Intermittency in Branching Processes

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\textbf{ABSTRACT}

We study the intermittency properties of two branching processes, one with a uniform and another with a singular splitting kernel. The asymptotic intermittency indices, as well as the leading corrections to the asymptotic linear regime are explicitly computed in an analytic framework. Both models are found to possess a monofractal spectrum with $\varphi_q = q - 1$. Relations with previous results are discussed.

July 1993

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1 Introduction.

The original proposal of intermittency in multiparticle production [1] consisted of a method of quantifying and measuring short-range correlations in multi-hadron final states, as well as a hypothesis about the self-similar nature of fluctuations. This fractal character is supposed to have dynamical origins and expected to provide valuable information about the underlying mechanism of hadron production. The subject of intermittency has been intensely pursued since its introduction in [1], both theoretically [2, 3] and experimentally [4, 5] and is by now a standard topic in soft hadronic physics.

On the theoretical side, the property of intermittency was found to hold in many models, most remarkably $\alpha$-models [1], of which detailed studies have been made ([6, 7] and references therein). A different class of systems where intermittency is expected to be present is branching processes. These have been applied to the phenomenological description of high-energy processes both of electromagnetic [8] and hadronic [9, 10, 11] nature. In the context of intermittency, branching processes provide a laboratory to study short-range correlations and fluctuations in multi-particle production either analytically or numerically.

Intermittency in branching processes has been studied in [12], where a proof of intermittent behavior is offered. It has also been considered in the more specific framework of QCD branching in [13, 14], where the problem of infrared divergences arises. In this paper we adopt a point of view complementary, in some sense, to that of [13, 14] and closer to [12]. We shall study the intermittent regime of two one-species branching models, characterized by a uniform and a singular splitting kernel, respectively. Our aim is to determine whether these mathematical models display intermittent behavior, of what kind, in what limit, and with what type of corrections. We are also interested in the relationship [15] between intermittency and KNO scaling [16]. Our mathematical approach is different and, we believe, simpler than that of [12], which allows us to obtain more specific results.

The price to pay for those results is that we have to give up generality, by considering two particular cases, and that we defer for future consideration the problem of infrared-divergent kernels.

The outline of the paper is as follows. In Section 2, we define the model, explain those general features of branching processes which are relevant to our purposes, and fix our notations and conventions. In Section 3, we study the particular models in terms of evolution equations for inclusive distributions. For both splitting kernels, a monofractal spectrum with maximal
intermittency indices [17] is found, as well as the leading corrections to the asymptotic behavior of scaled factorial moments and a precise characterization of the regime where intermittency appears. In section 4, a summary of results is given, together with our final remarks. An Appendix, finally, gathers some complementary material related to the mathematical approach used in the main body of the paper.

2 Model. Notations and Conventions.

The branching process under consideration consists of particles characterized by an energy fraction $x$ and a virtuality $t$, the latter being the evolution parameter of the system. Mathematically it is defined by the splitting kernel $P(x)$ entering the evolution equations for the probability $\mathcal{P}$ to having a branching at virtuality $t$,

$$\frac{d\mathcal{P}}{dt} = \int dx P(x)$$

(1)

This equation is inspired in the Altarelli-Parisi-Gribov-Lipatov equations for partonic evolution [18, 19], with the simplificatory assumption that the running coupling constant $\alpha_s(Q^2)$ does not depend on $x$, $Q^2 = Q^2(t)$, so that it can be absorbed in the evolution variable $t$ [20, 21]. In this way, a stationary Markov branching process is defined and evolution equations for the generating functional of transition probabilities can be found. We shall omit the derivation here, which is given in detail in, e.g., [20, 21, 22] (see also the Appendix, where the relevant equations are summarized).

We shall be interested, as will be made clear later in this section, in inclusive transition probabilities $D_n(x_1, \ldots, x_n; t)$ which represent the probability of observing particles with energy fractions $x_1, \ldots, x_n$, as well as particles with any other values of $x$, at virtuality $t$, and starting with one particle with $x = 1$ at $t = 0$. These inclusive distributions are normalized to the factorial moments of multiplicity,

$$\int_0^1 dx_1 \ldots dx_k D_k(x_1, \ldots, x_k; t) = \langle n(n - 1) \cdots (n - k + 1) \rangle(t)$$

(2)

where the multiplicity $n$ is the total number of particles in the system at virtuality $t$. This normalization condition follows naturally from the definition of $D_n$ in terms of the generating functional of the process (see Appendix and [20, 21]).
We shall further assume that the splitting kernel \( P(x) \) is normalized to unity, which implies no loss of generality in the one-species case; that it is symmetric \( P(x) = P(1 - x) \), which follows from energy conservation; and that its support is contained in the interval \([0, 1]\). Moreover, we shall adopt the convention that all probability densities, inclusive or exclusive of any order, vanish outside \([0, 1]\).

The inclusive densities of order \( n \), \( D_n(x_1, \ldots, x_n, t) \), satisfy the forward equation (see Appendix),

\[
\frac{\partial D_n}{\partial t}(x_1, \ldots, x_n, t) =
\]

\[-n D_n(x_1, \ldots, x_n, t) + 2 \int_0^1 dz P(z) \sum_{k=1}^n \frac{1}{z} D_n(x_1, \ldots, \frac{x_k}{z}, \ldots, x_n, t) \]

\[+ 2 \sum_{k>j=1}^n P \left( \frac{x_j}{x_j + x_k} \right) \frac{1}{x_j + x_k} D_{n-1}(x_1, \ldots, \underbrace{x_j + x_k, \ldots, x_k, \ldots, x_n, t}_{j}) \]

where the hat indicates that the variable \( x_k \) is omitted and we used the symmetry of \( P(z) \). The initial condition is given by,

\[ D_n(x_1, \ldots, x_n, t = 0) = \delta_1 \delta(x_1 - 1) \]  

We now define the factorial moment densities of order \( n \), (cf., eq. (2))

\[ d_n(x, t) = \int_0^1 dx_2 \cdots dx_n D_n(x, x_2, \ldots, x_n, t) \]  

Separating \( x_1 \) in the equation for \( D_n \) and integrating \( dx_2 \cdots dx_n \) we obtain the forward evolution equation for \( d_n \),

\[
\frac{\partial d_n}{\partial t}(x, t) = (n - 2) d_n(x, t) + (n - 1)(n - 2) d_{n-1}(x, t) \\
+ 2 \int_0^1 dz P(z) \frac{1}{z} d_n \left( \frac{x}{z} \right) + 2(n - 1) \int_0^1 dz P(z) \frac{1}{z} d_{n-1} \left( \frac{x}{z} \right) 
\]

with initial condition,

\[ d_n(x, t = 0) = \delta_1 \delta(x - 1) \]  

The Mellin transformed equation,

\[
\frac{\partial \tilde{d}_n}{\partial t}(s, t) = (n - 2 + 2 \tilde{P}(s)) \left( \tilde{d}_n(s, t) + (n - 1) \tilde{d}_{n-1}(s, t) \right) \\
\tilde{d}_n(s, t = 0) = \delta_1
\]  

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can be explicitly solved, to obtain,

\[
\tilde{d}_n(s, t) = e^{(n-2)t} \left(1 - e^{-t}\right)^{n-1} e^{2\bar{P}(s)t} \Gamma \left(\frac{n - 1 + 2\bar{P}(s)}{\bar{P}(s)}\right) \Gamma \left(\frac{n - 1}{\bar{P}(s)}\right) \tag{10}
\]

Substituting \(s = 1\) in this expression, we obtain the factorial moments corresponding to a pure-birth, binary-fission process, which are independent of the form of the splitting kernel \([23]\),

\[
\langle n(n - 1) \cdots (n - q + 1) \rangle = q! e^{qt}(1 - e^{-t})^{q-1} \tag{11}
\]

We shall find useful later the following expanded form of the solution,

\[
\tilde{d}_n(s, t) = e^{(n-2)t} \left(1 - e^{-t}\right)^{n-1} e^{2\bar{P}(s)t} \sum_{m=1}^{n-1} C_m^{n-1} \left(2\bar{P}(s)\right)^m \tag{12}
\]

where

\[
\begin{cases}
C_n^{n-1} = 1 \\
C_m^{n-1} = \sum_{i_1> \ldots > i_r=1}^{n-2} i_1 \cdots i_r
\end{cases} \tag{13}
\]

with \(r = n - 1 - m\) and \(n > 2\).

### 2.1 Intermittency indices.

Our goal in this paper is to obtain the intermittency indices of a branching process, defined as the logarithmic slope of reduced factorial moments as functions of bin size \([1]\). Reduced factorial moments are defined as,

\[
F_q \equiv \frac{U_q}{U_1} \tag{14}
\]

\(U_q \equiv \langle n \cdots (n - q + 1) \rangle\) being the factorial moment of the multiplicity within the given bin. Notice that factorial moments \(U_q\) are obtained from the above-defined densities as,

\[
U_q(x; t) = \int dx \; d_q(x; t) \tag{15}
\]

where the integration extends to the bin under consideration.

Due to the asymmetry of the process, \(\langle x \rangle(t)\) being a monotone decreasing function of \(t\) (see Appendix), the relevant bin in any partition of \([0, 1]\) is the one adjacent to the point \(x = 0\). For that reason, it is appropriate to apply vertical analysis (see, e.g., \([4, 24]\)) to that bin in order to find the
intermittency slopes. However, the particle density $U_1$ need not be uniform within the bin under consideration, so that we shall define intermittency indices in terms of $F_q$ as a function of $U_1$ [25],

$$
\frac{d \ln F_q}{d \ln U_1} = \frac{d \ln U_q}{d \ln U_1} - q \equiv -\varphi_q
$$

(16)

This is the definition of intermittency indices that we shall adopt in the sequel.

Before going to a particular case in the next section, we would like to mention another consequence of the asymmetry of the process. For long enough $t$, and a fixed resolution in $x$, almost all particles will be contained in the bin adjacent to $x = 0$, where the process will continue its evolution as a usual branching process with only multiplicity degrees of freedom. But this kind of processes KNO scale [10, 16, 23], leading to constant reduced factorial moments in the limit $t \rightarrow \infty$, and consequently to zero intermittency indices in that limit. We shall explicitly verify this fact below.

3 Two Particular Cases.

In this section, we shall compute the asymptotic form of reduced factorial moments with the expressions derived in the previous section. Factorial moment densities can be exactly found for $P(x) = 1$ and $P(x) = \delta(x - 1)$, which we shall exploit to obtain the asymptotic moments at a given resolution.

3.1 Uniform Splitting Kernel.

For the “uniform model”, we have $P(z) = 1$, $\bar{P}(s) = 1/s$. By expanding the exponential in eq. (12), we find

$$
\bar{d}_1(s,t) = e^{-t} \sum_{n=0}^{\infty} \frac{(2t)^n}{n!} \gamma \frac{1}{s^n}
$$

(17)

$$
\bar{d}_2(s,t) = 2(1 - e^{-t}) \sum_{n=0}^{\infty} \frac{(2t)^n}{n!} \gamma \frac{1}{s^{n+1}}
$$

(18)

$$
\bar{d}_n(s,t) = e^{(n-2)t}(1 - e^{-t})^{n-1} \sum_{m=1}^{n-1} \gamma C_{n-1}^{m-1} 2^m \sum_{k=0}^{\infty} \frac{(2t)^k}{k!} \gamma \frac{1}{s^{k+m}}
$$

(19)
Applying the inverse Mellin transform term by term leads to,

\[
d_1(x, t) = e^{-t} \sqrt{\frac{2t}{-\ln x}} I_1 \left( \sqrt{-8t \ln x} \right) + e^{-t} \delta(x - 1) \tag{20}
\]

\[
d_2(x, t) = 2(1 - e^{-t}) I_0 \left( \sqrt{-8t \ln x} \right) \tag{21}
\]

\[
d_n(x, t) = 2e^{(n-2)t}(1 - e^{-t}) \sum_{m=1}^{n-1} C_m^{n-1} \left( \sqrt{\frac{-2 \ln x}{t}} \right)^m I_{m-1} \left( \sqrt{-8t \ln x} \right) \tag{22}
\]

where \( I_n \) denotes a modified Bessel function of the first kind and order \( n \) [26]. Since these expressions are given by finite sums, in order to check that they are solutions to eq. (6) it is enough to see that their Mellin transform is correct. This is easily done by change of variables, turning one-sided-Mellin into Laplace transforms and using the known transforms of Bessel functions [27].

The factorial moments \( U_n(x, t) \equiv \int_0^x dz d_n(z, t) \) are then given by

\[
U_n(x, t) = 2e^{(n-2)t}(1 - e^{-t}) \sum_{m=1}^{n-1} C_m^{n-1} \int_{-\ln x}^{\infty} dy e^{-y} \left( \sqrt{\frac{2y}{t}} \right)^m I_{m-1} \left( \sqrt{8ty} \right) \tag{23}
\]

and analogously for \( U_1, U_2 \). We shall now assume that the argument of the Bessel functions is large, so that we can approximate them by their asymptotic expansion [26] to obtain,

\[
U_n(x, t) = 2e^{(n-2)t}(1 - e^{-t}) \sum_{m=1}^{n-1} C_m^{n-1} \int_{-\ln x}^{\infty} dy e^{-y} \times
\]

\[
\times \left( \sqrt{\frac{2y}{t}} \right)^{m-1} \frac{e^{\sqrt{8y}t}}{\sqrt{2\pi} \sqrt{8ty}} \left( 1 - \frac{4(m - 1)^2 - 1}{8\sqrt{8ty}} + \cdots \right) \tag{24}
\]

where we have retained the next-to-leading term in order to estimate the order of magnitude of corrections later.

3.1.1 Large \( t \).

We consider now a resolution \( x \) as small as desired but fixed, and \( t \rightarrow \infty \). We can then drop the correction term in eq. (24) and re-write the leading
term by means of the substitution $\eta = (y/2t)^{1/4}$ as,

$$U_n(x, t) = 2e^{nt}(1 - e^{-t})^{n-1} \sum_{m=1}^{n-1} C_m^{n-1} 2^m \sqrt{\frac{3t}{\pi}} \times$$

$$\times \int_{\left(\frac{2nt}{\sqrt{\pi}}\right)}^{\infty} d\eta \ e^{-2t(\theta-1)^2} \eta^{2m}$$

(25)

For $t \to \infty$ we can estimate this expression by saddle point in the form,

$$U_n(x, t) \approx 2e^{nt}(1 - e^{-t})^{n-1} \sum_{m=1}^{n-1} C_m^{n-1} 2^m \sqrt{\frac{3t}{\pi}} \times$$

$$\times \int_{\left(\frac{2nt}{\sqrt{\pi}}\right)}^{\infty} d\eta \ e^{-\theta^2(\eta-1)^2} \eta^{2m}$$

(26)

or, for $\rho = \sqrt{8t}(\eta - 1)$,

$$U_n(x, t) \approx 2e^{nt}(1 - e^{-t})^{n-1} \sum_{m=1}^{n-1} C_m^{n-1} 2^m \sqrt{\frac{3t}{\pi}} \times$$

$$\times \int_{-\infty}^{\infty} d\rho \ e^{-\rho^2} \left(\frac{\rho}{2\sqrt{2t}} + 1\right)^{2m}$$

(27)

Notice that we substituted $-\infty$ for $-\sqrt{8t}$ in the lower limit of the integral. Applying the binomial expansion to the bracket in the integrand and integrating term by term, the resulting sum shows that in the limit the integral takes the value $\sqrt{\pi}$. Then, by definition of $C_m^{n-1}$, eq. (13), we find $U_n(x, t) \approx n!e^{nt}$, which corresponds to the KNO value given by,

$$\frac{U_n}{U_1^n} = n! \quad (t \to \infty)$$

(28)

As discussed in the previous section, this leads to zero intermittency indices.

3.1.2 Small $x$.

For $t$ fixed, we now consider the regime of small $x$. Although the scale of smallness will be apparent as we proceed with the computation of factorial moments, we can in principle assume $x \ll \langle x \rangle$, where $\langle x \rangle = t/(e^t - 1)$ is the average energy fraction (see Appendix).
By changing the integration variable in eq. (24) to \( \eta = \sqrt{y} - \sqrt{2t} \), we obtain,

\[
U_n(x, t) = e^{nt}(1 - e^{-t})^{n-1} \frac{4}{\sqrt{2\pi}(4t)^{1/4}} \sum_{m=1}^{n-1} C_{n-1}^{m-1} \left( \frac{2}{t} \right)^{m-1/2} \times
\]

\[
\times \int_0^\infty d\eta \, e^{-\eta^2 - \eta^{m-1/2}/2} \left( 1 + \frac{\sqrt{2t}}{\eta} \right)^{m-1/2} \times
\]

\[
\times \left( 1 - \frac{(m-1)^2 - 1}{4\sqrt{2t}\eta} + \cdots \right)
\]

(29)

where we denoted \( a \equiv \sqrt{-\ln x} - \sqrt{2t} \) for brevity. By applying the series expansion, valid for \( a > \sqrt{2t} \),

\[
\int_a^\infty d\xi \, \xi^{p/2} \left( 1 + \frac{\sqrt{2t}}{\xi} \right)^{p/2} e^{-\xi^2} =
\]

\[
= \sum_{n=0}^\infty \frac{1}{2n!} \frac{\Gamma(p/2 + 1)}{\Gamma(p/2 - n + 1)} \left( \sqrt{2t} \right)^n \Gamma \left( \frac{p}{4} - \frac{n}{2} + 1, a^2 \right)
\]

(30)

to the previous expression of \( U_n \), we have,

\[
U_n(x, t) = e^{nt}(1 - e^{-t})^{n-1} \frac{4}{\sqrt{2\pi}(4t)^{1/4}} \sum_{m=1}^{n-1} C_{n-1}^{m-1} \left( \frac{2}{t} \right)^{m-1/2} \times
\]

\[
\times \sum_{k=0}^\infty \frac{\left( \sqrt{2t} \right)^k}{2k!} \frac{\Gamma(m + 1/2)}{\Gamma(m + 1/2 - k)} \Gamma \left( \frac{m}{2} - \frac{k}{2} + \frac{1}{4} a^2 \right) \times
\]

\[
\times \left[ 1 - \frac{(m-1)^2 - 1}{4t} \frac{k}{m-1/2} + \cdots \right]
\]

(31)

Some remarks about this expression are in order here,

i. The correction factor in brackets comes from the asymptotic expansion of the modified Bessel function. Since it is linear in \( k \), it does not affect the rate of convergence of the series as given, \( e.g. \), by the ratio test.

ii. This factor does not affect the zeroth-order term \( k = 0 \). Similarly, the next order term would not affect the \( k = 0 \) and \( k = 1 \) terms, \( e.t.c. \).
iii. As we shall see, only the leading term in the series need be kept. If we wanted to retain more terms in the series, we would have to take into account the next corrective terms coming from the asymptotic expansion of the Bessel function. Such expansion is, however, divergent, so that only a finite number of terms may be kept.

Regarding the convergence of the series, we can use the following asymptotic expression, obtained by a saddle-point argument for large \( \omega \),

\[
\Gamma(-\omega, x) = \frac{e^{-x}x^{-\omega}}{\omega + x + 1}[1 + \cdots] \quad (\omega \to +\infty)
\]

(32)
to obtain the rate of convergence of the series as

\[
\left| \frac{u_{n+1}}{u_n} \right| \to \frac{\sqrt{2t}}{\sqrt{-\ln x - \sqrt{2t}}}
\]

(33)
We see, then, that the series is absolutely convergent for \( 8t < -\ln x \) and, furthermore, that for \( \sqrt{2t} \ll \sqrt{-\ln x} \) it is justified to truncate it. We shall retain the leading \((k = 0)\) and next-to-leading \((k = 1)\) terms, the second one in order to have an estimate of the corrections.

\[
U_n(x, t) \cong e^{nt} \left(1 - e^{-t}\right)^{n-1} \frac{4}{\sqrt{2\pi(8t)^{1/4}}} \times \\
\times \sum_{m=1}^{n-1} C_m^{n-1} \left(\frac{2}{t}\right)^{m-1} \frac{1}{2} \Gamma\left(\frac{m}{2} + 1, a^2\right) \times \\
\times \left[1 + \sqrt{2t}(m-1/2) \frac{\Gamma\left(\frac{m}{2} - \frac{1}{4}, a^2\right)}{\Gamma\left(\frac{m}{2} + \frac{1}{4}, a^2\right)} \left(1 - \frac{(m-1)^2 - 1}{8t} \frac{1}{m-1/2}\right) \right]^{34}
\]

We now use the asymptotic form \([26]\) for \( \omega > 0 \) fixed, \( x \to \infty \),

\[
\Gamma(\omega, x) = e^{-x} x^{\omega-1} \left(1 + \frac{\omega - 1}{x} + \cdots\right)
\]

(35)
to obtain

\[
U_n(x, t) \cong e^{nt} \left(1 - e^{-t}\right)^{n-1} \frac{4}{\sqrt{2\pi(8t)^{1/4}}} \times \\
\times \sum_{m=1}^{n-1} C_m^{n-1} \left(\frac{2}{t}\right)^{m-1} \frac{1}{2} \left(e^{-a^2} a^{m-3/2}\right) \times \\
\times \left[1 + (m-1/2) \frac{\sqrt{2t}}{a} \left(1 - \frac{(m-1)^2 - 1}{8t} \frac{1}{m-1/2}\right) \right]^{36}
\]

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or, recalling that \( a \equiv \sqrt{-\ln x} - \sqrt{2t} \),

\[
U_n(x, t) \equiv e^{(n-2)t} \left( 1 - e^{-t} \right)^{n-1} \frac{2e^{-st\ln x}}{\sqrt{2\pi(8t)^{1/4}}} \times \\
\times \sum_{m=1}^{n-1} C_m^{n-1} \left( \frac{2}{t} \right)^{m-1} \left( -\ln x \right)^{\frac{m-1}{2}} \times \\
\times \left[ 1 + \sqrt{\frac{2t}{-\ln x}} \left( 1 - \frac{(m-1)^2 - 1}{8t} \right) \right]
\]

(37)

The leading contribution in the limit \( x \to 0 \) comes from the \( m = n-1 \) term. Keeping only this and the next-to-leading one we finally have,

\[
U_n(x, t) = \\
\left( e^{(n-2)t} \frac{(1 - e^{-t})^{n-1}}{(8t)^{1/4}} \right) \left( \frac{2}{t} \right)^{n-1} \sqrt{\frac{2}{\pi}} \left( e^{-st\ln x} x (-\ln x)^{\frac{n-1}{2}} \right) \times \\
\times \left[ 1 + \sqrt{\frac{2t}{-\ln x}} \left( 1 - \frac{(n-2)^2 - 1}{8t} + \frac{(n-1)(n-2)}{4} \left( \frac{2}{t} \right)^{n-1} \right) \right]
\]

(38)

**Mean value** \( U_1 \) and **factorial moment** \( U_2 \). For the cases \( n = 1, 2 \) we have,

\[
U_1(x, t) = e^{-t} \int_0^x dx \, I_0(\sqrt{-8t\ln x}) \sqrt{\frac{2t}{-\ln x}}
\]

(39)

\[
U_2(x, t) = 2(1 - e^{-t}) \int_0^x dx \, I_0(\sqrt{-8t\ln x})
\]

(40)

Expanding the Bessel functions as before, we arrive at

\[
U_1(x, t) = \frac{e^t (8t)^{1/4}}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{1}{2n!} \Gamma\left( \frac{1}{2} - n \right) \Gamma\left( \frac{1}{4} - \frac{n}{2}, a^2 \right) \left[ 1 + \frac{3n}{2t} \right]
\]

(41)

\[
U_2(x, t) = \frac{(1 - e^t)^2 t^{1/4}}{(2t)^{1/4}} \sum_{n=0}^{\infty} \frac{1}{2n!} \Gamma\left( \frac{3}{2} - n \right) \Gamma\left( \frac{3}{4} - \frac{n}{2}, a^2 \right) \left[ 1 + \frac{n}{16t} \right]
\]

(42)

Retaining only the leading and next-to-leading terms and expanding the gamma functions results in,

\[
U_1(x, t) = \frac{e^{-t} (8t)^{1/4}}{2\sqrt{2\pi}} \frac{e^{-st\ln x} x}{(-\ln x)^{3/4}} \left[ 1 + \sqrt{\frac{2t}{-\ln x}} \left( 1 - \frac{3}{4t} \right) \right]
\]

(43)
\[ U_2(x,t) = \frac{(1 - e^{-t})}{(2t)^{1/4}} e^{\sqrt{-8\ln x}} x^{(\ln x)^{1/4}} \left[ 1 + \frac{1}{2} \sqrt{\frac{2t}{\ln x}} \times \left( \frac{1}{2} + \frac{1}{16t} \right) \right] \]

\[ \varphi = 1 + \frac{1}{\ln x} - \frac{1}{2(-\ln x)^{3/2}} \left( \frac{-11}{8} \sqrt{2t} + \frac{25}{64} \sqrt{\frac{2t}{t}} \right) + \cdots \]  

\[ \varphi_n = (n - 1) + \frac{(n - 1)}{2(-\ln x)} - \frac{A}{\sqrt{t(-\ln x)^{3/2}}} + \cdots \]  

\[ A = (n - 1)(n - 2) \frac{2^{n+1}}{16} t^{2n/2} - \frac{(n - 1)t}{\sqrt{2}} - \frac{\sqrt{2}}{16}(n^2 - 4n - 3) \]

We see, then, that for $\sqrt{2t} \ll \sqrt{-\ln x}$ and $n \ll -\ln x$ we have intermittency with a monofractal spectrum of indices, with logarithmic corrections that, to first order, do not depend on $t$.

### 3.2 Singular Splitting Kernel.

For the “twin” model [28], we have $P(z) = \delta(z - 1/2)$, $\bar{P} = 1/2^a$. We shall proceed here as before, by expanding the exponential in (12),

\[ \tilde{d}_1(s,t) = e^{-t} \sum_{n=0}^{\infty} \frac{(2t)^n}{n!} \gamma - \gamma \left( \frac{1}{2^{s-1}} \right)^n \]  

\[ \tilde{d}_2(s,t) = 2(1 - e^{-t}) \sum_{n=0}^{\infty} \frac{(2t)^n}{n!} \gamma - \gamma \left( \frac{1}{2^{s-1}} \right)^{n+1} \]  

\[ \tilde{d}_n(s,t) = e^{(n-2)t} \frac{(2t)^n}{n!} \gamma - \gamma - C^{n-1} \gamma 2^n \sum_{k=0}^{\infty} \frac{(2t)^k}{k!} \gamma \times \left( \frac{1}{2^{s-1}} \right)^{k+m} \]  

and applying the inverse Mellin transform term by term to find,

\[ d_1(x,t) = e^{-t} \sum_{n=0}^{\infty} \frac{(2t)^n}{n!} \gamma - \gamma \left( x - \frac{1}{2^n} \right) \]  

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\[ d_2(x,t) = 2(1 - e^{-t}) \sum_{n=0}^{\infty} \left( \frac{2t}{n!} \right)^n \gamma - \delta \left( x - \frac{1}{2n+1} \right) \]  
\[ d_n(x,t) = e^{(n-2)t} \left( 1 - e^{-t} \right)^{n-1} \sum_{m=1}^{n-1} \gamma - C_m^{n-1} 2^m \sum_{k=0}^{\infty} \left( \frac{2t}{k!} \right)^k \times \]  
\[ \times \gamma - \delta \left( x - \frac{1}{2k+m} \right) \]  

(51)  

(52)  

For the factorial moment densities we have,

\[ U_1(x,t) = e^{-t} \sum_{n=0}^{\infty} \frac{(2t)^n}{n!} \gamma - H \left( x - \frac{1}{2} \right) = e^{-t} \sum_{n=\lceil -\log_2 x \rceil}^{\infty} \frac{(2t)^n}{n!} \]  
\[ U_2(x,t) = 2(1 - e^{-t}) \sum_{n=0}^{\infty} \frac{(2t)^n}{n!} \gamma - H \left( x - \frac{1}{2n+1} \right) \]  
\[ = 2(1 - e^{-t}) \sum_{n=\lceil -\log_2 x \rceil - 1}^{\infty} \frac{(2t)^n}{n!} \]  
\[ U_n(x,t) = e^{(n-2)t} \left( 1 - e^{-t} \right)^{n-1} \sum_{m=1}^{n-1} \gamma - C_m^{n-1} 2^m \sum_{k=0}^{\infty} \left( \frac{2t}{k!} \right)^k \gamma - H \left( x - \frac{1}{2k+m} \right) \]  
\[ = e^{(n-2)t} \left( 1 - e^{-t} \right)^{n-1} \sum_{m=1}^{n-1} \gamma - C_m^{n-1} 2^m \sum_{k=\lceil -\log_2 x \rceil - m}^{\infty} \left( \frac{2t}{k!} \right)^k \]  

(53)  

(54)  

(55)  

where \( \lceil x \rceil \) is the smallest integer \( n \) such that \( n \geq x \) and we denoted \( H \) the right continuous Heaviside step function.

We shall use now the fact that,

\[ \sum_{n=1}^{\infty} \frac{r^n}{n!} = \frac{e^r}{\Gamma(N)} \gamma(N, r) \]  

\( \gamma \) being an incomplete gamma function [26], so that,

\[ U_1(x,t) = e^{-t} \frac{\gamma([-\log_2 x], 2t)}{\Gamma([-\log_2 x])} \]  
\[ U_2(x,t) = 2e^{2t} (1 - e^{-t}) \frac{\gamma([-\log_2 x] - 1, 2t)}{\Gamma([-\log_2 x] - 1)} \]  
\[ U_n(x,t) = e^{nt} (1 - e^{-t})^{n-1} \sum_{m=1}^{n-1} C_m^{n-1} 2^m \frac{\gamma([-\log_2 x] - m, 2t)}{\Gamma([-\log_2 x] - m)} \]  

(56)  

(57)  

(58)
3.2.1 Large $t$.

For fixed $x$ and $t \to \infty$ the ratio of gamma functions goes to 1 and we have KNO scaling, $U_q/U_1^q \to q!$, by definition of the coefficients $C_m^{-1}$, as in the previous case.

3.2.2 Small $x$.

The functions $U_q$ have a stairway shape superimposed to the general profile obtained by eliminating the integer part from the arguments. We shall work with this smoothed form in what follows. The density $U_q(U_1)$ is unaltered by these device, as can be explicitly checked, and it is in this density that we are interested.

We shall now consider the form of these functions for large $y \equiv -\log_2(x)$. To that end we use the following asymptotic expansion for large values of the parameter, obtained by a saddle-point approximation,

\[
\gamma(y, x) = \frac{x^y e^{-x}}{y-1} \left[ 1 + \frac{x-1}{y} + \frac{x(x-2)}{y^2} + \cdots \right] \\
\Gamma(y) = \sqrt{2\pi} e^{-y} y^{y-1/2} \left[ 1 + \frac{1}{12y} + \frac{1}{288y^2} + \cdots \right]
\]

This way, we obtain, to first order in $1/y$,

\[
\log_2 U_1 = -y \log_2(y) + y \log_2(2te) - \frac{1}{2} \log_2(y) + \log_2 \left( \frac{e^{-t}}{\sqrt{2\pi}} \right) + \\
+ \frac{2t - 1/2}{y} + O \left( \frac{1}{y^2} \right) \\
\log_2 U_2 = -y \log_2(y) + y \log_2(2te) + \frac{1}{2} \log_2(y) + \log_2 \left( \frac{1 - e^{-t}}{\sqrt{2\pi t}} \right) + \\
+ \frac{2t - 1/2}{y} + O \left( \frac{1}{y^2} \right)
\]

The case of $U_n$ is slightly more complicated since it is a sum of several terms. Assuming $y \gg 2t + n$ we have, for $k \leq n$,

\[
\frac{\gamma(y-k, 2t)}{\Gamma(y-k)} \frac{\Gamma(y-k+1)}{\gamma(y-k+1, 2t)} \approx \frac{y-k}{2t} \gg 1
\]
and then we can retain only the leading and next-to-leading terms $n = n - 1$, $n - 2$, so that,

$$U_n \simeq e^{nt}(1 - e^{-t})^{n-1}\left[2^{n-1}\gamma(y - n + 1, 2t) + \frac{(n-1)(n-2)}{2} 2^{n-2}\gamma(y - n + 2, 2t) + \cdots \right]$$

(64)

We shall use a first order approximation for gamma functions in the first term and zeroth order for the ones in the second. Up to higher order terms we finally have,

$$\log_2 U_n = -y \log_2 y + y \log_2 (2t) + \left(n - \frac{3}{2}\right) \log_2 y + \text{cst.} + \frac{1}{y} \left[2t - \frac{13}{12} + \frac{(n - 1)(n - 2)}{4} 2t - \frac{n(n - 3)}{2} \right]$$

(65)

Proceeding as in the previous case, we arrive at the following expressions for the intermittency indices,

$$\varphi_2 = 1 + \frac{1}{y(\ln y - \ln(2t))} + O\left(\frac{1}{y^2}\right)$$

(66)

$$\varphi_n = (n-1) + \frac{n-1/2}{y(\ln y - \ln(2t))} + O\left(\frac{1}{y^2}\right)$$

(67)

Taking into account that $y = -\log_2 x$, we see that for $-\log_2 \gg 2t + n$ we have the same spectrum of asymptotic intermittency indices as in the case of a uniform $P(x)$, with logarithmic corrections independent of $t$ to first order.

4 Final Remarks.

In the previous sections we considered two particular cases of one-species branching processes of the pure-birth type with binary fission, characterized by a uniform and a Dirac-delta splitting kernel, respectively. We exploited the fact that for these special kernels explicit solutions to the evolution equations for inclusive distributions can be found. We have shown that for a fixed resolution in $x$, and $t \to \infty$ a KNO scaling regime is reached, while for $t$ fixed and $x \to 0$ factorial moments grow without bound, giving asymptotic intermittency indices $\varphi_q = q - 1$ in both cases. Leading corrections to this
behavior for finite $x$ are proportional to $1/\ln x$, and independent of $t$. Furthermore, they are positive, \textit{i.e.}, the asymptotic linear region is approached from above.

The intermittent regime is defined as $\sqrt{2t} \ll \sqrt{-\ln x}$ for the uniform and as $2t \ll -\log_2 x$ for the singular kernel, for the first few indices with $n \sim 1$. It follows that $x \ll \langle x \rangle(t)$ is a necessary condition, which seems difficult to fulfill in practical applications such as numerical simulations. In [28] a Monte Carlo experiment with three branching models is reported, of which only the “twin” model has been considered here. Their results, as shown in Figure 2 of [28], are not inconsistent with our prediction for $n \leq 4$, within numerical uncertainties in their computation, although it should be noticed that for $q \geq 4$ their indices are systematically larger than $q - 1$, which is the maximum possible value [17].

Even though our approach does not appear to be easy to extend to more complicated functional forms of the kernel $P(x)$, we conjecture that the spectrum of intermittency indices is independent of it within the class of models considered in this paper. Our results may be useful to tune up numerical studies of these models, along the lines of [28]. We believe that a richer intermittent behavior, as observed in experimental data, may be obtained with infrared divergent kernels, about which results have already been obtained [14].

\textbf{Acknowledgments.}

I would like to thank Prof. R. Peccei for encouragement and support during completion of this work. I benefitted also from many discussions with Dr. P. Van Driel.

This work was supported by ICSC-World Laboratory.

\textbf{References}


Appendix.

We present here some basic equations and definitions from the theory of branching processes as used in sections 2 and 3.

The starting point is the generating functional of the process, \( \phi[W, t; f] \) [20, 21], which is a function of the evolution parameter \( t \) and of the energy \( W \) of the initial particle, and a functional of the dummy function \( f(\omega) \). Probability densities will be obtained by functional differentiation with respect to \( f \). Notice that \( \omega \) has dimension of energy, like \( W \).

The evolution of \( \phi \) is given by the equations [20, 21],

\[
\frac{\partial \phi}{\partial t}[W, t; f] = \int_0^1 dz \ P(z) \left( \phi[zW, t; f] \frac{\phi[(1-z)W, t; f] - \phi[W, t; f]}{\delta f(\xi W)} \right)
\]

(A1)

\[
\frac{\partial \phi}{\partial t}[W, t; f] = \int_0^1 dz \ \int_0^1 d\xi \ P(z) W \frac{\delta \phi[W, t; f]}{\delta f(\xi W)} \times
\]

\[
\times \left( f(\xi W) ((1-z)\xi W) - f(\xi W) \right)
\]

(A2)

The first being the backward and the second the forward Kolmogorov evolution equations [29]. These two equations have the same set of solutions [29],

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and only the forward equation is used above. We use the initial condition 
\( \phi[W; t = 0; f] = f(W) \), corresponding to one particle in the initial state.

Inclusive probability densities are defined as,

\[
D_n(x_1, \ldots, x_n; t) = W^n \frac{\delta^n \phi[W; t; f]}{\delta f(x_1 W) \cdots \delta f(x_n W)} \bigg|_{f=1} \tag{A3}
\]

where we have already used the fact that since the splitting kernel \( P(x) \) depends only on energy fraction, and not on energy, the process is scale
invariant and probability densities depend only on \( x_1, \ldots, x_n \) [21]. Furthermore, we see from this definition that \( D_n \) is completely symmetric in its
arguments.

From the evolution equations for \( \phi \) we can derive the corresponding
equations for \( D_n \). The forward one is given by (3) and the backward equation is,

\[
\frac{\partial D_n}{\partial t}(x_1, \ldots, x_n; t) = \frac{1}{(1 - z)^n} \frac{1}{z^n} D_{n-k} \left( \frac{x_{i_1}}{z}, \ldots, \frac{x_{i_k}}{z}; t \right) \times \left( \sum_{\pi(n,k)} D_k \left( \frac{x_{i_1}}{z}, \ldots, \frac{x_{i_k}}{z}; t \right) - D_n(x_1, \ldots, x_n; t) \right) \tag{A4}
\]

where the sum runs over all possible partitions \( \pi(n, k) \) of \( (1, \ldots, n) \) into two
sets \( (i_1, \ldots, i_k), (i_{k+1}, \ldots, i_n) \) such that \( i_1 > i_2 > \cdots > i_k \) and \( i_{k+1} > \cdots > i_n, \) with \( k = 0, \ldots, n; \) and we used the relation,

\[
W \frac{\delta \phi[zW; t; f]}{\delta f(zW)} \bigg|_{f=1} = \frac{1}{z} D_1 \left( \frac{x}{z}; t \right) \tag{A5}
\]

and its generalizations to higher order densities \( D_n \).

Analogously, exclusive distributions are defined by,

\[
E_n(x_1, \ldots, x_n; t) = W^n \frac{\delta^n \phi[W; t; f]}{\delta f(x_1 W) \cdots \delta f(x_n W)} \bigg|_{f=0} \tag{A6}
\]

and satisfy,

\[
\frac{\partial E_n}{\partial t}(x_1, \ldots, x_n; t) = \int_0^1 dz P(z) \left( \sum_{\pi(n,k)} \frac{1}{z^n (1 - z)^{n-k}} \times \right.
\]
\[ E_k \left( \frac{x_{i_1}}{z}, \ldots, \frac{x_{i_k}}{z}; t \right) E_{n-k} \left( \frac{x_{i_{k+1}}}{z}, \ldots, \frac{x_{i_n}}{z}; t \right) - E_n (x_1, \ldots, x_n; t) \] (A7)

\[ \frac{\partial E_n (x_1, \ldots, x_n, t)}{\partial t} = -n E_n (x_1, \ldots, x_n, t) + \frac{2}{n} \sum_{k>j=1}^{n} P \left( \frac{x_j}{x_j + x_k} \right) \times \]

\[ \frac{1}{x_j + x_k} E_{n-1}(x_1, \ldots, x_j + x_k, \ldots, x_n, t) \] (A8)

The initial condition is given by \( E_n (x_1, \ldots, x_n; t = 0) = \delta_{i_1} \delta(x_1 - 1) \).

The relation between \( D_n \) and \( E_n \) may be explicitly written as,

\[ D_n (x_1, \ldots, x_n; t) = \sum_{N=n}^{\infty} N (N - 1) \cdots (N - n + 1) \times \]

\[ \int_0^1 dy_N \cdots dy_N E_n (x_1, \ldots, x_n, y_1, \ldots, y_n; t) \] (A9)

where, once again, we have made use of the symmetry with respect to permutations of the arguments of these distributions. Notice that our definition of \( E_n \) is slightly different from that of [21], since we normalize \( E_n \) to the multiplicity distribution,

\[ \int_0^1 dx_1 \cdots dx_n E_n (x_1, \ldots, x_n; t) = P_n (t) \] (A10)

from whence the normalization of \( D_n \) to factorial moments results.

It is not difficult to convince oneself that solutions to the evolution equations have the form,

\[ E_n (x_1, \ldots, x_n; t) = P_n (t) K_n (x_1, \ldots, x_n) \] (A11)

where \( K_n \) is defined by this equation. The evolution equations for \( P_n (t) \), the multiplicity distribution, can be obtained by integration of the equations for \( E_n \). Since these are well-known [11, 23], we shall quote only the solutions,

\[ P_n (t) = \frac{1}{\langle n \rangle} \left( \frac{\langle n \rangle - 1}{\langle n \rangle} \right)^{n-1} \] (A12)

\[ \langle n \rangle = e^t \] (A13)

From this expression, and the evolution equations for \( E_n \), recurrence relations (backward and forward) for \( K_n (x_1, \ldots, x_n) \) can be found. We shall
only write the forward one,

\[ K_n(x_1, \ldots, x_n) = \frac{2}{n(n-1)} \sum_{k \geq j=1}^{n} P \left( \frac{x_j}{x_j + x_k} \right) \frac{1}{x_j + x_k} \times K_{n-1}(x_1, \ldots, x_j + x_k, \ldots, x_1) \quad (A14) \]

with \( K_1(x) = \delta(x - 1) \). From this relation and \( P(x) = P(1 - x) \) it follows that,

\[ \int_0^1 dx_1 \cdots dx_n x_1 K_n(x_1, \ldots, x_n) = \frac{1}{n} \quad (A15) \]

and then,

\[ \langle x \rangle \equiv \sum_{n=1}^{\infty} \int_0^1 dx_1 \cdots dx_n x_1 E_n(x_1, \ldots, x_n; t) = \frac{\ln(n)}{\langle n \rangle - 1} \quad (A16) \]

independently of the detailed form of \( P(x) \).

In [12], it is shown that a branching process satisfying,

\[ E_n(x_1, \ldots, x_n; t) = \prod_{i=1}^{n} f(x_i; t) \delta \left( 1 - \sum_{j=1}^{n} x_j \right) \quad (A17) \]

is not intermittent. From the factorization property discussed above, we see that such a branching process does not exist, at least within the class of models considered here.