Quantum Fluctuations around the Electroweak Sphaleron

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Abstract
We present an analysis of the quantum fluctuations around the electroweak sphaleron and calculate the associated determinant which gives the 1–loop correction to the sphaleron transition rate. The calculation differs in various technical aspects from a previous analysis by Carson et al. so that it can be considered as independent. The numerical results differ also – by several orders of magnitude – from those of this previous analysis; we find that the sphaleron transition rate is much less suppressed than found previously.

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1 Introduction

The electroweak theory is known [1, 2] for quite some time to possess a topologically nontrivial solution which describes a saddlepoint between two topologically distinct vacua. The recent interest in this solution has been centered around its possible rôle in generating baryon number violating processes in the early universe or even at accelerator energies [3]–[5]. The rate of sphaleron transitions in the range of temperatures \( M_W(T) \ll T \ll M_W(T)/\alpha_w \) has been derived on the basis of the work of Langer [6] and Affleck [7] by Arnold and McLerran [4]. It is given by

\[
\Gamma = \frac{\omega_\perp}{2\pi} \mathcal{N} e^{-E_d/T} \kappa. \tag{1.1}
\]

Here \( \omega_\perp \) is the absolute value of the eigenvalue of the unstable mode, the prefactor \( \mathcal{N} \) refers to normalizations introduced by the translation and rotation zero modes and is given in detail below. \( E_d \) is the classical sphaleron energy and the factor \( \kappa \) takes account of the quantum fluctuations of the sphaleron. It is given by

\[
\kappa = \text{Im}(\frac{\det \Delta_{gf}^S}{\det \Delta_{gf}^0} \frac{\det \Delta_{gf}^0}{\det \Delta_{gf}^S})^{1/2} \tag{1.2}
\]

where the symbols \( \Delta \) denote the small fluctuation operators. They are obtained by expanding the gauge fixed action (gf) and the Fadeev–Popov action (FP) evaluated around the sphaleron (S) and the vacuum (0), respectively. If the fluctuation operators are diagonalized the determinants are formally given by the product of the squared eigenfrequencies \( \lambda_\perp^2 \). The determinant \( \det \Delta_{gf}^S \) of the gauge fixed action is to be evaluated without the zero modes and with the eigenvalue of the unstable mode, \( \lambda_\perp^2 = -\omega_\perp^2 \) replaced by its absolute value.

The first evaluation of \( \kappa \) by Akiba, Kikuchi and Yanagida [8] was restricted to the three lowest partial waves. The authors concluded from their results that the inclusion of this correction did not suppress the sphaleron transition rate. Of course they considered their conclusion as tentative and to be checked by a more precise evaluation. Subsequently Carson and McLerran [9] evaluated \( \kappa \) in an approximation scheme (referred to as DPY in the following) developed by Diakonov, Petrov and Yung [10], a complete exact numerical evaluation was presented first by Carson et al. [11] (referred to as CLMW in the following). The results of this exact calculation differ significantly from the DPY approximation and from a perturbative estimate. It is therefore of interest to repeat this exact evaluation and this is – besides a general analysis of the fluctuation Lagrangean and its partial wave reduction – the subject of our investigation.

Our investigation departs from the one of CLMW in various points. While we use the background gauge as CLMW, we use another angular momentum basis and a different scheme for evaluating the determinant.

The analysis of the small fluctuations around the sphaleron requires a partial wave decomposition with respect to the quantum number \( \vec{K} = \vec{J} + \vec{I} \) (\( K \)=spin). Our present
work is based on the analysis of Ref. [13] where the sphaleron stability was investigated. The small fluctuation equations obtained there had to be modified however. While that investigation was performed in the $A_0 = 0$ gauge we use here the background gauge. So a gauge fixing term had to be added and also we had to construct the small fluctuation Lagrangean for the Fadeev–Popov modes. The evaluation of the determinant has been performed using the Euclidean Green function technique in analogy to some recent investigations of one of the authors (J.B.) [14]. The results of our calculation have already been communicated previously in a short version [15]. Here we present an extensive version including the complete partial wave analysis.

The paper is organized as follows. We present the basic relations for the electroweak theory and the sphaleron solution in section 2. The small fluctuation expansion and its partial wave analysis are discussed in sections 3 and 4, respectively. The explicit equations of motion which constitute our first main result are presented in Appendix A. In section 5 we collect the main formulae for the 1–loop correction to the sphaleron transition rate and define more precisely the zero mode prefactors and the fluctuation determinant, some material being deferred to Appendix B. In section 6 we express the fluctuation determinant in terms of Euclidean Green functions, a formulation that is used as the basis of our numerical evaluation. Sections 7 and 8 contain the discussion of renormalization and of the treatment of the zero and unstable modes. Some aspects of the numerical calculations and the results are presented in section 9 and in Appendix C.

## 2 Basic Relations

The action of a pure SU(2) gauge theory with minimal Higgs sector is given in Minkowski space as

$$S = \int d^4x \mathcal{L}(\bar{x}),$$

where the Lagrangean density is given by

$$\mathcal{L} = -\frac{1}{4} \tilde{F}_{\mu\nu}^{a} \tilde{F}^{\mu\nu a} + \left( \tilde{D}_{\mu} \Phi \right)^{\dagger} \left( \tilde{D}^{\mu} \Phi \right) - \lambda \left( \Phi^{\dagger} \Phi - \frac{1}{2} v^{2} \right)^{2}.$$  \hspace{1cm} (2.2)

We have used a bar to denote the original fields and coordinates ($\bar{x}, \Phi, \bar{F}_{\mu\nu}^{a}, \ldots$), we will use the same letters without bar ($x, \Phi, F_{\mu\nu}^{a}, \ldots$) for a rescaled version of these quantities (see below).

The non-Abelian field strenght tensor and the covariant derivative of the Higgs field are given by

$$\tilde{F}_{\mu\nu}^{a} = \tilde{\partial}_{\mu} \tilde{W}_{\nu}^{a} - \tilde{\partial}_{\nu} \tilde{W}_{\mu}^{a} + g \varepsilon^{abc} \tilde{W}_{\mu}^{b} \tilde{W}_{\nu}^{c}$$

and

$$\tilde{D}_{\mu} = \tilde{\partial}_{\mu} - i \frac{g}{2} \gamma^{a} \tilde{W}_{\mu}^{a}$$  \hspace{1cm} (2.3)

respectively, where $\tau^{a}$ are the Pauli matrices.
After spontaneous symmetry breaking the vacuum expectation value of $\bar{\Phi}$ is $v/\sqrt{2}$ and the $W_\mu^a$ bosons get a mass $m_W = gv/2$. The mass of the physical Higgs boson becomes $m_H = \sqrt{2\lambda}v$. We define further, as usual, the weak fine structure constant as $\alpha_w = g^2/4\pi$.

In view of the renormalization of the theory at finite temperature it is suitable to rescale fields and coordinates as \(^2\)

$$x_\mu = \frac{x_\mu}{m_W} = \frac{2x_\mu}{gv}, \quad \bar{W}_\mu^a = gW_\mu^a \quad \text{and} \quad \bar{\Phi} = \frac{v}{\sqrt{2}}\Phi.$$  \hspace{1cm} (2.4)

As a result of this rescaling the action takes the form

$$S = \frac{1}{g^2} \int d^4x\mathcal{L}(x)$$  \hspace{1cm} (2.5)

with the Lagrangean density

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} + 2(D_\mu\Phi)\dagger(D^\mu\Phi) - \frac{1}{2}\xi^2(\Phi\dagger\Phi - 1)^2,$$  \hspace{1cm} (2.6)

Here we have introduced the ratio of Higgs and W masses $\xi$ via

$$\xi^2 \equiv \frac{m_H^2}{m_W^2} = \frac{8\lambda}{g^2}.$$  \hspace{1cm} (2.7)

Field strength and covariant derivative reduce to

$$F_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + \epsilon_{abc} W_\mu^b W_\nu^c$$

$$D_\mu = \partial_\mu - \frac{i}{2} \gamma^a W_\mu^a.$$  \hspace{1cm} (2.8)

As a consequence of this rescaling the action does not depend any more on the vacuum expectation value of the Higgs field and the couplings appear only in the mass ratio $\xi$. As explained in [9, 11] at high temperatures the fields can be considered as essentially static and the theory reduces to a three dimensional theory if the vacuum expectation value is rescaled to the temperature dependent one [16, 4]

$$v(T) = v(0)\sqrt{1 - T^2/T_c^2},$$  \hspace{1cm} (2.9)

where $T_c^2 = 2v(0)^2\left[1 + \frac{3g^2}{8\lambda}\right]$  \hspace{1cm} (2.10)

is the critical temperature for symmetry restoration. While the masses become then functions of the temperature, $M_W(T)$ and $M_H(T)$, this dependence cancels in their ratio $\xi$.

One obtains for the three dimensional Euclidean action

$$S_E = \frac{1}{g^3} \int d^3x\mathcal{L}_E(x),$$  \hspace{1cm} (2.11)

\(^2\)Our scale units differ from those of CLMW, we use $m_W$ instead of $gv$ as the basic unit.
with the Lagrangean density

\[ \mathcal{L}_E(x) = \frac{1}{4} F_{\mu \nu}^a F_{\mu \nu \alpha}^a + 2 (D_\mu \Phi)^\dagger (D_\mu \Phi) + \frac{1}{2} \xi^2 \left( \Phi^\dagger \Phi - 1 \right)^2. \]  

(2.12)

The effective coupling constant of the three dimensional theory \( g_3 \) which describes the interaction at high temperatures is given by

\[ g_3^2(T) = \frac{g^2 T}{M_W(T)}. \]  

(2.13)

For temperatures

\[ T \ll M_W(T)/\alpha_w \]  

(2.14)

this coupling is much smaller than 1 and an expansion with respect to \( g_3 \) should be reliable.

The variation of the action (2.5) with respect to \( W^a_\mu \) and \( \Phi^\dagger \) leads to the classical equations of motion

\[(D^\nu F_{\mu \nu})^a + i \left[ (D_\mu \Phi)^\dagger \tau^a \Phi - \Phi^\dagger \tau^a (D_\mu \Phi) \right] = 0 \]

and

\[ D^\mu D_\mu \Phi + \frac{1}{2} \xi^2 \left( \Phi^\dagger \Phi - 1 \right) \Phi = 0, \]  

(2.15)

where the covariant divergence of the field strength tensor is given by

\[(D^\nu F_{\mu \nu})^a = \partial^\nu F^a_{\mu \nu} + \varepsilon^{abc} W^{ab}_\nu F^c_{\mu \nu} \]  

(2.16)

In [2] a static saddle point solution to these equations has been constructed explicitly, the well known sphaleron. We choose here an Ansatz that differs from the one of Ref. [2] by a \( SU(2) \) rotation of the Higgs field by an angle \( \pi/2 \) (which is another special case of a more general parametrisation [8])

\[
\begin{align*}
W^a_0 &= 0 \\
W^a_j &= \frac{f_A(r) - 1}{r} \varepsilon_{j a m} \hat{x}_m \\
\Phi &= H_0(r) \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\end{align*}
\]  

(2.17)

with

\[
\hat{x}_m = \frac{x_m}{r}.
\]  

(2.18)

Our solution will then be a gauge rotated version of the one given in [2].

Inserting this Ansatz into the classical equations of motion (2.15) one obtains for the profile functions \( f_A \) and \( H_0 \) a coupled system of the form

\[
\begin{align*}
f_A'' &= (f_A - 1) H_0^2 + \frac{f_A (f_A^2 - 1)}{r^2} \\
H_0'' &= \frac{1}{2} \xi^2 H_0 (H_0^2 - 1) - \frac{2}{r} H_0' + \frac{(f_A - 1)^2}{2 r^2} H_0.
\end{align*}
\]  

(2.19)
The energy of the sphaleron configuration is given by

\[ E_{cl} = \frac{M_W(T)}{\alpha_w} \int dr \mathcal{H}(r), \]  

(2.20)

with the Hamiltonian density

\[ \mathcal{H}(r) = f_A^2 + \frac{1}{2r^2} (f_A^2 - 1)^2 + 2r^2 H_0^2 + (f_A - 1)^2 H_0^2 + \frac{1}{r^2} \right. \]

\[ \left. + \frac{1}{2} \xi^2 r^2 (H_0^2 - 1)^2. \right\]  

(2.21)

Requiring the finiteness of the energy yields the following boundary conditions for the profile functions

\[ f_A(0) = -1 \]
\[ f_A(\infty) = 1 \]
\[ H_0(0) = 0 \]
\[ H_0(\infty) = 1. \]  

(2.22)

The numerical solutions of the equations (2.19) used later on were obtained using a method developed in [18]. The classical sphaleron energy as a function of \( \xi \) is presented in Fig. 1 in units of \( M_W(T)/\alpha_w \). The classical Euclidean high temperature action as defined in Eq. (2.11) is then given by

\[ S_{cl}^E = E_{cl}/T. \]  

(2.23)

3 Small Fluctuations around the Sphaleron

In this section we will develop the small fluctuation expansion around the sphaleron. We will start with expanding the Lagrangian up to second order in the small fluctuations and fix the gauge. Subsequently we will expand the small fluctuations into partial waves.

The gauge and Higgs fields of the theory are expanded around the spaleron configuration via

\[ W_0^a = a_0^a \]
\[ W_i^a = A_i^a + a_i^a \]
\[ \Phi = (H_0 + \hbar) U(\varphi) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]
\[ = \Phi_{cl} + \Phi^{(1)} \]  

(3.1)

(3.2)

\( A_i^a \) and \( \Phi_{cl} \) denote here the classical solution (2.17)

\[ A_i^a = \frac{f_A - 1}{r} \varepsilon_{i\mu} \hat{x}_\mu \]
\[ \Phi_{cl} = H_0 \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \]  

(3.3)
and $U(\phi)$ is given by

$$U(\varphi) = \exp(i r^a \varphi^a).$$

So the Higgs field fluctuations are parametrized by the isosinglet $h$ and the isotriplet $\varphi^a$ and those of the gauge field by $a^a_\mu$. We note that $\Phi$ has to be expanded up to second order in the fluctuation fields if the second order Lagrangean is to be determined; so $\Phi^{(1)}$ includes second order terms $^3$.

Inserting these expansions into Eq. (2.6) and collecting terms of the same order in the small fluctuations we find in zeroth order the classical sphaleron action, the first order contribution vanishes since the sphaleron configuration is a saddle point of the action. The second order Lagrangean density in which we are interested here becomes

$$\mathcal{L}^{(2)} = -\frac{1}{2} \left( \mathring{\mathcal{D}}_\mu a_\nu \right)^a \left( \mathring{\mathcal{D}}^\mu a^a_\nu \right) + \frac{1}{2} \left( \mathring{\mathcal{D}}_\mu a_\nu \right)^a \left( \mathring{\mathcal{D}}^\mu a^a_\nu \right) + \frac{1}{2} H_0^2 a^a_\mu a^{a\mu}
+ 2 \partial_\mu h \partial^\mu h - \xi^2 \left( 3 H_0^2 - 1 \right) h^2 + 2 \partial_\mu \varphi^a \partial^\mu \varphi^a
- \frac{1}{2} \varepsilon^{abc} \mathring{F}^a_{\mu\nu} a_\mu^b a_\nu^c - 2 \varepsilon^{abc} A^a_\mu \partial^\mu \varphi^b \varphi^c - 2 H_0 a^a_\mu \partial^\mu \varphi^a
- 4 h A^a_\mu \partial^\mu \varphi^a + 2 H_0 h A^a_\mu a^{a\mu} + \frac{1}{2} A^a_\mu A^{a\mu} h^2 \quad (3.5)$$

The tildes on the covariant derivatives indicate that only the classical gauge fields are to be used:

$$\mathring{\mathcal{D}}_\mu = \partial_\mu A^a_\mu - \partial_\mu A^a_\mu + \varepsilon^{abc} A^b_\mu A^c_\nu$$

$$(\mathring{\mathcal{D}}_\mu a_\nu)^a = \partial_\mu a^a_\nu + \varepsilon^{abc} A^b_\mu a^c_\nu. \quad (3.6)$$

In order to eliminate the gauge degrees of freedom we have used the background gauge which restricts the fluctuating fields. In general form the three constraints are given by

$$\mathcal{F}_a = \left( \mathring{\mathcal{D}}_\mu a^a_\nu \right)^a - i \left[ \Phi^\dagger_a \mathbb{r}^a \Phi^{(1)} - \Phi^{(1)} \mathbb{r}^a \Phi^\dagger_a \right] = 0 \quad (3.7)$$

for $a = 1, 2, 3$ where $\Phi^{(1)}$ is defined in Eq (3.2). With our parametrization of the Higgs field they take the form

$$\mathcal{F}_a = \left( \mathring{\mathcal{D}}_\mu a^a_\nu \right)^a + 2 H_0 \varphi^a. \quad (3.8)$$

Here only the linear terms in $\Phi^{(1)}$ had to be taken into account since only the square of $\mathcal{F}_a$ appears in the gauge-fixed Lagrangean (see below).

The Fadeev–Popov determinant for this gauge condition is given by

$$\det \Delta_{a\bar{b}}^{FP} = \det \left[ \mathring{\mathcal{D}}^2 + \Phi^\dagger_a \Phi_a \right]_{a\bar{b}}$$

$$= \det \left[ \mathring{\mathcal{D}}^2 + H_0^2 \right]_{a\bar{b}}. \quad (3.9)$$

$^3$This has been done in exactly the same way in [13], but wrongly stated there in Eqs. (3.1) and (3.2) which should be replaced by our Eqs. (3.2) and (4.2).
Finally one obtains the gauge fixed Lagrangean density

\[
\mathcal{L}_{p}^{(2)} = \mathcal{L}^{(2)} - \frac{1}{2} F^a F^a
\]

\[
= - \frac{1}{2} \left( \tilde{\mathcal{D}}_\mu a_0 \right)^a \left( \tilde{\mathcal{D}}^\mu a_0 \right)^a + \frac{1}{2} H_0^2 a_0^a a_0^a + \frac{1}{2} \tilde{\mathcal{D}}_\mu a_i \left( \tilde{\mathcal{D}}^\mu a_i \right)^a - \frac{1}{2} H_0^2 a_i^a a_i^a
\]

\[
+ 2 \partial_\mu h \partial^\mu h - \xi^2 \left( 3 H_0^2 - 1 \right) h^2 - \frac{1}{2} A_i^a \dot{A}_i^a h^2
\]

\[
+ 2 \partial_\mu \varphi^a \partial^\mu \varphi^a - 2 H_0^2 \varphi^a \varphi^a
\]

\[
- \frac{1}{2} e^{abc} \tilde{F}^b_{ij} a_i^a a_j^a + 2 e^{abc} A_i^a \partial_i \varphi^b \varphi^c + 2 H_0 e^{abc} A_i^a \dot{a}_i^a \varphi^c
\]

\[
+ 4 h A_i^a \partial_i \varphi^a - 2 H_0 h A_i^a \dot{a}_i^a
\]

(3.10)

The background gauge has led to a decoupling of the time components of the gauge fields. Their contribution to the Lagrangean density

\[
\mathcal{L}^{(2)}_{a_0} = \frac{1}{2} a_0^a \left[ \tilde{\mathcal{D}}^2 + H_0^2 \right] a_0^b
\]

\[
= \frac{1}{2} a_0^a \Delta F^a_{ab} a_0^b
\]

(3.11)

leads to a fluctuation operator identical to that of the Fadeev–Popov ghost fields.

4 Partial Wave Expansion of the Fluctuations

In order to arrive at a suitable basis for our numerical computations we have to decompose our system of small fluctuations into partial waves with respect to the K spin \( \vec{K} = \vec{J} + \vec{I} \) combining angular momentum and isospin, which is conserved on the sphaleron background. We avoid a large amount of Clebsch–Gordan algebra by choosing a basis of tensor spherical harmonics which is constructed using cartesian vector operators (see e.g. [19]).

We define the following dimensionless operators

\[
\begin{align*}
\mathbf{J}_a^1 &= \hat{x}_a \\
\mathbf{J}_a^2 &= r \nabla_a \\
\mathbf{J}_a^3 &= e_{abc} x_b \nabla_c \equiv \Lambda_a \\
\mathbf{I}_j^b &= \hat{x}_j \lambda_b \\
\mathbf{I}_j^a &= x_j \partial^a \\
\mathbf{I}_j^a &= r \nabla_j \hat{x}_a \\
\mathbf{F}_j^a &= x_j \Lambda_a \\
\mathbf{F}_j^a &= \Lambda_j \hat{x}_a \\
\mathbf{F}_j^a &= r \nabla_j \Lambda_a \\
\mathbf{F}_j^a &= \Lambda_j r \nabla_a \\
\mathbf{F}_j^a &= \Lambda_j \Lambda_a
\end{align*}
\]

(4.1)

These can be used to obtain a suitable basis of tensor spherical harmonics with the spin–isospin transformation properties appropriate to the different fields:

\[
a_0^a = \sum_{K,M} \sum_{\nu=1}^3 s_{\nu}(r) \mathbf{J}_a^\nu Y_{K\nu}(\hat{x})
\]
\[
\begin{align*}
  a_{ja} &= \sum_{K,M} \sum_{\alpha=1}^{9} t_{\alpha}(r) \mathbf{I}^\alpha_{ja} Y_{KM}(\hat{x}) \\
  \hat{h} &= \sum_{K,M} h_{1}(r) Y_{KM}(\hat{x}) \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \\
  \varphi_{a} &= \sum_{K,M} \sum_{\alpha=1}^{3} p_{\alpha}(r) \mathbf{J}^{\alpha}_{a} Y_{KM}(\hat{x}) \left( \begin{array}{c} 0 \\ 1 \end{array} \right) .
\end{align*}
\] (4.2)

The algebraic properties of the operators \( \mathbf{J} \) and \( \mathbf{I} \) and of the tensor basis are discussed extensively in [20] and will not be repeated here. All manipulations have been performed using REDUCE, starting with the basic Lagrangian density (2.6) and using the substitutions (3.1), (3.2) and (4.2) and the gauge fixing based on (3.8).

The Lagrangean decomposes into contributions for each \( K \) spin. Varying these Lagrangeans one obtains a set of 16 linear coupled differential equations for the radial functions \( s_{a}, t_{a}, h_{a} \) and \( p_{a} \). According to their parity transformation properties the 16 amplitudes fall into two groups which do not couple between each other, an electric and a magnetic sector. The electric sector consists of the amplitudes \( t_{1}, t_{5}, t_{7}, t_{8}, p_{3}, h_{1} \) and \( s_{3} \) with parity \( \pi = (-1)^{K+1} \) and the magnetic sector contains the amplitudes \( t_{2}, t_{4}, t_{6}, t_{9}, p_{1}, p_{2}, s_{1} \) and \( s_{2} \) with parity \( \pi = (-1)^{K} \). Furthermore the ghost amplitudes \( s_{1}, s_{2} \) and \( s_{3} \) are not coupled to the other fields. They form the Fadeev–Popov sector. So altogether we have a \( 6 \times 6 \) and a \( 7 \times 7 \) subsystem for the gauge and Higgs fields. Only \( s_{1} \) and \( s_{2} \) form a coupled system in the Fadeev–Popov sector but we will not divide this sector furthermore. We will refer to the three sectors by a symbol \( \sigma \) which can have the values \( E, M \) and \( FP \) and the different coupled channels will be referred to by the two quantities \( \sigma \) for the fields and \( K \) for the partial wave.

The \( K \) spins \( K = 0 \) and \( K = 1 \) have to be considered separately: In the case \( K = 0 \) the \( \mathbf{J} \)– and \( \mathbf{I} \)– operators act on constant spherical harmonic \( Y_{00} \) so that only the operators \( \mathbf{I}^{1}, \mathbf{I}^{3}, \mathbf{P} \) and \( \mathbf{J}^{1} \) contribute; the amplitudes \( t_{2}, t_{4}, t_{6}, t_{7}, t_{8}, p_{2}, p_{3}, s_{2} \) and \( s_{3} \) have to be discarded and we have only 6 fluctuation equations. In the case \( K = 1 \) the action of the operators \( \mathbf{P}^{5} \) and \( \mathbf{P}^{8} \) on the spherical harmonics \( Y_{1M} \propto \hat{x} \) yields identical tensors and the same is true for \( \mathbf{I}^{5} \) and \( \mathbf{I}^{8} \):

\[
\begin{align*}
\mathbf{I}^{5}_{ia} \hat{x}_{j} &= -\mathbf{I}^{6}_{ia} \hat{x}_{j} = \delta_{ia} \hat{x}_{j} + \delta_{ij} \hat{x}_{a} - 2 \hat{x}_{i} \hat{x}_{a} \hat{x}_{j} \\
\mathbf{I}^{8}_{ia} \hat{x}_{j} &= -\mathbf{I}^{9}_{ia} \hat{x}_{j} = (\varepsilon_{iha} \hat{x}_{j} + \varepsilon_{ikj} \hat{x}_{a}) \hat{x}_{k} .
\end{align*}
\] (4.3)

One of the amplitudes \( t_{3} \) and \( t_{8} \) and one of the amplitudes \( t_{5} \) and \( t_{6} \) have to be discarded. We have chosen

\[
t_{6} = t_{8} = 0, \quad \text{for } K = 1 .
\] (4.4)

This elimination cannot be done simply in the general fluctuation equation but has to be performed already in the Lagrangean.

For the following general developments we will design by \( \tilde{\Psi} \) a column vector which is formed by the amplitudes of one of the coupled sectors. The fluctuation equations can
then be written in the general form

\[- \ddot{\Psi}_i + \dddot{\Psi}_i'' + \frac{2}{r} \dot{\Psi}_i' - m_i^2 \Psi_i = \tilde{V}_{ij} \dot{\Psi}_j.\]  \hspace{1cm} (4.5)

They are given explicitly in Appendix A. As one can see there the fluctuation equations have some shortcomings: the amplitudes do not display a unique centrifugal term, the potential \( \tilde{V}_{ij} \) is not symmetric and in the vacuum, i.e. for \( (f_A \rightarrow 1, H_0 \rightarrow 1, f_A' \rightarrow 0, H_0' \rightarrow 0) \) the amplitudes are still coupled. One has to find therefore a transformation

\[ \tilde{\Psi}_i = C_{ij} \Psi_j \]  \hspace{1cm} (4.6)

of the fields that brings them to a form

\[- \ddot{\Psi}_i + \dddot{\Psi}_i'' + \frac{2}{r} \dot{\Psi}_i' - m_i^2 \Psi_i - \frac{l_i(l_i + 1)}{r^2} \Psi_i = V_{ij} \Psi_j \]  \hspace{1cm} (4.7)

where the potential satisfies

\[ V_{ij} = V_{ji} \quad \text{und} \quad \lim_{r \to \infty} V_{ij} = 0. \]  \hspace{1cm} (4.8)

Such a transform must obviously exist since the Lagrangean (3.10) is a symmetric bilinear form in the fields.

The manipulations leading to this transform have been performed using REDUCE. The new amplitudes, the associated angular momenta and masses and the potential in the new basis are also given in Appendix A. The equations in this form constitute the basis of our further developments.

5 The Fluctuation Determinant \( \kappa \)

The rate for sphaleron transitions – based on the general theory of Langer – has been obtained by various authors ([7, 4, 8]) to be given by

\[ \Gamma/V = \frac{\omega}{2\pi} T^{-3} N_{tr} N_{rot} V_{rot} \exp(-E_{cl}/T) \kappa. \]  \hspace{1cm} (5.1)

Here \( \kappa \) and the other prefactors arise from the Gaussian functional integration of the quadratic fluctuations around the classical saddle point solution. \( \kappa \) is essentially (see below) the determinant corresponding to the fluctuation operator

\[ \Delta_{ij} = \left. \frac{\delta^2 S_f^{(2)}(\psi)}{\delta \psi_i \delta \psi_j} \right|_{\psi = \psi_{cl}}. \]  \hspace{1cm} (5.2)

The calculation of \( \kappa \) is, besides the general partial wave analysis, the main subject of this publication.
The prefactors labelled $tr$ and $rot$ are related to the existence of six zero modes due to the invariance of the theory with respect to translations and rotations which is broken by the sphaleron solution. These modes satisfy

$$\Delta_{ij} \Psi_j^{rot} = \Delta_{ij} \Psi_j^{tr} = 0. \quad (5.3)$$

The functional integration of these modes has to be treated separately (see [6, 4, 9, 11]). They have to be excluded in evaluating the fluctuation determinant. These prefactors and their calculation are discussed in Appendix B. The results are displayed in Fig. 2. They agree within a few percent with those of CLMW apart from factors $2^{3/2}$ arising from the different scale factors $M_W$ resp. $g_P$ used to make the radial variable dimensionless (see Eq. (2.4)).

As obvious from the discussion in the previous section K spin and parity invariance lead to a decomposition of the fluctuation operator into a direct sum of fluctuation operators within the different coupled channels $\Delta_{\sigma K}$. On account of Eq. (4.7) these operators take the form

$$\Delta_{(\sigma K)ij} \equiv \left( \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - m_i^2 - \frac{l_i(l_i + 1)}{r^2} \right) \delta_{ij} - V_{(\sigma K)ij}. \quad (5.4)$$

The angular momenta $l_i$, masses $m_i$, and potentials $V_{(\sigma K)ij}$ of these operators in the various channels are given in Appendix A. The fluctuation determinant decomposes accordingly. Therefore

$$\ln \kappa = \frac{1}{2} \sum_{K=0}^{\infty} (2K + 1) \left[ \ln \frac{\det \Delta^{(0)}_{MK}}{\det' \Delta^{(0)}_{MK}} + \ln \frac{\det \Delta^{(0)}_{EK}}{\det' \Delta^{(0)}_{EK}} - \ln \frac{\det \Delta^{(0)}_{FPK}}{\det' \Delta^{(0)}_{FPK}} \right]. \quad (5.5)$$

Here it has been used that the sector consisting of the time components of the gauge field has the same fluctuation operator as the Fadeev–Popov sector. The minus sign in front of the Fadeev–Popov contribution arises from a factor $(1 - 2)$ where the 1 correspond to the time components of the (real) gauge field and the $-2$ to the (complex) Fadeev–Popov ghosts. The apostrophes $'$ indicate that in the evaluation of the determinants the zero modes have to be removed and the negative squared frequency of the unstable mode has to be replaced by its absolute value.

### 6 The Numerical Procedure

Consider the contribution of one of the coupled channels $(\sigma K)$ to the logarithm of the fluctuation determinant $\kappa$

$$(\ln \kappa)_{\sigma K} \equiv \frac{1}{2} \ln \frac{\det \Delta^{(0)}_{\sigma K}}{\det' \Delta^{(0)}_{\sigma K}} \quad (6.1)$$

with the operators $\Delta_{\sigma K}$ and $\Delta^{(0)}_{\sigma K}$ as defined in Eq. (5.4). Since our discussion will be confined just to one channel the indices $K$ and $\sigma$ are omitted in the following from the operators $\Delta$, their Green functions, eigenfunctions and eigenvalues.
Let us define the Green function \( G_{ij}(r,r',\nu) \) by
\[
[\Delta_{ij} - \nu^2 \delta_{ij}] G_{jk}(r,r',\nu) = -\frac{1}{r^2} \delta(r-r') \delta_{ik}
\]  
(6.2)
and the analogous Green function \( G^{(0)} \) for the operator \( \Delta^{(0)} \). The solution to this equation can be written (using discrete notation formally) as
\[
G_{jk}(r,r',\nu) = \sum_{\alpha} \frac{\phi_{ji\alpha}(r) \phi_{k\alpha}^*(r')}{\lambda_\alpha^2 + \nu^2}
\]  
(6.3)
in terms of the orthonormalized eigenfunctions \( \phi_{ji\alpha} \) of the fluctuation operator \( \Delta \). The latin letters refer to the field components and the greek letters label the different eigenmodes. Analogous relations hold for the vacuum Green function. We define now a function \( F_{\sigma K}(\nu) \) by
\[
F_{\sigma K}(\nu) = \int drr^2 (G_{\alpha\alpha}(r,r,\nu) - G^{(0)}_{\alpha\alpha}(r,r,\nu)).
\]  
(6.4)
Using the expression (6.3) it can be written in terms of the eigenfrequencies \( \lambda_\alpha \) as
\[
F_{\sigma K}(\nu) = \sum_{\alpha} \left( \frac{1}{\lambda_\alpha^2 + \nu^2} - \frac{1}{\lambda^{(0)}_\alpha^2 + \lambda^2} \right).
\]  
(6.5)
Integrating this expression over \( \nu \, d\nu \) from \( \epsilon \) to \( \Lambda \) yields
\[
\int_{\epsilon}^{\Lambda} d\nu \, \nu F_{\sigma K}(\nu) = \frac{1}{2} \sum_{\alpha} \left( \ln \left( \frac{\lambda^{(0)}_\alpha^2 + \epsilon^2}{\lambda^2_\alpha + \epsilon^2} \right) - \ln \left( \frac{\lambda^{(0)}_\alpha^2 + \Lambda^2}{\lambda^2_\alpha + \Lambda^2} \right) \right).
\]  
(6.6)
Taking into account the fact that the operators \( \Delta \) and \( \Delta^{(0)} \) and therefore their eigenvalues \( \lambda^2 \) are linear in the \( m^2_q \) this expression is a Pauli–Villars regulated version of \( (\ln \kappa)_{\sigma K} \), the second term in the sum being obtained by replacing all \( m^2_q \) by \( m^2_q + \Lambda^2 \). Of course we should let \( \Lambda \to \infty \) after the expressions have been renormalized and the integral has become ultraviolet convergent. Renormalization will be discussed in section 7.

The lower limit \( \epsilon \) should of course be set equal to zero. We have introduced it because for \( K = 1 \) we have six zero modes and the limit \( \epsilon \to 0 \) then obviously does not exist. We will deal with this problem below (see section 8) when we discuss the removal of these modes.

A further problem arises from the unstable mode which leads to a pole in the region of integration and requires a precise definition of the integration contour. This problem will be tackled also in section 8.

Leaving aside these problems for the time being we have the “naive” relation
\[
(\ln \kappa)_{\sigma K} = \int_{0}^{\infty} d\nu \, \nu F_{\sigma K}(\nu).
\]  
(6.7)
In order to evaluate \( F_{\sigma K}(\nu) \) the Green functions have to be determined. We will not use their expansion with respect to eigenfunctions (see Eq. (6.3)), but use another standard method:
Let $f_n^{\alpha \pm}$ be the solutions of the homogeneous differential equations

$$
\left( \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{l_n(l_n+1)}{r^2} - \kappa_n^2 \right) f_n^{\alpha \pm}(r) = V_{nn'}(r) f_n^{\alpha \pm}(r);
$$

(6.8)

where $\kappa_n$ has been defined as

$$
\kappa_n \equiv \sqrt{\nu^2 + m_n^2}
$$

(6.9)

with $m_n = 1$ or $\xi$ depending on the field component $n$. $f_n^{\alpha +}$ designs the solution regular as $r \to \infty$ and $f_n^{\alpha -}$ the solution regular as $r \to 0$. The index $n$ labels again the field components, the greek letters will in the following label a set of linearly independent solutions of the system; there of course as many such solutions as there are field components.

For the vacuum ($V_{nn'} = 0$) these equations are solved by modified Bessel functions with the argument $z_n = \kappa_n r$, which are defined here (slightly deviating from [17] ) as

$$
i_l(z) = \sqrt{\frac{\pi}{2z}} I_{l+\frac{1}{2}}(z)$$

$$
k_l(z) = \sqrt{\frac{2}{\pi z}} K_{l+\frac{1}{2}}(z).
$$

(6.10)

Their Wronskian is given by

$$
W(i_{l_n}, k_{l_n}) = \kappa_n \left(i_{l_n}(z_n) k'_{l_n}(z_n) - k_{l_n}(z_n) i'_{l_n}(z_n) \right) = -\frac{1}{\kappa_n r^2}.
$$

(6.11)

The behaviour of the solutions $f_n^{\alpha \pm}$ for $r \to 0$ and $r \to \infty$ is analogous to the one of these free solutions:

$$
i_l \propto z^l \quad \text{for } z \to 0 \quad \text{and} \quad i_l \propto \frac{e^{-\xi z}}{z} \quad \text{for } z \to \infty.
$$

(6.12)

It is convenient to split off the Bessel functions from the solutions $f_n^{\alpha \pm}$ via

$$
f_n^{\alpha \pm}(r) = \left[ k_n^\alpha + h_n^{\alpha \pm}(r) \right] b_l^{\pm}(z_n)
$$

(6.13)

with

$$
b_l^- = i_l \quad \text{and} \quad b_l^+ = k_l.
$$

(6.14)

On account of the boundary conditions for the functions $f_n^{\alpha \pm}$ the functions $h_n^{\alpha \pm}$ tend to a constant as $r \to \infty$. The choice

$$
\lim_{r \to \infty} h_n^{\alpha \pm}(r) = \lim_{r \to \infty} h_n^{\alpha \pm}(r) = 0
$$

(6.15)

\footnote{The letter $\kappa$ has been introduced previously to denote the fluctuation determinant. Since $\kappa_n$ as defined here will always appear with an index there should be no confusion.}

\[12\]
determines the normalization of the amplitudes \( f_n^{a\pm} \) in such a way that their Wronskian satisfies
\[
W_{\alpha\beta}^{a}(r) = \sum_{n} \left( f_n^{a+} \frac{d}{dr} f_n^{\beta-} - f_n^{\beta-} \frac{d}{dr} f_n^{a+} \right) = \frac{1}{\kappa_{n} r^2} \delta_{\alpha\beta}.
\] (6.16)

For \( r \to 0 \) the centrifugal barriers for the \( f_n^{a\pm} \) are different from the ones for the free solutions. Effectively they differ at most by two units from those, so that the functions \( \delta_{n}^{a} + h_{n}^{a\pm} \) behave as \( r^{0}, r^{\pm1} \) or \( r^{\pm2} \) in this limit; most of the typical \( r^{1n} \) behaviour that is dangerous for numerical calculations in high partial waves has however been taken out by splitting off the free solutions.

For our new functions \( h_{n}^{a\pm} \) equations (6.8) yield the inhomogeneous differential equations
\[
\left[ \frac{d^2}{dr^2} + 2 \left( 1 + \kappa_{n} \frac{b_{n}^{a'}(z_n) d}{b_{n}(z_n) dr} \right) \right] h_{n}^{a\pm}(r) = V_{n,n'} \left[ \delta_{n}^{a} + h_{n}^{a\pm}(r) \right] \frac{b_{n'}(z_{n'})}{b_{n}(z_{n})}.
\] (6.17)

They can be integrated numerically using the Nyström method (Runge Kutta integration for second order differential equations [21]). Some numerical details are discussed in Appendix C.

From these solutions the Green function defined in Eq. (6.2) is now obtained as
\[
G_{nn'}(r, r'; \nu) = \Theta(r - r') f_{n}^{a'}(r) C_{\alpha\beta}^{-1} f_{n'}^{\beta'}(r') + \Theta(r' - r) f_{n}^{a'}(r) C_{\alpha\beta}^{-1} f_{n'}^{\beta'}(r')
\] (6.18)

where the coefficients \( C_{\alpha\beta} \) are related to the Wronskian of the amplitudes \( f_{n}^{a\pm} \) as
\[
C_{\alpha\beta} = r^2 W_{\alpha\beta}^{a}(r) = \frac{1}{\kappa_{a} \delta_{\alpha\beta}}.
\] (6.19)

With these preliminaries we obtain for the trace of the Green functions at \( r' = r \):
\[
G_{nn}(r, r, \nu) = \sum_{n} \kappa_{n} i_{n}(z_{n}) k_{n}(z_{n})
\]
\[
G_{nn}(r, r, \nu) = \sum_{a,n} \kappa_{a} f_{n}^{-a}(r) f_{n}^{a}(r)
\] (6.20)

Therefore, using Eq. (6.13), the function \( F_{\sigma K}(\nu) \) can be calculated for each system of coupled channels (\( \sigma K \)) as
\[
F_{\sigma K}(\nu) = \int_{0}^{\infty} dr' \sum_{a,n} \kappa_{a} \left[ \delta_{n}^{a} \left( h_{n}^{a^{-}} + h_{n}^{a^{+}} \right) + h_{n}^{a^{-}} h_{n}^{a^{+}} \right] i_{n} k_{n}
\] (6.21)

once the functions \( h_{n}^{a\pm} \) for that channel have been found numerically.

The total fluctuation determinant is then obtained as
\[
\ln \kappa = \int_{0}^{\infty} d\nu \nu F(\nu)
\] (6.22)

with
\[
F(\nu) = \sum_{K=0}^{\infty} (2K + 1)(F_{EK}(\nu) + F_{MK}(\nu) - F_{FPK}(\nu))
\] (6.23)

This expression is still formal, we have to discuss the treatment of zero and unstable modes and of renormalization.
7 Renormalization

As has been discussed by Carson et al. [11] renormalization requires, in the $T \to \infty$ limit discussed here, the replacement of the Higgs vacuum expectation value by its temperature dependent value and the subtraction of the tadpole graphs with Higgs fields as external legs.

In order to do this we discuss first the relation of our expression for $\ln \kappa$ of Eqs. (6.21)–(6.23) to a Feynman graph expansion. The function $F(\nu)$ is obtained as the trace of the Green function of the "small fields" $\eta_i$, $\eta$ and the Fadeev–Popov fields in the external potential generated by the classical fields. It has been obtained here by decomposing this Green function first into partial waves and then summing over the individual partial wave contributions. This Green function may also be expanded with respect to the external potential; formally

$$G_{ij}(\tilde{x}, \tilde{x}', \nu) = \langle \tilde{x}, i | \sum_{n=0}^{\infty} \left[ \frac{-1}{p^2 + m^2 + \nu_2} V(x) \right]^n \frac{1}{p^2 + m^2 + \nu_2} | \tilde{x}', j \rangle$$  \hspace{1cm} (7.1)

where bold face letters indicate operators and/or matrices. Then

$$F(\nu) = \int d^3 \nu G_{\nu}(\tilde{c}, \tilde{x}, \nu)$$  \hspace{1cm} (7.2)

and therefore the fluctuation determinant is obtained via

$$\ln \kappa = \int d\nu \nu F(\nu)$$

$$= \sum_{n=1}^{\infty} \frac{1}{2n} \int d^3 x < \tilde{x}, i | \left[ \frac{-1}{p^2 + m^2} V(x) \right]^n | \tilde{x}, i >$$  \hspace{1cm} (7.3)

which is just the normal Feynman graph expansion of the 1-loop effective action for a 3 dimensional theory.

The tadpole contributions to $F(\nu)$ are therefore obtained as that part of the first order contribution in the external potential that is generated by the classical Higgs field. These terms are easily recognized in the second order Lagrangian $L^{(2)}_{\phi \bar{\phi}}$ presented in Eq. (3.10) as those containing $\Phi^{\dagger}_{\eta i}$, $\Phi_{\eta i}^{\dagger}$ and $\Phi_{\eta i}^{\dagger} P \phi_{\eta i}$. In the partial wave potential given explicitly in Appendix A these terms appear in the diagonal elements and are proportional to $(H_0^2 - 1)$.

The tadpole terms can therefore be subtracted either in the single partial waves as the first order perturbative contribution generated be the $(H_0^2 - 1)$ terms or directly from $F(\nu)$.

For each partial wave the differential equation for the radial wave function may be transformed into an integral equation and this integral equation can be used for a perturbative expansion; this has been discussed extensively in [14]. The first order contribution of such an expansion is obtained as

$$\langle \ln \kappa \rangle^{(1)}_{\phi R} = - \int_0^\infty d\nu \nu^2 \int_0^\infty dr r^2 \int_0^\infty dr' r'^2 \tilde{V}_{\nu}(r \nu) i_{n}(\kappa \nu r' \nu) k_{n}^2(\kappa \nu r' \nu)$$  \hspace{1cm} (7.4)
where \( r' \approx \min(r, r') \) and \( r'' \approx \max(r, r') \). The tilde over the potential indicates that only the terms proportional to \((H_0^2 - 1)\) are to be included.

One of the radial integrations can be performed analytically so that

\[
(ln \kappa)^{(1)}_{\nu K} = \frac{1}{2} \int_0^\infty d\nu \nu \int_0^\infty dr \, r^2 \tilde{V}_n(r) \left[ i_{n-1} k_n - k_{n+1} i_n \right].
\]  

(7.5)

Either by summing up the partial wave contributions or by calculation of the Feynman type graphs according to Eq. (7.1) the partial tadpole contribution to \( F(\nu) \) is obtained as

\[
F_{\text{tad}}(\nu) = -\frac{1}{2} \int_0^\infty dr \, r^2 (H_0^2(r) - 1) \left[ \frac{9}{\kappa W} + \frac{3\xi^2}{2\kappa_H} + \frac{3\xi^2}{2\kappa_W} \right]
\]

(7.6)

where \( \kappa_n \) has been defined in Eq. (6.9). The renormalized value of \( \ln \kappa \) is therefore obtained from

\[
F_{\text{ren}}(\nu) = F(\nu) - F_{\text{tad}}(\nu)
\]

(7.7)

where the subtraction can be done either in the partial waves using Eq. (7.5) or in the full amplitude using Eq. (7.6). The \( \nu \) integration of \( F_{\text{ren}}(\nu) \) is now ultraviolet convergent, i.e. the upper limit of integration \( \Lambda \) introduced in Eq. (6.6) which serves also as a Pauli-Villars regulator can be sent to \( \infty \).

8 Zero and Unstable Modes

Besides the question of renormalization we have also postponed the treatment of the zero and unstable modes.

The zero modes should be removed from the fluctuation determinant. Equation (6.6) shows that their contribution to \((\ln \kappa)_{\varepsilon 1}\) and \((\ln \kappa)_{\lambda 1}\) is in both cases \((-1/2) \ln(\lambda_0^2 + \epsilon^2)\) before \( \epsilon \) goes to 0. Of course \( \lambda_0 = 0 \) here and the limit does not exist. But we have to remove just these contributions. Therefore the fluctuation determinant without the zero modes is obtained as

\[
\ln \kappa = \lim_{\epsilon \to 0} \left( \int_0^\infty d\nu \nu F_{\text{ren}}(\nu) + 6 \ln(\epsilon) \right).
\]

(8.1)

The function \( F_{\text{ren}}(\nu) \) will of course behave as \( 6/\nu^2 \) due to the zero modes so that the limit exists. It has to do so also in the numerical evaluation; this represents a good cross check. Of course \( \epsilon \) is a quantity of dimension energy and indeed removing the zero modes makes \( \kappa \) a quantity of dimension \((\text{energy})^6\). This has to be taken into account when comparing results if different length and energy units are used. We have used units of \( M_W^{-3} \) for the radial variable; our unit for the eigenvalues of the second order differential operators \( \Delta \) is therefore \( M_W^0 \), their eigenfrequencies and therefore also the variables \( \nu \) and \( \epsilon \) are therefore in units \( M_W \).

The unstable mode is not to be removed but, according to the general theory [6, 7] to be replaced by its absolute value. If its eigenvalue is denoted by \( \lambda_0^2 = -\omega_0^2 < 0 \) then it leads to a pole in \( F(\nu) \) at \( \nu^2 = \omega_0^2 > 0 \) and therefore in the region of integration of Eq. (6.22). It would contribute a term \((-1/2) \ln(-\omega_0^2)\) and the minus sign in the logarithm
should be removed. This can be done simply by evaluating the integral as a principal value integral.

Since doing singular principal value integrals numerically is a delicate operation we have subtracted the pole from the integrand and done its integration analytically. We have used the identity

\[
(\ln \kappa) = \int_0^\infty d\nu \nu F(\nu) \\
= \int_0^\infty d\nu \nu \left[ F_{\text{ren}}(\nu) - \frac{1}{\nu^2 - \omega_1^2} + \frac{1}{\nu^2 + \sigma^2} \right] - \ln(\omega_1/\sigma).
\] (8.2)

The third term in the parenthesis was added in order not to spoil the ultraviolet convergence of the integral. Since (8.2) is an identity the value of \( \sigma \) is in principle arbitrary. We have chosen \( \sigma = 1 \) in units of \( M_W \) for convenience.

For easier notation have treated the removal of the zero modes and of the unstable mode singularity separately. It is of course understood that both prescriptions Eqs. (8.1) and (8.2) are applied simultaneously.

9 Results

We have now discussed the principles of our numerical procedure for calculating the fluctuation determinant. Before presenting the results we will discuss some specific details of our numerical evaluation.

The first step was the Runge–Kutta integration of the partial wave differential equations (6.17) in each channel and the numerical integration of the exact trace as presented in Eqs. (6.21)–(6.23). The numerical details are discussed in Appendix C.

In order to obtain the function \( F(\nu) \) we had to sum over all partial waves \( K \) within the various sectors \( \sigma \) and then to perform the summation over \( \sigma \). In order to have a check on the \( K \) summation we have considered the asymptotic behaviour at large \( K \) of the terms in this sum. For this purpose it is sufficient to consider the perturbative contribution of first order in the potentials \( V_{ij} \); higher order contributions will decrease faster. The first order contributions to the sum in Eq (6.23) have the form

\[
(2K + 1)F_{\sigma K}^{(1)} = (2K + 1) \int_0^\infty dr \frac{r^3}{2} V_{nn}(r) [i_{i_{n-1}k_{i_{n}} - k_{i_{n+1}i_{n}}]}.
\] (9.1)

Using the uniform asymptotic expansion of the Bessel functions at large \( K \) [17] one finds that this expression behaves as \( K^{-2} \) asymptotically. This determines then the convergence behaviour of the \( K \) summation. It has been checked numerically to a good accuracy. Since we know the leading behaviour, we can extrapolate the terms to arbitrary value of \( K \). We have done so by fitting the last five calculated values with a power behaviour

\[
(2K + 1)F_{\sigma K}^{(1)} \simeq \frac{C_2}{(2K + 1)^2} + \frac{C_3}{(2K + 1)^3}
\] (9.2)
To the sum extended to some value $K_{max}$

$$F(K_{max}, \nu) = \sum_{K=1}^{K_{max}} \sum_{\sigma} (2K + 1) F_{\sigma K}$$

(9.3)

we have then added the sum from $(K_{max} + 1)$ to $\infty$ using the fit (9.2). Fig. 3 shows
the partial sums as well as the sums completed using the extrapolated values. One sees
that the complete sum becomes independent of $K_{max}$ already at moderate values of this
variable. This shows that the extrapolation procedure is reliable.

The function $F(\nu)$ is displayed in Fig. 4 for $\nu = 1$. The pole contribution of the unstable
mode is already subtracted here according to Eq. (8.2). The dashed line shows the full
function $F(\nu)$, the dash-dotted line the tadpole contribution. Note that this contribution
is determined analytically in absolute normalization. So the fact that both curves approach
each other as $\nu \to \infty$ checks also the absolute normalization of the unsubtracted $F(\nu)$.
The full line shows $F_{\text{ren}}(\nu)$ together with the asymptotic estimates at small and large $\nu$
(dotted lines). The behaviour at small $\nu$ is normalized absolutely; it is determined by the
zero modes to be $6/\nu^2$.

The $\nu$ integration was performed numerically up to $\nu_{max} \simeq 2.5$, then an asymptotic
part was added to the integral by extrapolating $F_{\text{ren}}$ as $C_i/\kappa_i^3 + D_i/\kappa_i^5$. The results for $\ln \kappa$
are given in Table 1 for various values of $\xi = M_H/M_W$ in the scale $M_W$. They are plotted
in Fig. 5 together with previous results and estimates in the scale $g v$, i.e. after subtracting
$6 \ln 2$ from the values of Table 1. Our calculation stops at $M_H/M_W = 2$ for a technical
reason: Above this value the leading asymptotic behaviour $\exp(-\kappa_H v)$ of the fluctuation
$h_1$ of the Higgs field becomes dominated – through a cross term in the potential – by a
gauge field contribution which behaves as $\exp(-(M_W + \kappa_W) v)$. So for $M_H > 2M_W$ the
boundary conditions for this function have to be modified. If the Higgs mass should turn
out to be larger than this value we would have to deal with this complication.

We think that the main uncertainty of our results comes from the extrapolation of
$F_{\text{ren}}(\nu)$ to $\nu = \infty$. We estimate the error of this asymptotic contribution to be around
10% yielding an typical error of 0.3 for $\ln \kappa$ and we think that this is a conservative estimate.

Our results differ considerably from the ones of CLMW [11]. There is one point which
could lead to a difference on physical grounds: we have used a different gauge in the classical
 sphaleron ansatz (see below Eq. (2.17)). While the sphaleron transition rate must be gauge
invariant, a difference in the classical configuration could modify the zero mode prefactors
and then require also a compensating change in the fluctuation determinant. However
– as mentioned above – the prefactors agree and therefore should also the fluctuation
determinants.

The methods used by CLMW and by us differ also considerably. Since we work with
the Euclidean Green function, we have no difficulties with the fact that the spectrum of
the various fluctuation operators is continuous (except for zero and unstable modes and a
few further bound states). The Schwinger proper time method as used by CLMW requires
a discrete spectrum which has to be created artificially by introducing a space boundary

\footnote{We thank L. McLerran for pointing this out to us.}
(a kind of “bag”) of radius $R$. The calculation requires then a limit $R \to \infty$. This limiting procedure is absent in our method. This makes the algorithm much faster. Furthermore the method contains more internal consistency checks. Of course this does not necessarily imply that our results are the correct ones but at present we have not found any reason to doubt them. We hope that the discrepancy can be settled in the near future, possibly by another calculation.

Comparing to the DPY approximation we find that our results come, at $M_H > M_W$, much closer to this approximation than the ones of CLMW; they show a different trend for small Higgs masses, however. The validity of the DPY approximation has been discussed by Carson [12] for a one-dimensional sphaleron where the exact fluctuation determinant is known analytically. He finds only fair agreement between the exact results and the approximation. Furthermore (see [22]) it is not obvious in which way the convergence of gradient expansions (and the DPY approximation falls under this category) in $1 + 1$ and $3 + 1$ dimensions can be compared.

In conclusion we have presented here a new set of fluctuation equations for the the electroweak sphaleron at $\Omega W = 0$. We have used it to evaluate the fluctuation determinant. We obtain the result that the quantum corrections lead to an enhancement of the sphaleron transition with respect to the estimate obtained from the classical saddle point solution.
A The Partial Wave Fluctuation Equations

The equations of motion have been obtained as follows: the Ansatz for the fluctuation amplitudes was inserted into the second order fluctuation Lagrangean which itself had been obtained using REDUCE from the general gauge-fixed Lagrangean. A second REDUCE step used the various algebraic properties of the operators $I$ and $J$ given in [20] in order to obtain a Lagrangean quadratic in the amplitudes $t_a$, $s_a$, $p_a$ and $h$. The Lagrangean before gauge fixing had already be obtained in [13], Checking its gauge invariance and the presence of all zero modes presents a good test on its correctness.

The equations of motion obtained in this way are – for $K > 1$ – given by

$$r^2 t_1'' + 2r t_1' - r^2 t_1 = t_1(r^2 H_0^2 + 2 f_A^2 + K^2 + 2) - 2t_2 f_A K^2 + 2t_3(2r f_A' - 2 f_A - K^2)$$
$$- 2t_4 K^2(r f_A' - f_A - 1) - 2t_5 K^2(r f_A' - f_A + 1) + 2p_1 r^2 H_0'$$

$$r^2 t_2'' + 2r t_2' - r^2 t_2 = -2t_1 f_A + t_2(r^2 H_0^2 + f_A^2 + K^2 + 1) - 2t_3(r f_A' - f_A - 1)$$
$$+ 2t_4(r f_A' - f_A - K^2 + 1) - 2t_5(r f_A' - f_A + 1) + 2p_2 r^2 H_0'$$

$$r^2 t_3'' + 2r t_3' - r^2 t_3 = -2t_1 - 2t_2(r f_A' - f_A + 1) + t_3(r^2 H_0^2 + 2 f_A^2 + K^2)$$
$$- t_6[(3f_A - 1)(f_A - 1) + 2f_A K^2] + t_6(3f_A - 1)(f_A - 1)$$
$$- p_3 r H_0(f_A - 1)$$

$$r^2 t_6'' + 2r t_6' - r^2 t_6 = \{ -2t_1(r f_A' - f_A + 1) - 2t_2(r f_A' - f_A + K^2 + 1)$$
$$- t_3(r f_A' - f_A' + 2f_A K^2 - 1) + t_6 [2K^2 f_A(f_A - 1) + (3f_A - 1)(f_A - 1)]$$
$$+ t_6[K^2(r^2 H_0^2 + f_A^2 + K^2 - 1) - (3f_A - 1)(f_A - 1)]$$
$$- p_3 r H_0(f_A - 1) - p_2 r H_0(f_A - 1)]/K^2$$

$$r^2 t_9'' + 2r t_9' - r^2 t_9 = \{ -2t_1(r f_A' - f_A + 1) - 2t_2(r f_A' - f_A + 1) - t_3(f_A^2 - 1)$$
$$+ t_6[2K^2 f_A(f_A - 1) - (3f_A - 1)(f_A - 1)]$$
$$+ t_6[K^2(r^2 H_0^2 + f_A^2 + K^2 - 1) + (3f_A - 1)(f_A - 1)]$$
$$- p_3 r H_0(f_A - 1) - p_2 r H_0(f_A - 1)]/K^2$$

$$r^2 p_1'' + 2r p_1' - r^2 p_1 = \{ 4t_1 r^2 H_0^2 + 4t_3 r H_0(f_A - 1) - 2t_5 r H_0 K^2(f_A - 1)$$
$$- 2t_4 r H_0 K^2(f_A - 1) - 2p_2 K^2(f_A + 1)$$
$$+ p_1[r^2 H_0^2 - 1 + 2r^2 H_0^2 + (f_A + 1)^2 + 2K^2])/2$$

$$r^2 p_2'' + 2r p_2' - r^2 p_2 = \{ 4t_1 r^2 H_0^2 - 2t_3 r H_0(f_A - 1) + 2t_5 r H_0(f_A - 1)$$
$$- 2t_4 r H_0(f_A - 1) - 2p_1(f_A + 1)$$

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\begin{equation}
+ p_2 [r^2 \xi^2 (H_0^2 - 1) + 2r^2 H_0^2 + f_A^2 + 2 \mathcal{K}^2 - 1] / 2
\end{equation}

for the magnetic sector, by

\begin{align*}
r^2 t_0'' &+ 2r t_4' - r^2 t_4 \\
&= t_4 (r^2 H_0^2 + f_A^2 + \mathcal{K}^2 + 1) - 2t_5 (r f_A' - f_A + 1) \\
&+ 2t_7 (r f_A' - f_A - \mathcal{K}^2 + 1) + 2t_6 (r f_A' - f_A + 1) + 2p_3 r^2 H_0^2 \\
r^2 t_5'' &+ 2r t_5' - r^2 t_5 \\
&= -2t_4 (r f_A' - f_A + 1) + t_5 (r^2 H_0^2 + 2 f_A^2 + \mathcal{K}^2) \\
&- t_7 (3 f_A - 1)(f_A - 1) - t_6 [(3 f_A - 1)(f_A - 1) + 2 f_A \mathcal{K}^2] \\
&- p_5 r H_0 (f_A - 1) \\
r^2 t_7'' &+ 2r t_7' - r^2 t_7 \\
&= \{2t_4 (r f_A' - f_A - \mathcal{K}^2 + 1) + t_6 (f_A^2 - 1) \\
&+ t_7 [2 \mathcal{K}^2 f_A (f_A - 1) - (3 f_A - 1)(f_A - 1)] \\
&- p_5 r H_0 (f_A - 1) - h_1 r H_0 (f_A - 1)] / \mathcal{K}^2 \\
r^2 t_8'' &+ 2r t_8' - r^2 t_8 \\
&= \{ -2t_4 (r f_A' - f_A + 1) - t_5 (f_A^2 + 2 f_A \mathcal{K}^2 - 1) \\
&- t_7 [2 \mathcal{K}^2 f_A (f_A - 1) + (3 f_A - 1)(f_A - 1)] \\
&- p_5 r H_0 (f_A - 1) + h_1 r H_0 (f_A - 1)] / \mathcal{K}^2 \\
r^2 p_3'' &+ 2r p_3' - r^2 p_3 \\
&= \{4t_4 r^2 H_0^2 - 2t_5 r H_0 (f_A - 1) + 2t_7 r H_0 (f_A - 1) \\
&+ t_6 [2 \mathcal{K}^2 (H_0^2 - 1) + 2r^2 H_0^2 + f_A^2 + 2 \mathcal{K}^2 - 1] \\
&+ 2h_1 (f_A - 1)] / 2 \\
r^2 h_1'' &+ 2r h_1' - r^2 h_1 \\
&= \{ -4t_5 r H_0 (f_A - 1) - 2t_7 r H_0 \mathcal{K}^2 (f_A - 1) + 2t_8 r H_0 \mathcal{K}^2 (f_A - 1) \\
&+ 2p_5 \mathcal{K}^2 (f_A - 1) + (f_A - 1)^2 + 2 \mathcal{K}^2] / 2 \}
\end{align*}

for the electric sector and by

\begin{align*}
r^2 s_1'' &+ 2r s_1' - r^2 s_1 \\
&= s_1 (r^2 H_0^2 + 2 f_A^2 + \mathcal{K}^2) - 2s_2 f_A \mathcal{K}^2 \\
r^2 s_2'' &+ 2r s_2' - r^2 s_2 \\
&= -2s_1 f_A + s_2 (r^2 H_0^2 + f_A^2 + \mathcal{K}^2 - 1) \\
r^2 s_3'' &+ 2r s_3' - r^2 s_3 \\
&= s_3 (r^2 H_0^2 + f_A^2 + \mathcal{K}^2 - 1) \}
\end{align*}

for the Fadeev–Popov sector.
As one sees these equations do not have the form of Eq. (4.7) with a symmetric potential $V_{ij}$. Such a form is however needed on order to apply the Green function formalism in a convenient way.

The transformations that bring the equations of motion to a symmetric form have been found by an – in fact very simple – educated guessing. They are

\[
\begin{align*}
t_1 &= \frac{K(K + 1)}{\sqrt{2K + 1}} \left[ \sqrt{\frac{(K + 2)(K + 1)}{2K + 3}} T_1 - \frac{K}{\sqrt{2K + 1}} T_2 \right. \\
& \quad - \left. \frac{K + 1}{\sqrt{2K + 3}} T_3 - \frac{K}{\sqrt{2K - 1}} T_6 \right] \\
t_2 &= \frac{K}{\sqrt{(2K + 1)(2K + 3)}} \left[ (K + 1) T_3 - \sqrt{(K + 1)(K + 2)} T_1 \right] \\
& \quad + \frac{K + 1}{\sqrt{(2K - 1)(2K + 1)}} \left[ \sqrt{K(K - 1)} T_6 - K T_2 \right] \\
t_3 &= \frac{K(K + 1)}{\sqrt{(2K - 1)(2K + 1)}} \left[ T_2 + \sqrt{\frac{K}{K - 1}} T_6 \right] \\
& \quad - \frac{K(K + 1)}{\sqrt{(2K + 1)(2K + 3)}} \left[ T_3 + \sqrt{\frac{K + 1}{K + 2}} T_1 \right] \\
t_6 &= \frac{K + 1}{\sqrt{(2K + 1)(2K - 1)}} \left[ T_2 + \sqrt{\frac{K}{K - 1}} T_6 \right] \\
& \quad + \frac{K}{\sqrt{(2K + 1)(2K + 3)}} \left[ T_3 + \sqrt{\frac{K + 1}{K + 2}} T_1 \right] \\
t_9 &= T_9 \\
p_1 &= \frac{K(K + 1)}{\sqrt{2K + 1}} \left[ \sqrt{K} P_2 - \sqrt{K + 1} P_1 \right] \\
p_2 &= \frac{K(K + 1)}{2K + 1} \left[ \sqrt{K} P_1 + \sqrt{K + 1} P_2 \right]
\end{align*}
\]  

for the magnetic amplitudes and

\[
\begin{align*}
t_4 &= \frac{1}{\sqrt{2K + 1}} \left[ \frac{1}{\sqrt{K + 1}} T_5 - \frac{1}{K} T_4 \right] \\
t_5 &= \frac{1}{\sqrt{2K + 1}} \left[ \sqrt{K - 1} T_8 - \frac{1}{\sqrt{K + 2}} T_7 \right] \\
t_7 &= \frac{K(K + 1)\sqrt{2K + 1}}{1} \left[ \sqrt{K T_4} + \sqrt{K + 1} T_5 \right] \\
t_8 &= \frac{1}{\sqrt{2K + 1}} \left[ (K + 1)\sqrt{K + 1} T_7 + \frac{1}{K \sqrt{K - 1}} T_8 \right] \\
p_3 &= \frac{1}{\sqrt{K(K + 1)}} P_3 \\
h_1 &= H_1 
\end{align*}
\]
The n-tuples formed by the amplitudes of the various sectors for the electric amplitudes. For the Faddeev-Popov amplitudes the transformations are

\[
\begin{align*}
\psi_M^1 &= (T_1, T_2, T_3, T_4, T_5), \\
\psi_M^2 &= (T_1, T_2, T_3, T_4, T_5), \\
\psi_M^3 &= (T_1, T_2, T_3, T_4, T_5).
\end{align*}
\]
\[
V_{13}^M = -\sqrt{\frac{K+2}{K+1}} \left\{ \frac{2f_A'}{r(2K+3)} \right. \\
- \frac{f_A-1}{r^2(2K+3)(2K+1)} [4(f_A+1)K + f_A + 3] \left. \right\}
\]

\[
V_{14}^M = 0
\]

\[
V_{15}^M = -\sqrt{\frac{K+2}{(2K+1)(2K+3)(K+1)}} \\
\times \left\{ \frac{2f_A'K}{r} - \frac{f_A-1}{r^2} [2(f_A+1)K - f_A + 1] \right\}
\]

\[
V_{16}^M = -\sqrt{\frac{K+2}{2K+3}} \frac{1}{r} [2rH_0' - (f_A-1)H_0]
\]

\[
V_{17}^M = 0
\]

\[
V_{23}^M = \sqrt{\frac{2(f_A-1)^2}{(2K+3)(2K-1)r^2(2K+1)}}
\]

\[
V_{24}^M = \sqrt{\frac{K-1}{K}} \left\{ \frac{2f_A}{r(2K-1)} - \frac{f_A-1}{r^2(4K^2-1)} [4(f_A+1)K + 3f_A + 1] \right\}
\]

\[
V_{25}^M = \frac{1}{\sqrt{4K^2-1}} \left\{ \frac{2f_A'(K+1)}{rK} \\
+ \frac{f_A-1}{r^2K} [2(f_A-1)K^2 + (f_A-3)K - 3f_A + 1] \right\}
\]

\[
V_{26}^M = 0
\]

\[
V_{27}^M = -\frac{1}{\sqrt{K(2K-1)r^3}} [2rH_0'K + (f_A-1)H_0(K-1)]
\]

\[
V_{34}^M = -\sqrt{\frac{K(K-1)(2K+3)}{2K-1}} \frac{1}{r^2(2K+1)}
\]

\[
V_{35}^M = \frac{1}{\sqrt{(2K+1)(2K+3)}} \times \left\{ \frac{2f_A'K}{r} + \frac{f_A-1}{r^2(K+1)} [2(f_A-1)K^2 + (3f_A-1)K - 2f_A + 2] \right\}
\]

\[
V_{36}^M = \frac{1}{\sqrt{(K+1)(2K+3)r}} \frac{1}{r} [2rH_0'(K+1) + (f_A-1)H_0(K+2)]
\]

\[
V_{37}^M = 0
\]

\[
V_{45}^M = -\sqrt{\frac{K-1}{K(4K^2-1)}} \left\{ \frac{2f_A'(K+1)}{r} - \frac{f_A-1}{r^2} [(f_A+1)K + 3f_A + 1] \right\}
\]
\[ V_{46}^M = 0 \]
\[ V_{47}^M = \sqrt{\frac{K-1}{2K-1}} \frac{1}{r} [2rH'_0 - (f_A - 1)H_0] \]
\[ V_{56}^M = \frac{K}{\sqrt{(K+1)(2K+1)}} \frac{1}{r} (f_A - 1)H_0 \]
\[ V_{57}^M = -\frac{K+1}{\sqrt{K(2K+1)}} \frac{1}{r} (f_A - 1)H_0 \]
\[ V_{67}^M = 0. \]

(\text{A.8})

For the electric sector we obtain

\[ V_{11}^E = (H_0^2 - 1) - \frac{4f'_A}{r(2K+1)} \]
\[ + \frac{f_A - 1}{r^2(2K+1)(K+1)} [2(f_A + 1)K^2 + (3f_A + 7)K + 4f_A + 4] \]
\[ V_{22}^E = (H_0^2 - 1) + \frac{4f'_A}{r(2K+1)} \]
\[ + \frac{f_A - 1}{r^2K(2K+1)} [2(f_A + 1)K^2 + (f_A - 3)K + 3f_A - 1] \]
\[ V_{33}^E = (H_0^2 - 1) \]
\[ + \frac{f_A - 1}{r^2(2K+1)(K+1)} [4K^3 + (3f_A + 11)K^2 + 9(f_A + 1)K + 3f_A + 3] \]
\[ V_{44}^E = (H_0^2 - 1) - \frac{f_A - 1}{r^2K(2K+1)} [4K^3 - (3f_A - 1)(K^2 - K - 1)] \]
\[ V_{55}^E = \frac{1}{2} \xi^2 + 2(H_0^2 - 1) + \frac{1}{2r^2}(f_A^2 - 1) \]
\[ V_{66}^E = \frac{3}{2} \xi^2(H_0^2 - 1) + \frac{1}{2r^2}(f_A - 1)^2 \]
\[ V_{12}^E = \frac{1}{\sqrt{K(K+1)}} \frac{2rf'A - (f_A - 1)(3f_A + 1)}{r^2(2K+1)} \]
\[ V_{13}^E = -\frac{\sqrt{K(K+2)}}{2K+1} \left\{ \frac{2f'_A}{r} + \frac{f_A - 1}{r^2(K+1)} [2(f_A - 1)K - f_A - 1] \right\} \]
\[ V_{14}^E = \frac{\sqrt{K-1}}{K} \frac{1}{r^2(2K+1)} [2rf'_A(K + 1) \]
\[ - (f_A - 1) [2(f_A + 1)K + (3f_A + 1)] \}
\[ V_{15}^E = -\frac{1}{\sqrt{(K+1)(2K+1)r}} [2rH'_0(K + 1) - (f_A - 1)H_0] \]
\[\begin{align*}
V_{16}^E &= -\sqrt{\frac{K}{2K+1}} \left(\frac{1}{r} (f_A - 1) H_0\right)
V_{25}^E &= \sqrt{\frac{K+2}{K+1}} \left(\frac{1}{r^2 (2K+1)} \{2r f_A' K - (f_A - 1) [2(f_A + 1)K - f_A + 1]\}\right)
V_{24}^E &= -\sqrt{\frac{K^2 - 1}{K(2K+1)}} \left(\frac{1}{r^2} \{2r f_A' K + (f_A - 1) [2(f_A - 1)K + 3f_A - 1]\}\right)
V_{25}^E &= \sqrt{\frac{K}{2K+1}} \left(\frac{1}{r^2} \{2r H_0' K + (f_A - 1) H_0\}\right)
V_{26}^E &= -\sqrt{\frac{K+1}{2K+1}} \left(\frac{1}{r^2} (f_A - 1) H_0\right)
V_{34}^E &= -\sqrt{\frac{(K-1)(K+2)}{(2K+1)^2}} \left(\frac{1}{r^2} (f_A - 1)^2\right)
V_{35}^E &= \sqrt{\frac{K(K+2)}{(K+1)(2K+1)}} \left(\frac{1}{r^2} (f_A - 1) H_0\right)
V_{36}^E &= \sqrt{\frac{K+2}{2K+1}} \left(\frac{1}{r^2} (f_A - 1) H_0\right)
V_{45}^E &= -\sqrt{\frac{K^2 - 1}{K(2K+1)}} \left(\frac{1}{r^2} (f_A - 1) H_0\right)
V_{46}^E &= \sqrt{\frac{K-1}{2K+1}} \left(\frac{1}{r^2} (f_A - 1) H_0\right)
V_{56}^E &= \sqrt{\frac{K(K+1)}{r^2}} \left(\frac{1}{r^2} (f_A - 1)\right)
\end{align*}\]

and finally for the Fadeev–Popov sector
\[\begin{align*}
V_{11}^{FP} &= (H_0^2 - 1) + \frac{f_A - 1}{(2K+1)r^2} \left[4K^2 + (3f_A + 7)K + 2f_A + 2\right]
V_{22}^{FP} &= (H_0^2 - 1) - \frac{f_A - 1}{(2K+1)r^2} \left[4K^2 - (3f_A - 1)K - f_A - 1\right]
V_{33}^{FP} &= (H_0^2 - 1) + \frac{1}{r^2} (f_A^2 - 1)
V_{12}^{FP} &= -\sqrt{\frac{K(K+1)}{(2K+1)^2}} \left(\frac{1}{r^2} (f_A - 1)^2\right)
V_{13}^{FP} &= 0
V_{23}^{FP} &= 0.
\end{align*}\]

We have again used REDUCE to obtain these potentials from the equations of motion.
for the transformed amplitudes.

In the case $K = 0$ the $\mathbf{I}$- and $\mathbf{J}$-operators act on the constant spherical harmonic $Y_{00}$. Therefore only the operators $\mathbf{1}$, $\mathbf{1}'$, $\mathbf{1}''$ and $\mathbf{J}$ contribute to the fluctuation Lagrangean. We can account for this by setting $-$ in the Lagrangean $-$ the amplitudes $t_2$, $t_4$, $t_6$, $t_7$, $t_8$, $t_9$, $p_2$, $p_3$, $s_2$ and $s_3$ equal to zero. For the remaining amplitudes $t_1$, $t_5$, $t_6$, $p_1$, $h_1$ and $s_1$ one obtains then the Euler–Lagrange–equations

$$
\begin{align*}
    r^2 t_1'' &= 2rt_1' - r^2 t_1 \\
    &= t_1(r^2 H_0^2 + 2f_A^2 + 2) + 4t_3(r f_A' - f_A) + 2p_1 r^2 H_0^0 \\
    r^2 t_3'' &= 2rt_3' - r^2 t_3 \\
    &= 2t_3(r f_A' - f_A) + t_6(r^2 H_0^2 + 3f_A^2 - 1) + p_1 r H_0 (f_A - 1) \\
    r^2 t_5'' &= 2rt_5' - r^2 t_5 \\
    &= t_5(r^2 H_0^2 + 3f_A^2 - 1) - h_1 r H_0 (f_A - 1) \\
    r^2 p_1'' &= 2rp_1' - r^2 p_1 \\
    &= \{4t_1 r^2 H_0^0 + 4t_3 r H_0 (f_A - 1) \\
    &+ p_1 [r^2 \xi^2 (H_0^2 - 1) + 2r^2 H_0^2 + (f_A + 1)^2] \}/2 \\
    r^2 h_1'' &= 2rh_1' - r^2 h_1 \\
    &= \{-4t_5 r H_0 (f_A - 1) + h_1 [r^2 \xi^2 (3H_0^2 - 1) + (f_A - 1)^2] \}/2 \\
    r^2 s_1'' &= 2rs_1' - r^2 s_1 \\
    &= s_1 (r^2 H_0^2 + 2f_A^2). \\
\end{align*}
$$

(A.11)

Changing the basis as

$$
\begin{align*}
    t_1 &= \sqrt{2}T_3 - 2T_1 \\
    t_3 &= \sqrt{2}T_3 + T_1 \\
    s_1 &= S_1 \\
    p_1 &= \sqrt{6}P_1 \\
    t_5 &= \sqrt{2}T_3 \\
    h_1 &= 2H_1 \\
    \end{align*}
$$

(A.12)

and introducing the $n$-tuples

$$
(\eta_i^M) = (T_1, T_3, P_1), \quad (\eta_i^E) = (T_5, H_1) \quad \text{and} \quad \eta^{FP} = S_1
$$

(A.13)

for the fields the differential equations take the required form of Eq. 4.7. The masses $m_i$ and angular momenta $l_i$ are shown in table 3.

The potentials in the three sectors are

$$
\begin{align*}
    V_{11}^M &= (H_0^2 - 1) - \frac{1}{3r^2} [8r f_A' - (f_A - 1)(7f_A + 15)] \\
    V_{22}^M &= (H_0^2 - 1) + \frac{8}{3r^2} [r f_A' + (f_A - 1)f_A] \\
    V_{33}^M &= \frac{1}{2} (\xi^2 + 2)(H_0^2 - 1) + \frac{1}{2r^2} (f_A - 1)(f_A + 3) \\
    V_{12}^M &= -\frac{\sqrt{2}}{3r^2} [2rf_A' - (f_A - 1)(f_A + 3)] \\
\end{align*}
$$

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\[ V_{13}^M = \sqrt{\frac{2}{3 \pi}} [(f_A - 1) H_0 - 2r H'_0] \]
\[ V_{33}^M = \frac{2}{\sqrt{3 \pi}} [(f_A - 1) H_0 + r H'_0] \]
\[ V_{11}^E = (H_0^2 - 1) + \frac{3}{r^2} (f_A^2 - 1) \]
\[ V_{22}^E = \frac{3}{2} \xi^2 (H_0^2 - 1) + \frac{1}{2r^2} (f_A - 1)^2 \]
\[ V_{12}^E = -\sqrt{\frac{2}{r}} (f_A - 1) H_0 \]
\[ V^{FP} = (H_0^2 - 1) + \frac{2}{r^2} (f_A^2 - 1). \]  
(A.14)

Another exceptional case arises for \( K = 1 \). Here the J- and I-operators act on the spherical harmonics \( Y_{1M} \propto \hat{x} \). The action of \( \mathbf{P}^1 \) becomes equal up to a sign to the one of \( \mathbf{P}^6 \) and similarly the one of \( \mathbf{P}^1 \) to the one of \( \mathbf{P}^6 \) (see Eq. (4.3)). Therefore a linear combination of the amplitudes \( t_5 \) and \( t_6 \) and of \( t_5 \) and \( t_8 \) respectively can be chosen to vanish. We have taken the choice
\[ t_6 = 0 \quad \text{and} \quad t_8 = 0. \]  
(A.15)

For the remaining 14 amplitudes we find
\[ r^2 t''_1 + 2r t'_1 - r^2 t_1 \]
\[ = t_1 (r^2 H_0^2 + 2f_A^2 + 4) - 4t_2 f_A + 4t_3 (r f_A' - f_A - 1) \]
\[ - 4t_6 (r f_A' - f_A + 1) - 2p_1 r^2 H_0' \]
\[ r^2 t''_2 + 2r t'_2 - r^2 t_2 \]
\[ = -2t_1 f_A + t_2 (r^2 H_0^2 + f_A^2 + 3) - 2t_3 (r f_A' - f_A - 1) \]
\[ - 2t_9 (r f_A' - f_A + 1) + 2p_2 r^2 H_0' \]
\[ r^2 t''_5 + 2r t'_5 - r^2 t_5 \]
\[ = \{ 2t_1 (r f_A' - f_A - 1) - 2t_2 (r f_A' - f_A - 1) \}
\[ + t_3 (r^2 H_0^2 + 5f_A^2 + 4f_A + 3) - t_9 (f_A^2 - 1) + p_1 r H_0 (f_A - 1) \]
\[ - p_2 r H_0 (f_A - 1) \}/2 \]
\[ r^2 t''_6 + 2r t'_6 - r^2 t_6 \]
\[ = \{ -2t_1 (r f_A' - f_A + 1) - 2t_2 (r f_A' - f_A + 1) - t_3 (f_A^2 - 1) \}
\[ + t_9 (2r^2 H_0^2 + 5f_A^2 - 4f_A + 3) - p_1 r H_0 (f_A - 1) \]
\[ - p_2 r H_0 (f_A - 1) \}/2 \]
\[ r^2 p''_1 + 2r p'_1 - r^2 \bar{p}_1 \]
\[ = \{ 4t_1 r^2 H_0^2 + 4t_3 r H_0 (f_A - 1) - 4t_9 r H_0 (f_A - 1) \}
\[ + p_1 [r^2 \xi^2 (H_0^2 - 1) + 2r^2 H_0^2 + f_A^2 + 2f_A + 5] - 4p_2 (f_A + 1) \}/2 \]
\[ r^2 p''_2 + 2r p'_2 - r^2 \bar{p}_2 \]
\[ \begin{align*}
&= \{2t_3r^2H_0' - 2t_3rH_0(f_A - 1) - 2t_3rH_0(f_A - 1) - 2p_1(f_A + 1) \\
&+ p_2[r^2H_0^2 - 1] + 2r^2H_0^2 + f_A^2 + 3]\}/2 
\end{align*} \]

(A.16)

for the magnetic sector,

\[ \begin{align*}
r^2t_4'' &+ 2rt_4' - r^2t_4 \\
&= t_4(r^2H_0^2 + f_A^2 + 3) - 2t_3(r f_A' - f_A + 1) + 2t_4'(r f_A' - f_A - 1) \\
&+ 2p_3r^2H_0' \\
r^2t_5'' &+ 2rt_5' - r^2t_5 \\
&= \{2t_4(r f_A' - f_A + 1) + t_5(2r^2H_0^2 + 5f_A^2 + 4f_A + 3) \\
&+ t_4'(f_A^2 - 1) - p_3rH_0 (f_A - 1) - h_1rH_0(f_A - 1)\}/2 \\
r^2t_7'' &+ 2rt_7' - r^2t_7 \\
&= \{2t_4(r f_A' - f_A - 1) + t_7(2r^2H_0^2 + 5f_A^2 - 4f_A + 3) \\
&+ p_3rH_0 (f_A - 1) - h_1rH_0(f_A - 1)\}/2 \\
r^2p_3'' &+ 2rp_3' - r^2p_3 \\
&= \{4t_3r^2H_0' - 2t_3rH_0(f_A - 1) + 2t_7rH_0(f_A - 1) \\
&+ p_3[r^2H_0^2 - 1] + 2r^2H_0^2 + f_A^2 + 3] + 2h_1(f_A - 1)\}/2 \\
r^2h_1'' &+ 2rh_1' - r^2h_1 \\
&= \{4t_3rH_0(f_A - 1) - 4t_7rH_0(f_A - 1) + 4p_3(f_A - 1) \\
&+ h_1[r^2H_0^2 - 1] + f_A^2 - 2f_A + 5]\}/2 
\end{align*} \]

(A.17)

for the electric sector and

\[ \begin{align*}
r^2s_1'' &+ 2rs_1' - r^2s_1 \\
&= s_1(r^2H_0^2 + 2f_A^2 + 2) - 4s_2f_A \\
r^2s_2'' &+ 2rs_2' - r^2s_2 \\
&= -2s_1f_A + s_2(r^2H_0^2 + f_A^2 + 1) \\
r^2s_3'' &+ 2rs_3' - r^2s_3 \\
&= s_3(r^2H_0^2 + f_A^2 + 1) 
\end{align*} \]

(A.18)

for the Fadeev–Popov sector. The amplitudes are transformed as

\[ \begin{align*}
t_1 &= \frac{2}{\sqrt{5}} \left( 2T_1 + \sqrt{5}T_2 + \sqrt{6}T_3 \right) \\
t_2 &= \frac{1}{\sqrt{5}} \left( \sqrt{6}T_3 + 2\sqrt{5}T_2 - 2T_1 \right) \\
t_3 &= \frac{1}{\sqrt{3}} \left( \sqrt{6}T_3 + 3T_1 \right) \\
t_9 &= \frac{1}{\sqrt{3}} \left( \sqrt{6}T_3 + 3T_1 \right) \\
t_7 &= \frac{1}{\sqrt{3}} \left( T_4 + \sqrt{2}T_5 \right) \\
t_4 &= \frac{1}{\sqrt{3}} \left( \sqrt{2}T_5 - 2T_4 \right) \\
s_1 &= S_2 - \sqrt{2}S_1 \\
s_2 &= S_2 + \frac{1}{\sqrt{2}}S_1 \\
s_3 &= S_3 \\
p_3 &= \sqrt{2}P_3 \\
p_1 &= 2(P_2 - \sqrt{2}P_1) \\
p_2 &= 2P_2 + \sqrt{2}P_1 \\
h_1 &= 2H_1 
\end{align*} \]

(A.19)

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in order to get a symmetric potential and asymptotic decoupling. With the n-tuples

\[
(\eta_i^M) = (T_1, T_2, T_3, T_9, P_1, P_2)
\]

\[
(\eta_i^F) = (T_4, T_5, T_7, P_3, H_1)
\]

\[
(\eta_i^{FP}) = (S_1, S_2, S_3)
\]  \hspace{1cm} (A.20)

the fluctuation equations take the form of Eq. (4.7). The masses \( m_i \) and angular momenta \( l_i \) are given in Table 4.

The elements of the symmetric potential are given by

\[
V_{11}^M = (H_0^2 - 1) + \frac{12 f_A}{5 r} + \frac{(f_A - 1) (65 f_A + 61)}{30 r^2}
\]

\[
V_{22}^M = (H_0^2 - 1) + \frac{4 (f_A - 1)^2}{3 r^2}
\]

\[
V_{33}^M = (H_0^2 - 1) - \frac{12 f_A}{5 r} + \frac{(f_A - 1) (10 f_A + 34)}{5 r^2}
\]

\[
V_{44}^M = (H_0^2 - 1) + \frac{(f_A - 1) (5 f_A + 1)}{2 r^2}
\]

\[
V_{55}^M = \frac{1}{2} (2 + \xi^2) (H_0^2 - 1) + \frac{(f_A - 1) (f_A + 5)}{2 r^2}
\]

\[
V_{66}^M = \frac{1}{2} (2 + \xi^2) (H_0^2 - 1) + \frac{(f_A - 1)^2}{2 r^2}
\]

\[
V_{12}^M = \frac{2 (f_A - 1)^2}{3 \sqrt{5} r^2}
\]

\[
V_{13}^M = \frac{1}{5 \sqrt{6} r^2} [(f_A - 1) (5 f_A + 7) - 6 r f_A']
\]

\[
V_{14}^M = -\frac{1}{2 \sqrt{15} r^2} [(f_A - 1) (3 f_A - 1) + 4 r f_A']
\]

\[
V_{15}^M = -\frac{1}{\sqrt{10} r} [3 (f_A - 1) H_0 + 4 r H_0']
\]

\[
V_{16}^M = 0
\]

\[
V_{23}^M = -\sqrt{\frac{2}{15}} \frac{(f_A - 1)^2}{r^2}
\]

\[
V_{24}^M = \frac{4}{\sqrt{3} r^2} [(f_A - 1) - r f_A']
\]

\[
V_{25}^M = 0
\]

\[
V_{26}^M = 2 H_0'
\]

\[
V_{34}^M = -\frac{1}{\sqrt{10} r^2} [(f_A - 1) (f_A + 3) - 2 r f_A']
\]

\[
V_{35}^M = -\sqrt{\frac{3}{5} r} [(f_A - 1) H_0 - 2 r H_0']
\]
\[ V_{36}^M = 0 \]

\[ V_{45}^M = \frac{1}{\sqrt{6r}} (f_A - 1) H_0 \]

\[ V_{46}^M = -\frac{2}{\sqrt{3r}} (f_A - 1) H_0 \]

\[ V_{56}^M = 0 \]  \hspace{1cm} (A.21)

for the magnetic sector, by

\[ V_{11}^E = (H_0^2 - 1) - \frac{4 f_A'}{3r} + \frac{(f_A - 1)(9 f_A + 13)}{6r^2} \]

\[ V_{22}^E = (H_0^2 - 1) + \frac{4 f_A'}{3r} + \frac{(f_A - 1)(6 f_A - 2)}{3r^2} \]

\[ V_{33}^E = (H_0^2 - 1) + \frac{(f_A - 1)(5 f_A + 9)}{2r^2} \]

\[ V_{44}^E = \frac{1}{2} (2 + \xi^2) (H_0^2 - 1) + \frac{(f_A - 1)(f_A + 1)}{2r^2} \]

\[ V_{55}^E = \frac{3}{2} \xi^2 (H_0^2 - 1) + \frac{(f_A - 1)^2}{2r^2} \]

\[ V_{12}^E = -\frac{1}{3 \sqrt{2r^2}} [2r f'_A - (f_A - 1)(3 f_A + 1)] \]

\[ V_{13}^E = \frac{1}{2 \sqrt{3r^2}} [4r f'_A + (f_A - 1)(f_A - 3)] \]

\[ V_{14}^E = -\frac{1}{3 \sqrt{6r}} [4r H_0' - (f_A - 1) H_0] \]

\[ V_{15}^E = -\frac{1}{\sqrt{3r}} (f_A - 1) H_0 \]

\[ V_{23}^E = -\frac{1}{\sqrt{6r^2}} [2r f'_A - (f_A - 1)(f_A + 3)] \]

\[ V_{24}^E = \frac{1}{\sqrt{3r}} [2r H_0' + (f_A - 1) H_0] \]

\[ V_{25}^E = -\frac{1}{\sqrt{3r}} (f_A - 1) H_0 \]

\[ V_{34}^E = -\frac{1}{\sqrt{2r}} (f_A - 1) H_0 \]

\[ V_{35}^E = -\frac{1}{r} (f_A - 1) H_0 \]

\[ V_{45}^E = \frac{\sqrt{2}}{r^2} (f_A - 1) \]  \hspace{1cm} (A.22)
for the electric sector and by

\[ V_{11}^{FP} = (H_0^2 - 1) + \frac{(f_A - 1)(5f_A + 13)}{3r^2} \]
\[ V_{22}^{FP} = (H_0^2 - 1) + \frac{4(f_A - 1)^2}{3r^2} \]
\[ V_{33}^{FP} = (H_0^2 - 1) + \frac{(f_A - 1)(f_A + 1)}{r^2} \]
\[ V_{12}^{FP} = -\frac{\sqrt{2}}{3r^2}(f_A - 1)^2 \]
\[ V_{13}^{FP} = 0 \]
\[ V_{23}^{FP} = 0 \]

(A.23)

for the Fadeev–Popov amplitudes.
B Zero Mode Prefactors

The prefactors $N_{tr}$ and $N_{rot}$ are determined [6] by the normalization of the translation and rotation zero modes. We have included factors $1/2\pi$ which otherwise would appear with the prefactor $T^{-3}$. We have then (cf. [4])

$$\begin{align*}
N_{tr} &= N_{rot}^3 \\
N_{rot} V_{rot} &= 8\pi^2 N_{rot}^3
\end{align*}$$

(B.1)

where the normalization factors $N$ are given by

$$\frac{1}{2\pi} \int d^3x \psi_n^{tr,rot} \psi_n^{tr,rot}.$$  \hspace{1cm} (B.2)

The $\psi_n$ are the zero mode wave functions, the different field components are assumed to have canonical normalization (i.e. appearing as $\frac{1}{2} \partial_\mu \psi_n^2 \partial^\mu \psi_n$ for each real component in the Lagrangean density). The rotation and translation modes have been determined explicitly in [13] for the sphaleron solution in the form (2.17). The zero mode amplitudes were found to be

$$\begin{align*}
t_4 &= -\frac{f_A - 1}{r^2} \\
t_5 &= \frac{f_A + 1}{r^2} - \frac{f_A'}{2r} \\
t_7 &= -\frac{f_A'}{2r} \\
h_1 &= H_0'
\end{align*}$$

(B.3)

for the translation mode and

$$t_9 = \frac{f_A - 1}{r}$$

(B.4)

for the rotation mode.

For proper normalization (which was irrelevant in [13]) all these contributions have to be multiplied by $\sqrt{4\pi/3}$ which comes from the different normalization of the $Y_{1M}$ and the $\tilde{x}_M$ used in that calculation. Also the translation mode amplitudes have to be multiplied by $M_W$ if they are generated by the ordinary gradient, i.e. the derivatives w. r. t. $\tilde{x}_\mu$. We will include these additional factors at the end (see Eqs. (B.11) and (B.12)).

In this form the modes are not normalizable and do not satisfy the background gauge condition (3.7). The general form of the infinitesimal gauge transformations (which also applies for finite ones since we have expanded the fields linearly around the classical solution) has also been given in [13]. For the $K = 1$ channel it reads:

$$\begin{align*}
\delta t_1 &= g_1' \\
\delta t_2 &= g_2'
\end{align*}$$
\[
\begin{align*}
\delta t_3 &= \frac{g_1 + g_2}{r} - \frac{(f_A - 1)}{2r}(g_1 + 3g_2) \\
\delta t_4 &= g_3' \\
\delta t_5 &= \frac{3}{2} \frac{f_A - 1}{r} g_3 \\
\delta t_7 &= \frac{g_3}{r} + \frac{f_A - 1}{2r} g_3 \\
\delta t_9 &= -\frac{f_A - 1}{2r}(g_1 + 2g_2) \\
\delta p_1 &= H_0 g_1/2 \\
\delta p_2 &= H_0 g_2/2 \\
\delta p_3 &= H_0 g_3/2 \\
\delta h_1 &= H_0'.
\end{align*}
\] (B.5)

The background gauge conditions read in terms of the partial wave amplitudes
\[
\begin{align*}
rt_1' + 2t_1 &= 2(f_A + 1)t_3 - 2(f_A - 1)t_9 + 2rH_0 p_1 \\
rt_2' + 2t_2 &= -(f_A + 1)t_3 - (f_A - 1)t_9 + 2rH_0 p_2 \\
rt_4' + 2t_4 &= -(f_A - 1)t_5 + (f_A + 1)t_7 + 2rH_0 p_3.
\end{align*}
\] (B.6)

Inserting the amplitudes and gauge functions leads to three differential equations for the gauge functions
\[
\begin{align*}
g_1'' + \frac{2}{r}g_1' &= \left[H_0^2 + \frac{2(f_A^2 + 1)}{r^2}\right] g_1 - \frac{4f_A}{r^2} g_2 - \frac{2(f_A - 1)^2}{r^2} \\
g_2'' + \frac{2}{r}g_2' &= \left[H_0^2 + \frac{f_A^2 + 1}{r^2}\right] g_2 - \frac{2f_A}{r^2} g_1 - \frac{(f_A - 1)^2}{r^2} \\
g_3'' + \frac{2}{r}g_3' &= \left[H_0^2 + \frac{f_A^2 + 1}{r^2}\right] g_3 - \frac{(f_A - 1)^2}{r^3}
\end{align*}
\] (B.7)

which have to be solved with boundary conditions that ensure the normalizability of the zero modes. The solution for \( g_3 \) can be found explicitly:
\[
g_3 = \frac{1 - f_A}{r}
\] (B.8)

so that the translation mode becomes
\[
\begin{align*}
t_4 &= -\frac{f_A}{r} \\
t_5 &= \frac{f_A^2 - 1}{2r^2} - \frac{f_A'}{2r} \\
t_7 &= -\frac{f_A^2 - 1}{2r^2} - \frac{f_A'}{2r} \\
p_3 &= \frac{(f_A - 1)H_0}{2r} \\
h_1 &= H_0'.
\end{align*}
\] (B.9)
The gauge transformed rotation mode becomes

\[
\begin{align*}
t_1 &= g'_1 \\
t_2 &= g'_2 \\
t_3 &= \frac{g_1 - g_2}{r} + \frac{f_A - 1}{2r} (g_1 - g_2) \\
t_9 &= -\frac{f_A - 1}{2r} (g_1 + g_2) + \frac{f_A - 1}{r} \\
p_1 &= \frac{1}{2} H_0 g_1 \\
p_2 &= \frac{1}{2} H_0 g_2.
\end{align*}
\]

We have not been able, however, to solve the equations for the gauge functions \(g_1\) and \(g_2\) analytically. The normalization integrals \(N_{tr,rot}^2\) are obtained by inserting these amplitudes into the general expression amplitudes as

\[
N^2 = \frac{1}{2\pi} \frac{4\pi}{3} \int_0^\infty dr \int_0^\infty d\rho \left[ f_A^2 + \frac{(f_A - 1)^2}{2r^2} + r^2 H_0^2 + \frac{1}{2} H_0^2 (f_A - 1)^2 \right] = \frac{1}{3} \int_0^\infty dr \int_0^\infty d\rho \left[ f_A^2 + \frac{(f_A - 1)^2}{2r^2} + r^2 H_0^2 + \frac{1}{2} H_0^2 (f_A - 1)^2 \right].
\]

Here the first factor in front of the integral is a factor “borrowed” from the factors \(\sqrt{2\pi T}\) which arise for each extracted zero mode and which we have included here as it was done in [4]. The factor \(4\pi/3\) has been explained above. Finally the factors in front of the different amplitudes come additionally from the normalization of the tensors used in the expansion. Explicitly we obtain

\[
\begin{align*}
N_{tr}^2 &= \frac{8M_W}{3g^2} \int_0^\infty dr \left[ f_A^2 + \frac{(f_A - 1)^2}{2r^2} + r^2 H_0^2 + \frac{1}{2} H_0^2 (f_A - 1)^2 \right] + 2 \left[ \frac{f_A - 1}{2r} (g_1 + g_2) \right]^2 + \frac{1}{2} H_0^2 g_1^2 + H_0^2 g_2^2 \\
N_{rot}^2 &= \frac{4}{3M_W g^2} \int_0^\infty dr \int_0^\infty d\rho \left\{ \frac{1}{2} g_1^2 + g_2^2 + 2 \left[ \frac{g_1 - g_2}{r} + \frac{f_A - 1}{2r} (g_1 - g_2) \right]^2 + \frac{1}{2} H_0^2 g_1^2 + H_0^2 g_2^2 \right\}.
\end{align*}
\]

The expression obtained for the translation mode can be shown to be

\[
N_{tr}^2 = E_{cl} / 2\pi
\]

as expected from a general virial theorem. The rotation mode normalization should be related, by a similar virial theorem, to the moment of inertia of the sphaleron. We have evaluated the normalization integrals numerically; we find after taking into account factors \(2^{1/2}\) due to the different scales \(M_W\) and \(g\) used in the two publications – the same results as [9] within the numerical accuracy, though we have used a different gauge for the classical solution. Note that the scale factors cancel in the product \(N_{rot} N_{trans}\).
C Determination of the Amplitudes $h_n^{\alpha\pm}$

In this Appendix we will discuss briefly the numerical evaluation of the functions $h_n^{\alpha\pm}$, i.e. the solutions of the differential equations (6.17).

The numerical integration of the differential equations (6.17) is started, for $h_n^{\alpha+}$, at some sufficiently high $r = r_{\text{max}}$ with the initial condition $h_n^{\alpha+}(r_{\text{max}}) = 0$.

Starting the functions $h_n^{\alpha-}$ at $r = 0$ is by far more difficult: the behaviour of these functions as $r \to 0$ has to be determined analytically. This means that all the functions that enter the differential equations (6.17), i.e. the Bessel functions and their derivatives, the classical profiles $f_A$ and $H_0$ and the solutions $h_n^{\alpha-}$ have to be expanded, for $r \to 0$ into Taylor series. As a first step which proves to be nontrivial one has to find the leading behaviour of the solutions, since the centrifugal barrier at $r = 0$ differs from the vacuum sector one. If we write the leading behaviour as $r^\Delta$ we find

$$
\Delta = \begin{cases} 
2 & \text{for the amplitude } S_3 \\
1 & \text{for } P_2 \\
0 & \text{for } T_5, T_6, T_8, S_2 \\
-1 & \text{for } P_1, P_3, H_1 \\
-2 & \text{for } T_1, T_2, T_3, T_4, T_7, T_9, S_1.
\end{cases}
$$  \hfill (C.1)

With these parameters one obtains in each $n \times n$ sector a set of recursion relations for the next-to-leading coefficients and a set of starting conditions for $n$ independent solutions labelled by $\alpha = 1, n$.

This expansion which has been determined up to the second nonleading order in $r^2$ is used up to some suitable $r = r_{\text{min}}$ at which the Nystöm integration is then started. The solutions found in this way do not yet satisfy the boundary condition $h_n^{\alpha-} \to 0$ as $r \to \infty$. However a set of such solutions can now be found by a simple linear transformation.

A good check on the accuracy of the numerical integration consists in checking the constancy of the product $r^2W^{\alpha\beta}(r)$ related to the Wronskian (see Eq. (6.16)). For

$$
r > \begin{cases} 
5 \cdot 10^{-3} & \text{for the amplitude } S_3 \\
0.1 & \text{for all other sectors}
\end{cases}
$$  \hfill (C.2)

this expression was found to be constant to 5 significant digits. For smaller $r$ the numerical integration becomes delicate for all sectors except the amplitude $S_3$, since some of the amplitudes become singular as $r \to 0$. In this region we used the known leading behaviour of the exact Green function

$$
r^2G_{ii}(r, r, \nu) \propto \begin{cases} 
r^3 & \text{for the amplitude } S_3 \\
r & \text{for all other sectors}
\end{cases}
$$  \hfill (C.3)

with coefficients determined from the numerical results in the reliable region (C.2).
References


Figure Captions

Fig. 1 The classical sphaleron energy $E_{cl}$: The figure shows the classical sphaleron energy $E_{cl}$ as a function of $\xi = M_H/M_W$ in units of $M_W(T)/\alpha_w$.

Fig. 2 The zero–mode normalization factors: The solid line shows the normalization factor $N_\text{tr}$ of the translation mode and the dashed line the normalization factor $N_\text{rot}$ of the rotation mode as a function of $\xi = M_H/M_W$. The units are $(M_W/g^2)^{3/2}$ and $(M_W g^2)^{-3/2}$ respectively (see Eqs.(B.12)).

Fig. 3 The convergence of the K summation: We show the partial sums $F(K_{max}, \nu)$ as defined in the text as a function of $K_{max}$ for different values of $\nu$. The dashed lines are the values obtained by including the sum from $(K_{max} + 1)$ to $\infty$ using the fit of Eq. (9.2). These values are seen to become independent of $K_{max}$ already around $K_{max} \approx 10$.

Fig. 4 The function $F(\nu)$ for $\xi = M_H/M_W = 1$: The solid line shows $\nu F_{\text{ren}}(\nu)$, the dashed line the unrenormalized $\nu F(\nu)$. The pole contribution of the unstable mode has been removed (see Eq. (8.2)). The dotted lines show the expected power behaviours at small and large $\nu$ and the dash–dotted line the analytically known tadpole contribution to $\nu F(\nu)$.

Fig. 5 The fluctuation determinant: The circles are our results, the crosses those of CLMW. The full line is the estimate of Carson and McLerran based on the DPY approximation and the dashed line a perturbative estimate.
Tables

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<th>( \xi )</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.8</th>
<th>1.0</th>
<th>1.5</th>
<th>2.0</th>
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<td>( \omega_- )</td>
<td>1.32</td>
<td>1.36</td>
<td>1.39</td>
<td>1.45</td>
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<td>1.62</td>
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<td>( \ln \kappa )</td>
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<td>5.89</td>
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<td>5.50</td>
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Table 1: The results for \( \ln \kappa \) for various values of \( \xi = M_H/M_W \) together with the frequencies \( \omega_- \) of the unstable mode

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<th>( M )</th>
<th>( m_i )</th>
<th>( l_i )</th>
<th>( E )</th>
<th>( m_i )</th>
<th>( l_i )</th>
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<td>1</td>
<td>( K + 1 )</td>
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<td>1</td>
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<td>( K )</td>
<td>2</td>
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<td>1</td>
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<td>( K )</td>
</tr>
<tr>
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</tr>
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<td>1</td>
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<td>1</td>
<td>( K )</td>
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</tr>
<tr>
<td>6</td>
<td>1</td>
<td>( K + 1 )</td>
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<td>( \xi )</td>
<td>( K )</td>
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<td></td>
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<tr>
<td>7</td>
<td>1</td>
<td>( K - 1 )</td>
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Table 2: Masses and Angular Momenta of the Amplitudes for \( K > 1 \)

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<th>( m_i )</th>
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<td>2</td>
<td>( \xi )</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 3: Masses and Angular Momenta of the Amplitudes for \( K = 0 \)
| $M$ | $m_i$ | $l_i$ |  | $E$ | $m_i$ | $l_i$ |  | $FP$ | $m_i$ | $l_i$ |
|-----|-------|-------|  |     |-------|-------|  |     |-------|-------|
| 1   | 1     | 1     |  | 1   | 1     | 2     |  | 1   | 1     | 2     |
| 2   | 1     | 1     |  | 2   | 1     | 0     |  | 2   | 1     | 0     |
| 3   | 1     | 3     |  | 3   | 1     | 2     |  | 3   | 1     | 1     |
| 4   | 1     | 1     |  | 4   | 1     | 1     |  | 4   | 1     | 1     |
| 5   | 1     | 2     |  | 5   | $\xi$ | 1     |  | 5   | $\xi$ | 1     |
| 6   | 1     | 0     |  |     |       |       |  |     |       |       |

Table 4: Masses und Angular Momenta of the Amplitudes for $K = 1$