Phase structure of a quantized chiral soliton on $S^3$

Akizo Kobayashi and Shoji Sawada*

Faculty of Education, Niigata University, Niigata 950-21, Japan

* Department of Physics, Nagoya University, Nagoya 464-01, Japan

A quantization of a breathing motion of a rotating chiral soliton on $S^3$ is performed in terms of a family of trial functions for a profile function of the hedgehog ansatz. We determine eigenenergies of the quantized $S^3$ skyrmion by solving the Schrödinger equation of the breathing mode for several lower spin and isospin states varying the Skyrme term constants $\epsilon$. When $S^3$ radius is smaller than $2/\epsilon f_\pi$, where $f_\pi$ is the pion decay constant, we always obtain a conformal map solution as the lowest eigenenergy state. In the conformal map case, allowed states have only symmetric or anti-symmetric wave function under inversion of a dynamical variable describing the breathing mode. As the $S^3$ radius increases the energy splitting between the symmetric and anti-symmetric states rapidly decreases and two states become completely degenerate state. When the $S^3$ radius larger than $3/\epsilon f_\pi$, for the small Skyrme term constant $\epsilon$ the lowest eigenenergy states are obtained with the profile function given by an arccosine form which is almost the same to those of usual $R^3$ skyrmion. When the effects of the Skyrme term are weak, i.e. large $\epsilon$, the lowest energy states are obtained by the profile function of conformal map, which correspond to the “frozen states” for the $R^3$ skyrmion as the limit of $S^3$ radius $\to \infty$. 

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§1. Introduction

A minimal extension of the nonlinear chiral Lagrangian can admit soliton solutions called skyrmions which have been considered as baryons, and the Skyrme model has been shown to provide a reasonable phenomenology for static properties of baryons.\textsuperscript{1) }Since Klebanov applied the skyrmion to the extended system of nuclear matter with a finite baryon density, a variety of numerical studies of the periodic arrays of the cubic lattice skyrmion in the flat space \( \mathbb{R}^3 \) have been made.\textsuperscript{2)\textendash 5) These approaches, however, are technically so complicated and require tedious numerical works due to the huge number of dense mesh points. The other approach is initiated by Manton where the flat space \( \mathbb{R}^3 \) is replaced by the 3-sphere.\textsuperscript{3), 4) }In this approach a single baryon on a 3-sphere \( S^3(\rho) \) with radius \( \rho \) is just an approximation of baryonic matter with an average baryon density of \( 1/2\pi^2\rho^3 \) and provides very good approximation of the generic features including the phase transition of the \( R^3 \) lattice skyrmion approaches.\textsuperscript{5) }

The most striking fact that is found from these studies is existence of two phases. In the \( S^3 \) skyrmion case, one is a phase where both baryon number and energy densities are well localized around the north or south pole of \( S^3 \) corresponding to the low baryon density matter. The other is a phase where these densities are much more homogeneously distributed and corresponds to the high baryon density matter. It is expected that transitions between the localized states and the homogeneous states will be closely related to the breathing motion of skyrmion. So far, as we know, no studies of the \( S^3 \) skyrmion on quantum level including breathing mode have been made. Recently, it has been shown by Yang and one of the author\textsuperscript{6) }that for a certain region of the Skyrme term constant the generic features of the \( R^3 \) skyrmion on quantum level is drastically changed from those on classical level and that the rotational and breathing modes are completely frozen. Therefore, it is very interesting to study whether quantum effects of the breathing motion may give rise to fundamental changes in energy levels and phase structures of the \( S^3 \) skyrmion. These generic features including the phase structure seem to show close relations and striking similarities to the baryonic matter in the flat
space $R^3$ as known on classical level.\(^5\)

In this paper we investigate the phase structure of a quantized chiral soliton on $S^3$, where the effects of the quantization of breathing and rotational modes are taken into account on a basis of a family of trial functions for the profile function of the hedgehog ansatz.\(^6\),\(^7\) This approach is also a natural extension of the analysis in Ref. 6 for the $R^3$ skyrmion to those for $S^3$ skyrmion system.

This paper is organized as follows. In Sec. II the standard skyrmion on $S^3$ is quantized by taking account of the breathing mode in addition to the spin-isospin rotation. In Sec. III on the basis of a family of trial functions for the profile functions of the $S^3$ skyrmion we obtain explicit expressions of the Lagrangian and the Schrödinger equation for the spin-isospin rotation and the breathing motion. In Sec. IV we investigate the effects of quantization of the breathing mode by solving numerically the Schrödinger equation for the $I = J = 0$ state of the $S^3$ skyrmion. In Sec. V we study the phase structure of the quantized $S^3$ skyrmion obtained by solving the Schrödinger equation for spin-isospin states $I = J = 1/2, 3/2$ and $5/2$. The final section is devoted to discussion and conclusions.

\section*{2. Breathing Motion of $S^3$ Skyrmion}

In the standard Skyrme model\(^3\),\(^4\) on $S^3$, the Lagrangian of a chiral $SU(2)_L \times SU(2)_R$ nonlinear $\sigma$ model is given in the massless pion limit as

\begin{align}
\mathcal{L} &= \mathcal{L}_A + \mathcal{L}_{SK}, \\
\mathcal{L}_A &= \frac{1}{4} f_\pi^2 \text{Tr}(g^{\mu \nu} \partial_\mu U(x) \partial_\nu U^\dagger(x)), \\
\mathcal{L}_{SK} &= \frac{1}{32e^2} \text{Tr}(g^{\mu \nu} g^{\kappa \lambda}[U^\dagger(x) \partial_\mu U(x), U^\dagger(x) \partial_\kappa U(x)][U^\dagger(x) \partial_\nu U(x), U^\dagger(x) \partial_\lambda U(x)]),
\end{align}

where $U(x) = \exp(2i\pi(x)/f_\pi)$ is the element of $SU(2)$ group, $\pi(x) = \pi^a(x)\tau^a/2$ describes the pion field, $f_\pi = 93\text{MeV}$ is the pion decay constant. The $\mathcal{L}_{SK}$ term, called
Skyrme term,\(^1\) is introduced to stabilize the soliton solutions where \(\epsilon\) is a free parameter. Here \(g^{\mu\nu}\) is the inverse of metric tensor \(g_{\mu\nu}\) in a curved space where \(g_{0\mu} = \delta_{0\mu}\). Using the standard spherical coordinate \(i = (\omega, \theta, \phi)\) on \(S^3(\rho)\), the metric tensor \(g_{ij}\) on the 3-sphere of radius \(\rho\) are given as

\[
g_{ij} = -\delta_{ij} g_i, \quad g_\omega = \rho^2, \quad g_\theta = \rho^2 \sin^2 \omega, \quad g_\phi = \rho^2 \sin^2 \omega \sin^2 \theta. \tag{2}
\]

The hedgehog ansatz for the classical static \(S^3\) skyrmion\(^1,3\) is given by

\[
U_0(\vec{x}) = \exp[i \hat{x} \cdot \vec{r} F(\omega)]
\]

\[
\hat{x} = \hat{e}_x \sin \theta \cos \phi + \hat{e}_y \sin \theta \sin \phi + \hat{e}_z \cos \theta, \tag{3}
\]

where \(F(0) = \pi\) and \(F(\pi) = 0\) for the baryon number \(B = 1\) sector. Substitution of \(U_0\) into \(U\) of (1) as

\[
M_0 = -\int \mathcal{L}(U_0(\vec{x})) \ dV,
\]

\[
dV = \sqrt{g} d^3 x = \rho^3 \sin^2 \omega d\omega d\phi \sin \theta d\theta, \quad g = \det g_{ij}, \tag{4}
\]

leads to the static energy functional;\(^3,4\)

\[
M_0[F] = 2\pi \rho \int_0^\pi d\omega \int_0^\pi \sin \theta d\theta [\mathcal{M}_A + \mathcal{M}_{SK}],
\]

\[
\mathcal{M}_A = \frac{f_0^2}{2} (F' \sin^2 \omega + 2 \sin^2 \omega), \quad \mathcal{M}_{SK} = \frac{1}{2c^2 \rho^2} \left\{ \frac{\sin^4 F}{\sin^2 \omega} + 2(F' \sin F)^2 \right\}, \tag{5}
\]

where \(F(\omega)\) and \(F' = dF(\omega)/d\omega\). The classical profile function \(F_{el}\) minimizes the functional \(M_0[F]\).

The quantization of the rotational and breathing modes is performed by the following replacement in the Lagrangian (1);\(^8\)

\[
U(\vec{x}, t) = A(t) \exp[i \hat{x} \cdot \vec{r} F(\omega, \chi(t))] A(t)^\dagger, \tag{6}
\]

\[\]

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where $A(t) \in SU(2)$ and $\chi(t)$ are the dynamical variables of the spin-isospin rotation and the breathing motion respectively. It is convenient to express the breathing motion as $F(\omega, \chi(t)) = F(\mu)$ by use of a dimensionless variable $\mu = \tan(\omega/2)/\chi(t)$ which assures invariant boundaries of the range of variable $\mu$ for any $\chi(t)$ as $\mu = 0$ at $\omega = 0$ and $\mu = \infty$ at $\omega = \pi$. If the boundary values of a given trial profile function $F(\mu)$ are taken as $F(0) = \pi$ and $F(\infty) = 0$, therefore, the breathing profile function $F(\omega, \chi(t))$ satisfies automatically the boundary conditions for the baryon number $B = 1$ sector on $S^3$, i.e. $F(\omega = 0, \chi(t)) = \pi$ and $F(\omega = \pi, \chi(t)) = 0$. It is worth noting that for the limit of the 3-sphere radius $\rho \rightarrow \infty$, $\mu$ is just equivalent to the dimensionless variable $r/R(t)$ of the flat spaces $R^3$, where $R(t)$ is regarded as a dynamical variable for the breathing motion of the $R^3$ skyrmion. For the simultaneous limit $\rho \rightarrow \infty$ and $\omega \rightarrow 0$, a radial distance $r$ between a point $(\omega, \theta, \phi)$ on $S^3$ and the origin of $R^3$ is approximated to $2\rho \tan(\omega/2)$ which is just the radial distance between the point $(\omega, \theta, \phi)$ and the north pole $(\omega = 0)$ of $S^3$. Thus, if we identify the dynamical variable $\chi(t)$ with $R(t)/(2\rho)$, where $R(t)$ is a dynamical variable of the breathing motion in the flat space $R^3$. The dimensionless variable $\mu = \tan(\omega/2)/\chi(t)$ behaves as

$$\lim_{\omega \rightarrow 0} \mu = \lim_{\omega \rightarrow 0} \frac{2\rho \tan(\omega/2)}{R(t)} = \frac{r}{R(t)},$$

(7)

Now the profile function $F(\mu) = F(r/R(t))$ describes the breathing motion of a localized skyrmion at the origin of $R^3$. These are the reason why we use the profile function $F(\mu)$ with $\mu = \tan(\omega/2)/\chi(t)$.

By substituting (6) to (1) we obtain the Lagrangian for the spin-isospin rotation and the breathing motion of the chiral solitons given by

$$L = a(\chi)\dot{\chi}^2 + \lambda(\chi)\text{Tr}[\dot{A}\dot{A}^\dagger] - V(\chi).$$

(8)

Here $a(\chi)$ and $\lambda(\chi)$ are the $\chi(t)$ dependent inertia of the breathing motion and the moment of inertia of the spin-isospin rotation, respectively, and are given by

$$a(\chi) = 2\pi \int_0^\pi d\omega \rho \left[ f^2 \rho^2 \sin^2 \omega + \frac{2}{e^2} \sin^2 F \right] \left( \frac{dF}{d\chi} \right)^2,$$

5
\[ \lambda(\chi) = \frac{8\pi f^2}{3} \int_0^{\pi} d\omega \rho^3 \sin^2 \omega \left[ 1 + \frac{1}{e^2 \rho^2} \left( F'' + \frac{\sin^2 F}{\sin^2 \omega} \right) \right] \sin^2 F, \]  

where \( F = F(\omega, \chi(t)) \) and \( F' = dF(\omega, \chi(t))/d\omega \). It is noted that \( dF/d\chi \) is rewritten as \(- (\mu/\chi) F(\mu)/d\mu \) by use of an expression \( F(\omega, \chi(t)) = F(\mu) \) with \( \mu = \tan(\omega/2)/\chi(t) \).

The \( V(\chi) = M_0[F] \) is the potential of the breathing motion obtained by replacement of \( F(\omega) \) by \( F(\omega, \chi(t)) \) in (5).

According to the standard procedure of the quantization based on the Lagrangian (8), after separation of the rotational mode, we obtain the Schrödinger equation for the wave function of the breathing mode, \( \psi_J(\chi) \) for a \( I = J \) spin-isospin eigenstate: \(^{6,8} \)

\[ \left[ -\frac{1}{4a(\chi)} \left( \frac{d^2}{d\chi^2} + h(\chi) \frac{d}{d\chi} \right) + V(\chi) + \frac{J(J + 1)}{2\lambda(\chi)} - E_J \right] \psi_J(\chi) = 0, \]

where \( E_J \) is an energy eigenvalue and

\[ h(\chi) = \frac{3\lambda'(\chi)}{2\lambda(\chi)} - \frac{\alpha'(\chi)}{2\alpha(\chi)}, \]

and prime means differentiation with respect to \( \chi \).

§3. Trial Profile Functions and Quantized \( S^3 \) Skyrmion

In order to obtain the quantized skyrmion solutions on \( S^3 \), we have to solve the Schrödinger equation of the breathing motion (10) for spin-isospin eigenstates varying the profile function \( F(\mu) \) which gives the most minimum eigenvalue \( E_J \). To investigate for the wide range of variation of the profile function we adopt the variational method in terms of a family of trial functions \( F(\mu; C) \) with a parameter \( C \) \( (C \geq 0) \), which is expressed by an integral form as \(^7\)

\[ F(\mu; C) = \frac{\pi}{N} \int_{\mu}^{\infty} dx \frac{dx}{(x^2 + 1)\sqrt{x^2 + C}}, \]
where

\[ N = \int_0^\infty \frac{dx}{(x^2 + 1)\sqrt{x^2 + C}}. \]

The trial profile functions \( F(\mu; C) \) satisfy the boundary condition \( F(0; C) = \pi \) corresponding to a soliton of the baryon number 1 and the asymptotic condition for finite energy \( F(\mu; C) \rightarrow 0 \) for \( \mu \rightarrow \infty \). \( F(\mu; C) \) with finite \( C \) has the asymptotic falloff \( \propto 1/\mu^2 \) for large \( \mu \) which is required by the Euler-Lagrange equation for the profile function in the chiral limit regardless of the stabilizer. This asymptotic falloff also agrees with that of the static meson theory in the massless pion limit.

In the usual \( R^3 \) skyrmion the function with \( C = 1 \), \( F(\mu; C = 1) \), is just the same as the one derived by Atiyah and Manton\(^9\) from the instanton solution and it is the nearest trial function to the classical numerical solution \( F_{cl}(\mu) \) among the family \( F(\mu; C) \). It gives a static energy only 0.9\% above that of numerical solution.\(^6\) The case with \( C = 2 \) is a trial function of arccosine form introduced by Igarashi, Otsu and two of the present authors,\(^10\) which gives also a very good approximation of the exact static profile function \( F_{cl}(\mu) \) and gives a static energy only 1.0\% above that of the numerical solution.

By use of the trial profile function (11) we obtain the inertias \( A(\chi) \) and \( \lambda(\chi) \) from (9) and the potential \( V(\chi) = M_0[F] \) from (5). These inertias and potential are dependent of the parameter \( C \). Varying this parameter \( C \) we solve the Schrödinger equations (10) for the breathing motion with definite spin and isospin \( I = J \).

In the limit of the \( S^3 \) radius \( \rho \rightarrow \infty \) the \( S^3 \) skyrmion tends to a usual \( R^3 \) skyrmion. In this case, when the Skyrme term contribution is small (i.e. when the Skyrme term constant \( \epsilon > 5.0 \)), there appear the “frozen states” in which all the freedoms of the spin-isospin rotation and the breathing motion are completely frozen. This is because the state with infinite inertias of the breathing motion and with infinite moment of inertia of spin-isospin rotation, so that the contributions from the kinetic energies of both modes become zero and the \( R^3 \) skyrmion energy comes only from the potential
energy of the breating mode. The profile function $F(\mu)$ in the frozen state is given by $F(\mu; C = \infty)$ which gives the most minimum eigenenergy $E_{\infty}$ in the region $\epsilon > 5.0$. In this region of $\epsilon$ the classical profile function $F_{cl}(\mu)$ or the trial function $F(\mu; C = 1)$ gives local maximum eigenenergies.

On the other hand when the Skyrme term constant $\epsilon$ is small the frozen state does not appear for the small spin-isospin states and the classical profile function which is approximated by $F(\mu; C = 1)$ gives most minimum eigenenergies.

We confirmed numerically that in the case of $S^3$ skyrmion a similar situation occurs as in the $R^3$ skyrmion. The eigenenergies of $S^3$ skyrmion with the definite spin-isospin $I = J$ become minimum at $C = 1$ or at $C = \infty$ depending on the parameter $\epsilon$. In the following, however, we calculate numerically for two special values of parameter $C$, i.e. $C = 2$ and $C = \infty$. The value $C = 2$ is a substitute of $C = 1$. The reason for solving with $C = 2$ instead of $C = 1$ lies on the fact that we can obtain analytically integrated results of the potential $V(\chi)$ and inertias $a(\chi)$ and $\lambda(\chi)$ for $C = 2$ as well as $C = \infty$ but not for $C = 1$. The difference between the eigenenergies for $C = 2$ and $C = 1$ is expected within $1\%$ as in the case of the $R^3$ skyrmion.

(i) Breathing motion, inertias and potential for $C = \infty$

The trial profile function for $C = \infty$

$$F(\mu; C = \infty) = \pi \left(1 - \frac{2}{\pi} \arctan \mu \right),$$

(12)

is rewritten as

$$\tan \frac{\omega}{2} = \cot \frac{F(\mu; C = \infty)}{2} = \mu.$$  

(12')

which is known as a conformal map.\textsuperscript{3,4)}

Now we investigate the breathing motions of the profile function and baryon density given by the conformal map:

$$F(\omega, \chi(t)) = 2 \arccot \mu.$$  

(12'')
In Fig. 1(a) we show the breathing profile functions for various values of \( \chi(t) \), i.e. \( \chi = 1/5, 1/3, 1, 3 \) and 5. The solid line represents the conformal map profile functions. The diagonal line represents the identity map \( F(\omega, 1) = \omega \). It is noted that the peak of the profiles \( F(\omega, \chi(t)) \) at \( \omega = 0 \) shrinks or spreads out as the values \( \chi(t) \) decreasing or increasing from 1.

The baryon density for the \( \chi(t) \) at the radial angle \( \omega \) on \( S^3 \) is defined as

\[
2\pi^2 j^0(\omega)\rho^3 = \frac{\sin^2 F(\omega, \chi(t))}{\sin^2 \omega} \frac{dF(\omega, \chi(t))}{d\omega}
\]  

(13)

where \( j^0(\omega) \) is the baryon number density. The solid line in Fig. 1(b) represents the baryon number density given by the conformal map for various \( \chi(t) \) values. The distributions of the baryon number density for the conformal map profile function have a exact symmetry. It is easy to prove by using the explicit form for \( F(\omega, \chi(t)) \) of (12") that the effect of an inversion \( \chi \to 1/\chi \) in (13) is equal to the one of a transformation \( \omega \to \pi - \omega \) by which the north semisphere of the \( S^3 \) is transformed on to the south semisphere. Thus, the peaked shapes of the baryon density around the north pole for \( \chi = 1/2 \) and 1/3 are exactly mirror symmetric to those of the corresponding one around the south pole for \( \chi = 2 \) and 3, respectively. The identity map \( F(\omega, 1) = \omega \) gives completely flat baryon density over the whole 3-sphere as shown by the solid line of \( \chi = 1 \) in Fig. 1(b), i.e. \( 2\pi^2 j^0(\omega)\rho^3 = 1 \). Thus, the breathing motion expressed by the dynamical variable \( \chi(t) \) means that distributions of the baryon density flip-flops between a higher peak at the north pole (e.g. \( \chi = 1/2 \)) through completely flat one (\( \chi = 1 \)) and to higher one at the south pole (\( \chi = 2 \)), and vice versa, as shown in Fig. 1(b).

Here, by substituting (12) to (9), we obtain integrated expressions of the inertias \( a(\chi) \) and \( \lambda(\chi) \) and the potential \( V(\chi) \) as follows,

\[
a(\chi) = \frac{4\pi^2}{f_\pi e^3} \frac{\Lambda(\chi)}{\chi^2},
\]

\[
\lambda(\chi) = \frac{16\pi^2}{3f_\pi e^3} \Lambda(\chi),
\]
\[
V(\chi) = \frac{6\pi^2 f_\pi}{\epsilon} \left( \frac{2L}{\chi + \frac{1}{\chi} + 2} + \frac{1}{4L} (\chi + \frac{1}{\chi}) \right),
\]

\[
\lambda(\chi) = \frac{1}{(\chi + \frac{1}{\chi} + 2)^2} (3L^3 + L(\chi + \frac{1}{\chi} + 4)).
\]

where \( L = \epsilon f_\pi \rho \) is the dimensionless \( S^3 \) radius and will be use instead of the radius \( \rho \) of the 3-sphere \( S^3 \) in the following.

We notice that under an inversion;

\[
\chi \rightarrow \frac{1}{\chi},
\]

there exists a symmetry as follows;

\[
V(\chi) = V\left(\frac{1}{\chi}\right),
\]

\[
\lambda(\chi) = \lambda\left(\frac{1}{\chi}\right),
\]

\[
a(\chi) \left( \frac{d\chi}{dt} \right)^2 = a\left(\frac{1}{\chi}\right) \left( \frac{d}{dt} \frac{1}{\chi} \right)^2.
\]

Therefore for the conformal map profile function (12), the Lagrangian (8) and the Schrödinger equation (10) are invariant under the inversion (15). Hereafter, we call this invariance as the inversion symmetry. In Fig. 2(a), the potential \( V(\chi) \) of the conformal map case (12) is shown for varying the radius \( L \). The potential exhibits inversion symmetry. In the region \( L \leq \sqrt{2} \) the potential \( V(\chi) \) has single valley symmetric for the variable \( \ln \chi \) and the potential \( V(\chi) \) takes the absolute minimum value at \( \chi = 1 \) for fixed value of \( L \). This conformal map with \( \chi = 1 \) called identity map realizes a homogeneous distribution of the baryon number density over the whole 3-sphere as shown in Fig. 1(b). In these \( L \) region, the minimum of potential for the conformal map profile function is given by

\[
V_{\text{min}}(\text{conformal}) = \frac{3\pi^2 f_\pi}{\epsilon}(L + \frac{1}{L}), \quad L \leq \sqrt{2},
\]

It is noted that this really gives the energy of exact solution, i.e. the static classical energy \( E_{\text{cl}} \), as shown in Ref. 3 and 4. Here, it is also worth noting that the theoretical
lower bound energy $6\pi^2 f_\pi/\epsilon$ is realized at $\chi = 1$ and $L = 1, 3, 4$ i.e. the absolute minimum of the potential valley in Fig. 2(a).

In the region $L \geq \sqrt{2}$, the potential $V(\chi)$ behaves as the double well potential of the variable $\ln \chi$ for a fixed $L$ and has local maxima at $\chi = 1$ and two local minima at $\chi = \chi_{\text{im}} = \sqrt{2}L - 1 - \sqrt{2(L^2 - \sqrt{2}L)} \leq 1$ and at $\chi = 1/\chi_{\text{im}} = \sqrt{2}L - 1 + \sqrt{2(L^2 - \sqrt{2}L)} \geq 1$. The minimum value of potential energy is given as a function of $L$ by $^{3, 4}$

$$V_{\text{min}}(\text{conformal}) = \frac{3\pi^2 f_\pi}{\epsilon} (2\sqrt{2} - \frac{1}{L}), \quad L \geq \sqrt{2}. \quad (18)$$

This minimum value is considerably larger than those of the exact solution$^{3, 4}$, which is approximated well by the solution for the arccosine profile function with $C = 2$ given in the following.

(ii) Breathing motion, inertias and potential for $C = 2$

The trial profile function for $C = 2$ in (11) is written by arccosine form;

$$F(\mu; C = 2) = \arccos \left(1 - \frac{2}{(1 + \mu^2)^2}\right). \quad (19)$$

The dotted line in Fig. 1(a) show the breathing profile functions of the arccosine form for values of $\chi(t) = 1/5, 1/3, 1, 3$ and 5. The dotted lines of Fig. 1(b) display the corresponding baryon number densities. The distributions of the baryon number density has no symmetry under the transformation $\chi \to 1/\chi$ and $\omega \to \pi - \omega$ in contrast to those for the conformal map profile function. The baryon density has a peak at the north pole for $\chi = 1/3, 1/2$ and 1, but has a bump and dip around the south pole for $\chi = 3$ and 2 ($\geq 1.6$) as shown by dashed line in Fig. 1(b).

Now, we obtain analytically integrated expressions by substituting (19) to (9) for the inertias $a(\chi)$ and $\lambda(\chi)$ and potential $V(\chi)$ as follows:

$$a(\chi) = \frac{2\pi^2}{f_\pi \epsilon^3} (L^3 A_1(\chi) + LB_1(\chi)),
$$

$$\lambda(\chi) = \frac{8\pi^2}{3f_\pi \epsilon^3} (L^3 A_2(\chi) + LB_2(\chi)), \quad (20)
$$

$$V(\chi) = \frac{2\pi^2 f_\pi}{\epsilon} (LA_3(\chi) + \frac{B_3(\chi)}{L}),$$
where

\[
A_1(\chi) = \frac{8\chi(32 - 20\sqrt{2} + (8 + \sqrt{2})\chi + 6\chi^2)}{\sqrt{2}(1 + \chi)^4(1 + \sqrt{2}\chi)^3},
\]

\[
B_1(\chi) = \frac{3 + 18\chi + 42\chi^2 + 42\chi^3 + 7\chi^4}{2\chi(1 + \chi)^6},
\]

\[
A_2(\chi) = \frac{\chi^3(7 + 12\chi)}{(1 + \chi)^6},
\]

\[
B_2(\chi) = \frac{\chi(157 + 1256\chi^2 + 4251\chi^3 + 7632\chi^4 + 7243\chi^5 + 3176\chi^6)}{128(1 + \chi)^8},
\]

\[
A_3(\chi) = \frac{8\chi(3 - 2\sqrt{2} + (2 - \sqrt{2})\chi)}{(1 + \chi)^2(1 + \sqrt{2}\chi)} + \frac{\chi(3 + 12\chi + 11\chi^2)}{2(1 + \chi)^4},
\]

\[
B_3(\chi) = \frac{23(11\chi + \frac{27}{\chi})}{512}. 
\]

In this arccosine profile function case, there exists no special symmetry in contrast with the conformal map case. As shown in Fig. 2(b), in the region \(L \leq \sqrt{2}\) the potential \(V(\chi)\) has also a single well but with no special symmetries and has a minimum at \(\chi \sim 1.6\). The minimum of the potential, however, is considerably larger than that of the conformal solution given by (17) in this small \(L\) region. This means that in classical level, the stable soliton solution for \(L \leq \sqrt{2}\) is given by the profile function of the identity map (the conformal map with \(\chi = 1\)).

In the region \(L \geq \sqrt{2}\), the \(V(\chi)\) behaves as a double well potential which has a local maximum at \(\chi \sim 1.6\) (saddle point) and two local minima with two different depths. The smaller one of these two minima exists in the region of \(\chi < 1.6\). This minimum value given by the arccosine profile function (19) gives a very good approximation of the exact solution within about 1% deviation. Thus, for \(L \geq \sqrt{2}\) the exact classical solution given by the \(F_{cl}\) is approximated well by the arccosine profile function \(F(\mu; C = 2)\).
§4. Effects of Quantization of Breathing mode

In order to examine the effects of quantization of the breathing mode we study here the solutions of the Schrödinger equation for the breathing motion of the \(I = J = 0\) state.

In Fig. 3 we display the arbitrarily normalized wave functions \(\psi_{J=0}(\chi)\) in a region \(0 \leq \chi \leq 1\) as a typical example of the solutions of conformal map with \(C = \infty\). The solid line and the dashed line represent the symmetric state and anti-symmetric state, respectively, for \(L = 1.3\), at which the eigenenergy of the symmetric state takes the lowest value among all \(L\) region. The value is \(E_{I=J=0} = 870\text{MeV}\) for \(\epsilon = 7\). The potentials \(V(\chi)\) for \(L = 1.3\) and \(L = 10\) with \(\epsilon = 7\) also shown by dotted lines in Fig. 3. The potentials and the wave functions in the region \(\chi \geq 1\) are given from those in \(0 \leq \chi \leq 1\) by \(V(1/\chi)\) and \(\pm \psi_{J=0}(1/\chi)\), respectively. For \(L = 10\) we get degenerate states as shown by dot-dashed line in Fig. 3. The absolute values of wave functions have sharp peaks at \(\chi = \chi_{lm} \sim 0.03\) and \(\chi = 1/\chi_{lm} \sim 33\) where the potential \(V(\chi)\) takes the minimum value as shown in Fig. 3 for \(0 \leq \chi \leq 1\). The probability distributions to find the \(\chi\) is so localized at \(\chi \simeq \chi_{lm}\) and \(\chi \simeq 1/\chi_{lm}\), therefore, the quantum tunneling effect between these two regions is considered to be negligible.

In Fig. 4, we show calculated results of eigenenergy \(E_{I=J=0}\) as a function of \(L\) for several values of the Skyrme term constant \(\epsilon\). Solid and dotted lines represent those of the symmetric and anti-symmetric solutions, respectively, for \(C = \infty\) (conformal map). Two lines of eigenenergies are completely degenerate for large \(L\)-region. Dot-dashed lines represent eigenvalues \(E_{J=I=0}\) for \(C = 2\) (arccosine form). For \(\epsilon = 7\), we display also in Fig. 3 by dashed lines the minimum values of the potential \(V_{\text{min}}(\text{conformal})\) given by (17) and (18) for the conformal map profile function with \(\epsilon = 7\). Therefore, the difference between the solid line and the dashed line represents contribution from the kinetic energy of breathing motion.

From these results for \(I = J = 0\) the effects of quantization of the breathing mode
are summarized as the followings:

(i) In classical level, there exist two phases.\(^3\) One is the phase where both baryon number and energy densities are well localized at the north or south pole of \(S^3\) in the region \(L > \sqrt{3}\), which corresponds to the \(R^3\) skyrmion with usual profile function numerically solved. The other is the phase where these densities are homogeneously distributed in the the region \(L < \sqrt{3}\). In this case the profile function is given by that of the identity map. In the quantum level, the quantum breathing motion in terms of the profile function for conformal map on \(S^3\) means quantum flip-flop between a localized distribution at north pole \((\chi < 1)\) and a localized one at the south pole \((\chi > 1)\) through a uniform distribution (identity map with \(\chi = 1\)).

(ii) As shown in the previous section and also in Fig. 2(a), the quantum breathing motion is described by a single-well potential in the region \(L \leq \sqrt{3}\), but it is described by a double-well potential in the region \(L \geq \sqrt{3}\). In the conformal map case the height of the potential barrier at \(\chi = 1\) from the bottom of potential increases from 0 to \(\infty\) as \(L\) increases from \(\sqrt{3}\) to \(\infty\). Thus, the ground state wave function changes smoothly from the one which has a peak at \(\chi = 1\) to the one with double peaks at around potential minimum points \(\chi_{\text{im}}\) and \(1/\chi_{\text{im}}\) as \(L\) increasing from 0 to \(\infty\).

(iii) The one of new features of quantization of the breathing mode is the appearance of pairs of eigenstates; symmetric and anti-symmetric under the inversion \((15)\) for the conformal map profile function. In the region of small \(L\) less than \(\sim 2\) the splitting between these two energy levels is rather large due to the lack of energy barrier between small and large \(\chi\) regions. On the other hand, in the region of large \(L\) larger than \(\sim 3\) this pair of states has an almost degenerate energy levels. This means the potential barrier around \(\chi = 1\) become high enough to forbid the penetration between the small and large regions of \(\chi\). As for the arccosine case, the potential \(V(\chi)\) shows a single-well or double-well behaviors depending on the sphere size \(L\) but does not show any symmetry such as given by \((15)\). Therefore, the eigenenergies for the arccosine case
have no degeneracy. In Fig. 4 only lower energy solutions are shown.

(iv) For small Skyrme term constant $\epsilon$ and in the large $L$ region the arc cosine profile function gives lower energy than the conformal map case. For $\epsilon = 3, 4$ and 5 the arc cosine case gives lower eigenenergy solution in a region $L > 3$. For $\epsilon = 7$ the energy of the arc cosine case is slightly lower than the one of the conformal map case in a region $3 < L < 20$, but the conformal map case gives the lowest energy solution for $L > 20$.

For the limit $L \to \infty$ we get the same results as Ref. 6 on $R^3$, i.e. for values of $\epsilon$ larger than a critical value $\epsilon_0 = 5.6$ all minimum energy values of spin-isospin $J$ states, $E_J$, degenerate to a value $\lim_{L \to \infty} E_{C=\infty}(\epsilon) = 7.788/\epsilon$ GeV of the conformal map case.

(v) As for the classical $S^3$ skyrmion, the lowest energy is always realized by the radius $L = 1$ for any $\epsilon$. For the quantized $S^3$ skyrmion, however, the radius $L$ giving the lowest energy $E_0^{min}$ increases slightly as $\epsilon$ increase. That is, the lowest energy is $E_0^{min} = 1.902$ GeV at $L = 1.1$ for $\epsilon = 3$, $E_0^{min} = 1.456$ GeV at $L = 1.1$ for $\epsilon = 4$, $E_0^{min} = 1.185$ GeV at $L = 1.2$ for $\epsilon = 5$ and $E_0^{min} = 0.870$ GeV at $L = 1.3$ for $\epsilon = 7$, respectively.

§5. Phase Structure of $S^3$ Skyrmion

Now we discuss the quantized $S^3$ skyrmion obtained by solving the Schrödinger equation with $I = J = 1/2, 3/2$ and $5/2$. In Fig. 5(a) ~ (d) we show the $L$-dependence of eigenvalues $E_J$ obtained for $J = 1/2, 3/2$ and $5/2$ and for the Skyrme term constant $\epsilon =3, 4, 5$ and 7. These eigenenergies of the symmetric and anti-symmetric states for conformal map profile function are represented respectively by the solid and dotted lines and almost degenerate in the large $L$ region. The dot-dashed lines in Fig. 5 represent the lowest eigenenergies for the arc cosine profile function.

For $\epsilon = 3$, as shown in Fig. 5(a), the arc cosine profile function realizes the lowest energy solutions for all spin-isospin states $I = J = 1/2, 3/2$ and $5/2$ in the region $L \geq 2$. 
In the region \( L \leq 2 \) the lowest energies are obtained by the conformal map profile function for all spin-isospin states. Then a phase transition occurs at \( L \simeq 2 \).

Fig. 5(b) exhibits the \( L \)-dependence of eigenenergies for \( \epsilon = 4 \). In this case, for spin-isospin states \( I = J = 1/2 \) and \( 3/2 \), there occurs a phase transition at \( L \simeq 2.5 \) i.e. the conformal and the arccosine profile functions give lower energies in the region \( L < 2.5 \) and \( L > 2.5 \), respectively. On the other hand, for \( I = J = 5/2 \) the conformal map profile function always gives the lowest energy.

For \( \epsilon = 5 \), as shown in Fig. 5(c), in a region \( L > 2.5 \), the arccosine profile function gives the lowest energy solution for \( I = J = 1/2 \) but the conformal map profile function gives the lowest one for \( I = 1/2 \) in the region \( L < 2.5 \) and for \( I = J = 3/2 \) and \( 5/2 \) in all region of \( L > 0 \). Then the phase transition occurs at \( L \simeq 2.5 \) only for \( J = 1/2 \), but not for \( J = 3/2 \) and \( 5/2 \).

Fig. 5(d) shows for the case \( \epsilon = 7 \). For all spin-isospin states with \( J = 1/2, 3/2 \) and \( 5/2 \), the conformal map profile function always gives the lowest energy solutions in whole \( L \) region.

We have considered hitherto the case where the radius of \( S^3 \) is fixed \textit{a priori} by some condition given from outside. Now we turn to a rather academic or fictitious problem where the isolated \( S^3 \) skyrmion is putting on the vacuum and its radius \( \rho \) or \( L \) is determined by itself. In this case we have to regard the sphere radius \( L \) as a dynamical variable and to find minimum eigenenergies by solving Shrödinger equation for both \( L \) and \( \chi \). If we assume that the radius \( L \) is an adiabatically variable of the \( S^3 \) skyrmion, then we have to search the lowest eigenenergies varying the radius \( L \). In Table I we list the local minimum energies \( E_J \) in the small \( L \) region of symmetric states and the values of \( L \) which give the minimum energies for \( \epsilon = 3, 4, 5 \) and 7. Fig. 6 shows \( \epsilon \)-dependence of these local minimum eigenenergies \( E_J \) for \( J = 0, 1/2, 3/2 \) and \( 5/2 \). As shown by solid lines of Fig. 5(a) \( \sim \) 5(d), in the limit of \( L = \infty \) the eigenenergies of the any spin-isospin states with conformal profile function converge to the same energy \( E_\infty = 7.788/\epsilon \) GeV.
which is the energy of the frozen state of the $R^3$ skyrmion. The classical $S^3$ skyrmion energy $E_{cl} = 6\pi^2 f_\pi/e$ and $E_\infty$ are also shown by dashed and dot-dashed lines. If the $S^3$ skyrmion with definite spin-isospin which is represented by the solid lines in Fig. 6 above the dot-dashed line of $E_\infty$, the radius of $S^3$ skyrmion increase infinitly and the skyrmion given by the conformal profile functions disperses. The energy of $I = J = 1/2$ state is always smaller than $E_\infty$ in the region $\epsilon$. For $\epsilon > 5$ the $S^3$ skyrmion with $I = J = 3/2$ disperses while for $\epsilon < 5$ the $I = J = 3/2$ skyrmion given by the conformal map profile function has most minimum energy. The $I = J = 5/2$ $S^3$ skymion of conformal map becomes most minimum energy state for $\epsilon < 4.1$. Occurrence of infinite dispersion of the quantized $S^3$ skyrmion is recognized by the $\epsilon$ and $L$ dependence of the kinetic energies of the rotational and breathing modes and the potential of breathing mode. As given in (14) and (20) both inertias $a(\chi)$ and $\lambda(\chi)$ are proportional to $\epsilon^{-3}$ and increase as $L$ increases. Then the kinetic energies of the rotational and breathing modes to $\epsilon^3$ and decrease as the $L$ increases. On the other hand the potential energy of the breathing mode is proportional to $\epsilon^{-4}$. Therefore in the large $\epsilon$ region where the contributions of kinetic energies dominate, increase of $L$ reduces total eigenenergy and eventually the $S^3$ skyrmion disperses as $L \to \infty$.

As listed in Table I, the values of minimum points $L$ which give the lowest eigenenergies $E_I$ are shifted toward larger values from that of the classical one, $L = 1$. This is also explained by the quantum effects of the kinetic energies of spin-isospin rotation and breathing motion.

§6. Discussion and Conclusions

In this paper we have investigated the quantum effects associated with rotation and breathing motions of a $S^3$ skyrmion making use of a family of trial profile function $F(\mu; C)$ with $\mu = \tan(\omega/2)/\chi(t)$ where $\chi(t)$ represents the dynamical variable
of breathing motion of skyrmion.

In the classical level, for $L \leq \sqrt{2}$, we have a $S^3$ skyrmion whose profile function is given by identity map $F(\omega) = \omega$ and its energy density is distributed homogeneously inside $S^3$. In this case the baryon number density is $1/2\pi^2 \rho^3 = e^3 f^2_\pi/2\pi^2 L^3 > e^3 f^2_\pi/4\sqrt{2}\pi^2$. If the Skyrme term constant $\epsilon$ is identified to the coupling constant $g_{\rho\pi\pi}=5.8$, then the baryon number density is larger than 0.37fm$^{-3}$ which corresponds to high baryon number density matter and considered as deconfined state.\(^3\),\(^4\) As shown in the preceding sections, in the region $L \leq \sqrt{2}$ where the potential $V(\chi)$ of the breathing motion has single valley at $\chi = 1$, the quantized $S^3$ skyrmion is realized by the conformal map profile function. The ground state wave function of breathing motion of the $I = J = 0$ $S^3$ skyrmion is shown by a full line in Fig. 3 which is symmetric under inverse transformation (15). As in the classical level this state is considered as the deconfined phase of baryonic matter, then the breathing motion obtained here can be regarded as a part of the breathing motion of the baryonic matter.

In the region $L \geq \sqrt{2}$, for most region of parameters $\epsilon$ and $L$ and for the states with lower spin-isospin, we obtain a localized skyrmion in the classical level whose profile function is given by arccosine form (19) and considered as the confined state.\(^3\),\(^4\) The arccosine profile function is an approximation of the Atiyah-Manton profile function given by (11) with $C = 1$. In this case the potential $V(\chi)$ has two valleys as shown in Fig. 2(b) and the one at $\chi$ smaller than 1.6 is deeper than the one at $\chi$ larger than 1.6. Then the wave function of the lowest energy is localized at very small $\chi$ and the quantized $S^3$ skyrmion is more sharply confined at the north pole of $S^3$. In the quantum level, however, there occurs for a certain region of the parameters $\epsilon$ and $L$ and for the spin-isospin states that the conformal profile function gives the lowest energy state of $S^3$ skyrmion. We take $L = 10$ as an example. The $I = J = 1/2$ $S^3$ skyrmion is given by the arccosine profile function in the region $\epsilon < 6.5$. Only in the region $\epsilon > 6.5$ the conformal profile function gives stable $S^3$ skyrmion. The $S^3$ skyrmion with $I = J = 3/2$ is realized by the arccosine and the conformal profile functions in the regions $\epsilon < 4.8$ and
$\epsilon > 4.9$, respectively. For $I = J = 5/2$ $S^3$ skyrmion state the conformal profile function gives the lowest energy in the region $\epsilon > 4$ and the arccosine profile function does in the region $\epsilon < 4$. In general in the region of small $\epsilon$ corresponding to strong stabilizer, the lower spin-isospin states are given by the arccosine (or Atiyah-Manton) form of profile function and localized at the north pole of $S^3$ for large $L$. When the Skyrme term constant $\epsilon$ increases, i.e. the contribution of stabilizer becomes weak, the $S^3$ skyrmions change to the ones given by the conformal map corresponding to deconfined system from higher spin-isospin states and eventually all spin-isospin states.

The conformal map profile function corresponds to the “frozen states” on the $R^3$ skyrmion where the inertias $a(\chi)$ and $\lambda(\chi)$ of the breathing and rotational modes are infinite.\textsuperscript{6,11} However, since in the case of the $S^3$ skyrmion the value $L$ is finite then the inertias $a(\chi)$ and $\lambda(\chi)$ are also finite contrary in the case of $R^3$ skyrmion, the “frozen states” do not appear any more.

In conclusion, there appears two types of breathing motions, one is the breathing motion of the profile function given by conformal map, and the other is those approximated by the arccosine profile function. For strong stabilizer the lowest eigenenergy state in large $L$ region is given by the breathing motion of arccosine profile function. For the weak stabilizer the eigen state of (10) for the conformal map profile function always gives the lowest energy in all $S^3$ radius $L$, which continuously connects to the “frozen states”\textsuperscript{6,11} on $R^3$ in the limit $L \to \infty$. However, in the small $L$ region, the lowest eigen state is always given by the breathing motion of the conformal map profile function.

Since the $S^3$ skyrmion approaches provide very good modeling of the periodic arrays of the cubic lattice skyrmion in flat space $R^3$ as shown by various studies in the classical level,\textsuperscript{5} above results are not specific ones of the $S^3$ skyrmion, but also essential features of the quantized lattice skyrmions in flat space $R^3$. For example, the quantum breathing motions on $S^3$ around the uniform distribution (identity map with $\chi = 1$), correspond
to the quantum fluctuations of the baryon density around the homogeneous distribution with the half-skyrmion symmetry\(^3\) of the periodic array of the cubic lattice skyrmion in the flat space \(R^3\). Thus, it may be possible to find these remarkable features in the dense nuclear matter.

**Acknowledgements**

We would like to thank H. Otsu for helpful discussions and comments.

**References**


Figure Captions

FIG. 1 (a) The breathing profile functions for $\chi = 1/5$, $1/3$, $1$, $3$ and $5$. The solid line represents the conformal map profile functions and the dashed line represents the arccosine profile function. (b) The baryon density as a function of the radial angle $\omega$ for $\chi = 1/3$, $1/2$, $1$, $2$ and $3$. The solid line (dashed line) represents the baryon density for the conformal map profile function (the arccosine profile function).

FIG. 2 The potential $V(\chi,L)$ as a function of two variable $\chi$ and $L$; (a) the potential for the conformal map profile functions. (b) the potential for the arccosine profile function.

FIG. 3 The potentials $V(\chi)$ in GeV and the arbitrarily normalized wave functions $\psi_0(\chi)$ of the spin-isospin state with $I = J = 0$ for the Skyrme term constant $e = 7$ plotted against the variable $\chi$. The dotted line shows the $\chi$ dependence of the potential $V(\chi)$ in GeV for $L = 1.3$ or for $L = 10$. The solid line (dashed line) represents symmetric (anti-symmetric) wavefunctions $\psi_0(\chi)$ for $L = 1.3$, and dot-dashed line represents a wavefunction for the almost degenerate state for $L = 10$.

FIG. 4 The eigenenergy $E_{I=J=0}$ in GeV of three types of eigen states as a functions of $L$ for various Skyrme term constants $e$; the symmetric states (solid line), anti-symmetric states (dotted line) and almost degenerate states (solid line) for conformal map profile function, and almost degenerate states (dot-dashed line) for the arccosine
profile functions. The dashed line represents the minimum value of the potential as a function of $L$ for the conformal map profile function with $\epsilon = 7$.

FIG. 5 The eigenenergy $E_{I=J}$ in GeV of the spin-isospin states with $I = J = 1/2, 3/2$ and $5/2$ as a function of $L$ for various Skyrme term constants $\epsilon$; (a) $\epsilon = 3$, (b) $\epsilon = 4$, (c) $\epsilon = 5$ and (d) $\epsilon = 7$. For the conformal map profile functions, the energy of the symmetric state, anti-symmetric state and almost degenerate state are represented by the solid line, the dotted line and the solid line, respectively. The dot-dashed line represents the lowest eigenenergy for the arccosine profile functions.

FIG. 6 The $\epsilon$-dependence of the lowest eigenenergies $E_J$ for $J = 0, 1/2, 3/2$ and $5/2$. The classical $S^3$ skyrmion energy $E_{cl} = 6\pi^2 f_\pi/\epsilon$ and $E_\infty$ are also shown by dotted and dot-dashed lines.
TABLE I. The lowest energies $E_{jm}^{\min}$ in MeV.

<table>
<thead>
<tr>
<th>$e$</th>
<th>$L$</th>
<th>$E_{0}^{\min}$</th>
<th>$L$</th>
<th>$E_{1/2}^{\min}$</th>
<th>$L$</th>
<th>$E_{3/2}^{\min}$</th>
<th>$L$</th>
<th>$E_{5/2}^{\min}$</th>
<th>$E_{\text{cl}}$</th>
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<td>1.8</td>
<td>1090</td>
<td>2.7</td>
<td>1455\dagger</td>
<td>3.5</td>
<td>1750\dagger</td>
<td>787</td>
<td>1113</td>
</tr>
</tbody>
</table>

The state a $\dagger$ is attached to is not stable and will infinitely disperse when $L$ is varied.