the cancellation conditions is unchanged from the untwisted case and the non-R symmetries satisfy eqs.(12)-(16).

In the case that there are twisted sectors the discrete symmetries are enlarged; the twisted states labeled by $p = 0, \cdots, (Q - 1)$ in eq.(20) transform as $\alpha^p$ under a $Z_Q$ group. It is straightforward to show that this discrete group satisfies the same cancellation conditions as the other discrete symmetries. We start with the orthogonal basis, eq.(19), of charges including $t_i$ and $\beta_i$. Taking a linear combination of the anomaly cancellation conditions of eq.(12) allows us to derive the result

$$\sum_{j=1}^{M} \Sigma_{i=1}^{r} t_i (q_i^j + 2p^j t_i)/(k_i + 2) = 0 \mod Z/2 .$$  \hspace{1cm} (22)

For a twisted model the states have to obey

$$\Sigma_{i=1}^{r} t_i (q_i + \vec{q}_i + 2pt_i)/(k_i + 2) = 0 \mod 2Z ,$$  \hspace{1cm} (23)

using the fact that $t_i$ commutes with $\beta_0$ and $\beta_i$ this implies

$$\Sigma_{i=1}^{r} t_i (q_i + pt_i)/(k_i + 2) = 0 \mod Z .$$  \hspace{1cm} (24)
Inserted in eq.(22) we now obtain

$$\sum_{j=1}^{M} \Sigma_{i=1}^{r} \frac{t_{ij}}{k+2} p' = 0 \mod Z/2. $$

(25)

Finally, if we denote by $<k+2>$ the smallest common multiple of the $(k+2)$ terms entering in eq.(25), we obtain the required cancellation condition

$$\sum_{j=1}^{M} \Sigma_{i=1}^{r} t_{ij} <k+2> p' = 0 \mod Q'/2 $$

(26)

where we have written $(k+2) <k+2> = <k+2>$ and $Q' = <k+2>$ is the discrete group factor left in the orthogonal basis commuting with supersymmetry. Eq.(26) is the mixed gravitational discrete symmetry cancellation condition eq.(12) for the new symmetry associated with twists. In a similar manner we may derive the remaining cancellation conditions of eqs.(13)-(16).

The cancellation conditions for the superconformal models have immediate implications for the symmetries found in the associated Calabi-Yau theories. For the $D_{odd}$, $E_6$ or $D_{even}$, $E_7$, $E_8$ modular invariants, the linear combinations of charges $\bar{q}_i + q_i$ generate a $Z_{2(k+2)} \otimes \tilde{Z}_{2(k+2)}$ or $Z_{(k+2)} \otimes \tilde{Z}_{(k+2)}$ group respectively. However the non-R symmetries which commute with $\beta_0$ in the right sector have charges $\bar{q} + q$ just twice the $\bar{q}$ charges while the combinations $q - \bar{q}$ vanish. Half the sum of the charges are the Gepner charges corresponding to the symmetry of the associated Calabi Yau theory and they satisfy the cancellation conditions eqs.(12) to (16).

In summary we have shown how the anomaly cancellation conditions of the underlying $U(1)$s of the Gepner construction lead to conditions on all the non-R discrete symmetries of the model, and not just those which are subgroups of the $U(1)$s. In many cases these conditions coincide with the discrete anomaly cancellation conditions which follow if the discrete symmetry is a subgroup of an underlying gauge symmetry. However we have not been able to prove these conditions in all cases (although we have not constructed any explicit counterexamples). It would certainly be interesting if the stronger conditions could be proved in general for this would strongly support the suggestion that all string symmetries are gauge symmetries.

The cancellation conditions we have found provide constraints on the structure of any effective low energy theory which can result from such four dimensional string theories and the related Calabi Yau theories. These constraints are quite non-trivial and can restrict the forms of possible symmetries capable of stabilising the proton[1] and of generating structure in the fermion mass matrix [14]. As such they represent a step forward in deriving general features following from an underlying stage of superstring compactification which may allow us to explore some of the phenomenological implications without relying on a choice of one out of the many string vacua that are now known to exist.
References


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