Relating the Lorentzian and exponential: Fermi’s approximation, the Fourier transform and causality

A. Bohm  
Physics Department  
University of Texas at Austin  
Austin, TX 78712

N.L. Harshman  
Department of Physics and Astronomy  
Rice University  
Houston, TX 77005

H. Walther  
Max-Planck Institut für Quantenoptik und Sektion Physik  
Universität München  
85748 Garching, Germany

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Abstract

The Fourier transform is often used to connect the Lorentzian energy distribution for resonance scattering to the exponential time dependence for decaying states. However, to apply the Fourier transform, one has to bend the rules of standard quantum mechanics; the Lorentzian energy distribution must be extended to the full real axis $-\infty < E < \infty$ instead of being bounded from below $0 \leq E < \infty$ (“Fermi’s approximation”). Then the Fourier transform of the extended Lorentzian becomes the exponential, but only for times $t \geq 0$, a time asymmetry which is in conflict with the unitary group time evolution of standard quantum mechanics. Extending the Fourier transform from distributions to generalized vectors, we are led to Gamow kets, which possess a Lorentzian energy distribution with $-\infty < E < \infty$ and have exponential time evolution for $t \geq t_0 = 0$ only. This leads to probability predictions that do not violate causality.
1 Introduction

In this paper we would like to draw a connection between two theoretical questions concerning the decay of quasistable states.

One question concerns the relation between the two experimentally-independent signatures of quasistable states, the Breit-Wigner or Lorentzian lineshape of resonances as measured in the cross section and the exponential decay rate measured for decaying states. Explicitly, how can the standard relation $\Gamma = h/\tau$ (based on the Weisskopf-Wigner approximation [1]), which connects the lineshape parameter $\Gamma$ to the lifetime $\tau$, be justified in a mathematically rigorous way? Only recently has this relation been experimentally tested to an accuracy that goes beyond the Weisskopf-Wigner approximation [2, 3].

The other question concerns how the quantum theory of quasistable states can incorporate the following notion: there can be no registration of decay products by a detector at times prior to the preparation of the decaying state. In other words, we expect that precursor events should be assigned zero probability in a theory of quasistable states. We refer to this commonsense idea as causality or the preparation-registration arrow of time [4]. Mathematical results within Hilbert space quantum mechanics challenge the applicability of this notion to quantum phenomena [5] and also challenge the validity of the exponential law [6].

In the following two sections, we address the above questions in reverse order and find a connection between them, aided by the classic example of Fermi's two atom problem [7]. Then the next two sections present a solution to these questions, defining the Gamow vectors for the representation of quasistable states. To make their definition rigorous, and to incorporate simple answers to the above questions, entails selecting separate boundary conditions for the space of states and for the space of observables. This requires a slight modification of the Hilbert space axiom of standard quantum mechanics. The consequences, physical and mathematical, of modifying this axiom so as to incorporate causal decaying state behavior are discussed in the conclusion.

2 Probabilities for Precursor Events

In his classic review of Dirac's quantum theory of radiation [7], Fermi postulated a problem to test whether the theory satisfied what he considered a sensible requirement, finite-velocity signal propagation (or Einstein causality as it is occasionally called). As he posed the problem:
Let $A$ and $B$ be two atoms; let us suppose at the time $t = 0$, $A$ is in an excited and $B$ in the normal [ground] state. After a certain time $A$ emits its energy which may in turn be absorbed by the atom $B$ which then becomes excited. Since the light needs a finite time to go from $A$ to $B$, the excitation of $B$ can take place only after the time $r/c$, $r$ being the distance between the two atoms. [7]

Fermi goes on to solve this problem, making the assumption that the mean life of the excited state of $A$ is short and that the mean life of the excited state of $B$ is very long. These facts are used to justify several simplifications; in particular he uses them to justify extending the range of integration for the emitted photon frequency from $[0, \infty)$ to $(-\infty, \infty)$. Because of this approximation, several integrals become exact and he achieves his desired result: there is zero probability that atom $B$ will be excited for $t < r/c$ or equivalently for $t - r/c < 0$.

As we would say today, Fermi achieved his result by analytically extending the photon energy range. Though this extension of energy appears a quantitatively minor modification, it has now become clear that precursor events have zero probability only because the integral over the physical values of energy $0 \leq E < \infty$ was extended to an integral over the range $-\infty, < E < \infty$. The absence of precursor events is an artifact of Fermi’s modification of the energy range, not a consequence of the quantum mechanics of his time.

The root of this problem is not that something “propagates faster than light” to give non-zero transition probabilities for $t < r/c$. In fact, if one does not extend the lower bound of the energy range to $-\infty$, the transition probability is non-zero even for $t < 0$, i.e., for times before the atom $A$ is excited. This result does not indicate a violation of Einstein causality so much as a violation of the basic notion of causality expressed by the preparation-registration arrow of time mentioned in the introduction.

This result is not specific to Fermi’s two-atom problem, but is the consequence of a general theorem by Hegerfeldt [5]: The transition probability between two Hilbert space vectors is either identically zero for all times (and therefore, there is no decay at all) or it is different from zero for all time $-\infty < t < \infty$. This means that excited states cannot be prepared at a finite time $t_0 > -\infty$ and subsequently decay, although this is the typical situation and the case considered by Fermi. This Hilbert space theorem has two consequences: 1. non-zero probabilities for precursor events, such as decay products detected at $t \leq t_0$ before the state was prepared, a result which violates causality, and 2. non-zero probabilities for detecting decay products for times $t < t_0 + r/c$ for any finite but arbitrary distance $r$, or equivalently non-zero probabilities for $r > c(t - t_0)$, which violates Einstein causality. This latter consequence was the main concern of [5] though the former consequence more directly conflicts with the concept of causality in general [4], since it does not also involve space translation and the constant $c$ of special relativity. Since Einstein causality involves the constant $c$ one can only do full justification to this problem in a relativistic theory using
Poincaré transformations, and not just time translations generated by the Hamiltonian. We shall briefly discuss this at the end of this paper and in the Appendix.

The theorem of [5] requires that the Hamiltonian be bounded from below, so Fermi circumvented the conditions of this theorem by extending the energy values for the spectrum of the Hamiltonian \( H \) from \( \{ E | 0 \leq E < \infty \} \) to \( \{ E | -\infty < E < \infty \} \) and obtained his result: the probability that atom B is excited before \( t_0 = 0 \), the time at which excited atom A had been originally created, is zero. Within standard quantum mechanics the concept of a time \( t_0 \) before which the Born probability is zero is precluded because the (Heisenberg or Schrödinger) equations of motion integrate under the Hilbert space boundary conditions to the unitary time evolution group, valid for \( -\infty < t < \infty \) [8].

Fermi’s procedure of extending the energy range can be justified as an analytic continuation of the energy, but this requires a modification of one of the axioms of quantum mechanics, the Hilbert space boundary condition. In fact, Fermi’s approximation points the way to selecting boundary conditions on the spaces of states and observables in which the Schrödinger and Heisenberg equations do not integrate to a group, but to a semigroup.

### 3 Cross Section and Exponential

There are two experimental signatures of quasistable states, the Breit-Wigner or Lorentzian for the energy dependence of the decay amplitude

\[
g(\omega) = \frac{i}{\omega - (E_R - i\Gamma/2)}
\]

and the exponential for the time dependence of the decay amplitude

\[
f(t) = e^{-iE_Rt}e^{-\Gamma_Rt/2}.
\]

From the point of view of physical observation, the two quantities \( \Gamma \) in (1) and \( \Gamma_R \) (or \( \Gamma_R/\hbar \) if the unit of time is not inverse energy) in (2) are fundamentally different quantities and are measured in different ways. The width \( \Gamma \) is measured by the Breit-Wigner lineshape as, for example, in the cross section of a resonance scattering process:

\[
\sigma_j^{BW}(E) \propto \frac{1}{(E - E_R)^2 + (\Gamma/2)^2}.
\]

The inverse lifetime \( \Gamma_R \equiv 1/\tau \), which is the initial decay rate (considering only one channel), can be measured by fitting the counting rate of decay products to the exponential law:

\[
\frac{1}{N} \frac{dN(t)}{dt} = \Gamma_R e^{-\Gamma_R t} \propto e^{-t/\tau},
\]

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where \( dN(t_i) \) is the number of decay products registered in the detector during the time interval \( dt \) around \( t_i \).

The energy scale of the particular quasistable state usually determines which of these methods is used to determine the characteristic lifetime or width parameter\(^1\). For long-lived states, e.g. \( \tau > 10^{-8} \) s (or \( h/\tau < 10^{-7} \) eV), it is reasonably easy to directly measure the lifetime by a fit to the exponential and such states are conventionally called decaying states. The range of experimentally accessible lifetimes can be extended by exploiting relativistic time dilation for fast-moving decaying states; for example, a direct lifetime measurement has been made of the decay of the \( \pi^0 \) [11], for which a value of \( \tau = 8.97 \pm 0.22 \pm 0.17 \times 10^{-17} \) s was extracted. This value, which corresponds to a ratio \( \Gamma/E_R \approx 10^{-7} \), is the limit on lifetime measurements. For broad resonances, \( \Gamma/E_R \approx 10^{-1} \) to \( 10^{-4} \), the lineshape is easy to measure and requires an energy resolution of the detection apparatus (and an energy distribution of the prepared quasistable states) comparable to \( \Gamma/E_R \). They can be resolved for much smaller ratios in particular physical systems, such as nuclear resonances, for which widths have been measured with the Mössbauer effect for at least \( \Gamma/E_R < 10^{-15} \) (see table 3.1 of [12] for early data).

Although if one can measure the exponential decay rate, the linewidth will be extremely narrow and similarly, although resonances broad enough for their true width to be measured have very short lifetimes, there have been quasistable states for which both width and lifetime have been measured and the lifetime-width relation \( \Gamma = \Gamma_R(\equiv h/\tau) \) can be experimentally tested. For example, using the Mössbauer effect 40 years ago, the width of the first excited state of Fe\(^{57} \) was found to be \( 4.7 \times 10^{-9} \) eV, which agreed within 10 percent with the direct lifetime measurement of \( \tau = 1.4 \times 10^{-7} \) s (see section 4.2 of [12]).

A recent precision measurement of \( \Gamma = 9.802(22) \) MHz = \( 4.0538(91) \times 10^{-8} \) eV for the natural linewidth of the \( 3p^2P_{3/2} \) excited state of Na (using trapped, ultracold atoms [2]) made it possible to compare for the same atomic state the lifetime calculated from \( \Gamma = h/\Gamma_R = 16.237(35) \) ns, with the lifetime measured directly \( \tau = h/\Gamma_R = 16.254(22) \) ns (using beam-gas-laser spectroscopy [3]). The agreement of those two values to an accuracy that exceeds the accuracy expected of the Weisskopf-Wigner approximation provides sufficient reason to seek a mathematical theory that justifies the relation \( \Gamma = \Gamma_R = 1/\tau \) as an exact equality\(^2\).

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\(^1\)There are other experimental methods to determine the lifetime of a quasistable state, such as using the Primakoff effect for photoproduction of neutral mesons [9] or mapping the resonant dipole-dipole interaction potential for atomic states [10]. We will not consider these methods and their theoretical connection to the lifetime and width measurements described here.

\(^2\)Note that \( \Gamma = \Gamma_R = 1/\tau \) has never been verified in the regime of high energy (relativistic) particle physics. This is of particular interest for the Z-boson resonance, where there is much debate about the parameterization of the lineshape in terms of mass and width and the relation of the width to the inverse
Theoretically, the width $\Gamma$ and inverse lifetime $\Gamma_R$ are often related to each other in the following heuristic way: the time evolution of a state vector $\phi$ in standard quantum mechanics with a self-adjoint Hamiltonian $H$ is given by

$$\phi(t) = e^{-iHt}\phi \text{ for } -\infty < t < \infty. \quad (5)$$

To obtain from (5) the exponentials of (2) and (4), one takes for the time evolution of a decaying state

$$\phi(t) = e^{-i(E_R-i\Gamma_R/2)t}\phi \text{ for } -\infty < t < \infty \quad (6a)$$

which would imply that

$$H\phi = (E_R - i\Gamma_R/2)\phi. \quad (6b)$$

Ignoring the fact that the statements (6) cannot hold for a self-adjoint $H$ in standard quantum mechanics, one can calculate the probability amplitude as a function of time for finding decay products described by $\psi$ in the decaying state $\phi(t)$ (6a) as

$$(\psi, \phi(t)) = (\psi, e^{-iHt}\phi) = f(t) = e^{-iE_Rt}e^{-\Gamma_Rt/2}(\psi, \phi), \text{ for } -\infty < t < \infty. \quad (7)$$

Although it has been known for some time that mathematical theorems of Hilbert space quantum mechanics forbid amplitudes with the exact exponential time dependence [6, 14], this heuristic approach is used to justify the exponential counting rate (4) for decaying states. However, more is required if the quasistable state is to have both the experimental signatures (1) and (2).

To relate the exponential decay amplitude (2) to the Lorentzian, the Fourier transform of the vector $\phi(t)$ is taken:

$$\chi(\omega) = \int_{-\infty}^{+\infty} dt \phi(t)e^{i\omega t} = \phi \int_{-\infty}^{+\infty} dt \ e^{-i(E_R-\frac{i\Gamma_R}{2})t}e^{i\omega t}$$

$$= \frac{i}{\omega - (E_R - i\Gamma_R/2)}\phi. \quad (8)$$

Since the vector $\chi(\omega)$ is the function (1) multiplied by the vector $\phi$, one argues that the relation between (1) and (2), and thus the equality $\Gamma = \Gamma_R = 1/\tau$, has been established theoretically [1].

From the point of view of mathematical rigor, there are problems with the above heuristic derivation: there is the conflict between the self-adjoint Hamiltonian and the complex lifetime (see [13] and references thereof).
eigenvalue in (6) and there is a problem with the boundaries of integration. The Fourier transformation is defined by

$$F[f(t)] = g(\omega) = \int_{-\infty}^{+\infty} dt \, f(t)e^{i\omega t}$$

(9a)

$$F^{-1}[g(\omega)] = f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega \, g(\omega)e^{-i\omega t}$$

(9b)

and this is what is usually used in (8). However, the exact mathematical relation between the Lorentzian and exponential is really given [15] by

$$\frac{i}{\omega - (E_R - i\Gamma_R/2)} = \int_{0}^{+\infty} dt \, e^{-iE_Rt}e^{-\Gamma_R t/2}e^{i\omega t}$$

(10a)

and its inverse is

$$\frac{i}{2\pi} \int_{-\infty}^{+\infty} d\omega \frac{e^{-i\omega t}}{\omega - (E_R - i\Gamma_R/2)} = \theta(t)e^{-i(E_R-i\Gamma_R/2)t} = \begin{cases} e^{-iE_Rt}e^{-\Gamma_R t/2} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}.$$  

(10b)

The ranges of integration for the variables in the mathematical relation (10) are not the accepted ranges for the corresponding physical quantities in standard quantum mechanics.

The energy, the spectrum of the self-adjoint Hamiltonian operator, has a lower bound, say $E_0 = 0$, because the scattering energy is a positive quantity and the stability of matter requires that the spectrum of the Hamiltonian be bounded from below. However the mathematical relation (10b) requires the energy range $-\infty < \omega < +\infty$. Further, the time variable in quantum mechanics extends over $-\infty < t < \infty$ because Hilbert space states evolve in time by a one parameter unitary group (5) and this time evolution is reversible. In the mathematical relation (10a), the values of $t$ range only from 0 to $\infty$. So the connection (10) given by the Fourier transform between the Lorentzian energy distribution and the exponential time dependence of the wave function requires ranges for time $t$ and energy $\omega$ that are incongruous with what standard quantum theory allows.

Tellingly, this energy range $-\infty < \omega < +\infty$ is exactly what Fermi needed for his derivation of zero probability for precursor events. Using the relation (10), the probability amplitude is not given by (7), but by

$$(\psi, \phi(t)) = \theta(t)f(t) = \begin{cases} (\psi, \phi)e^{-iE_Rt}e^{-\Gamma_R t/2} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases},$$

(11)

and the notion of time asymmetry, of an initial time $t_0 = 0$ before which there is no decay probability, emerges from the mathematics (10).
In order to define such an extension of the energy range as an analytic continuation of the energy, we must require certain analyticity properties of the energy wave functions, and that requires modifying the Hilbert space boundary condition hypothesis. This modification lead to the definition of the Gamow vector, a mathematical representation of unstable states, which we shall use in place of $\phi$ in (5). This Gamow vector has both a Breit-Wigner energy distribution and an exponential (and asymmetric as indicated by the $\theta(t)$ in (10b)) time evolution.

4 Gamow Vectors

In this section, we will define the Gamow vectors to be those states whose energy distributions and time dependence exactly fulfill the Fourier transform relations (10). The decaying Gamow vector has a Breit-Wigner energy wave function that extends over $-\infty < \omega < \infty$ and is related by Fourier transform to an exponential that starts at finite time $t_0 = 0$ and extends only into the future, $t_0 < t < \infty$, as required by causality. Although these Gamow vectors cannot be elements of the Hilbert space, they can be rigorously defined as continuous antilinear functionals on a dense subspace of the Hilbert space. They are, like Dirac kets, examples of generalized vectors and their use entails the generalization of the Hilbert space scalar product. In parallel with our development of the Gamow vector, we will show how the same concept of generalized vectors has been used to give meaning to the Dirac kets.

In standard (time symmetric) quantum theory, a state vector $\phi$ is an element of the Hilbert space, $\phi \in \mathcal{H}$ (and one also assumes a one-to-one correspondence between elements of $\mathcal{H}$ and pure physical states, c.f. (29) below). But in Dirac’s formalism of quantum mechanics, the state vector is expressed in terms of an energy wave function $\phi(E) = \langle E | \phi \rangle$ by Dirac’s basis vector expansion

$$\phi = \int_0^\infty d\omega \ |\omega \rangle \langle \omega | \phi \rangle,$$

where $|\omega \rangle$ are the Dirac kets which fulfill

$$H |E \rangle = E |E \rangle$$

for $E \in \text{spec}r.\text{tr}(H) = \{E | 0 \leq E < \infty \}$.  \(13\)

The spectrum$(H)$ is the positive real line $\mathbb{R}^+$; it is bounded from below by $E_0 = 0$.

The scalar product of a vector $\psi \in \mathcal{H}$ with $\phi \in \mathcal{H}$ is

$$\langle \psi | \phi \rangle = (\psi, \phi) = \int_0^\infty d\omega \ \langle \psi | \omega \rangle \langle \omega | \phi \rangle = \int_0^\infty d\omega \ \bar{\psi}(\omega) \phi(\omega).$$

(14)
In other words, the scalar product is given by a (Lebesgue) integral over the physical energy values $0 \leq E < \infty$ in the space $L^2(\mathbb{R}^+)$ of energy wave functions with $\langle E | \phi \rangle, \langle E | \psi \rangle \in L^2(\mathbb{R}^+)$. The space $L^2(\mathbb{R}^+)$ is the representation of the physical Hilbert space $\mathcal{H}$.

The Dirac ket $|E\rangle$ is not actually an element of $\mathcal{H}$. The Dirac kets can be made mathematically rigorous by defining them as continuous antilinear functions (often called functionals) on a dense nuclear subspace $\Phi$ of the Hilbert space, $\Phi \subset \mathcal{H}$. The Dirac basis vector expansion (12) is then the Nuclear Spectral Theorem and holds for all $\phi \in \Phi$. For example, a possible choice for $\Phi$ (and the most common) is such that $\phi(E) = \langle E | \phi \rangle \in \mathcal{S}(\mathbb{R}^+)$, where $\mathcal{S}(\mathbb{R}^+)$ is the Schwartz space of rapidly decreasing, infinitely differentiable (smooth) functions restricted to the positive semiaxis $\mathbb{R}^+$. Then the ket $|E\rangle$ has meaning as an element of $\Phi \times \mathcal{T}$, the space of continuous antilinear functionals on $\Phi$, and its delta function energy distribution $\langle E | \omega \rangle = \delta(\omega - E)$ is a element of $(\mathcal{S}(\mathbb{R}^+))^\times$, the space of tempered distributions. If $\phi$ and $\psi$ are elements of $\Phi$, then their wave functions are smooth and the integral in the scalar product (14) is a Riemann integral. If $\phi$ and $\psi$ are elements of $\mathcal{H}$, the integral in (14) is a Lebesgue integral because some elements of the complete Hilbert space $L^2(\mathbb{R}^+)$ will not be smooth.

With the above preparation, we now want to define a vector called $\psi^G$ ($G$ for Gamow [16]) whose energy wave function is a Breit-Wigner distribution,

$$\psi^G = \int d\omega \ |\omega\rangle \langle \omega | \psi^G \rangle = \int d\omega \ |\omega\rangle \left( i \sqrt{\frac{\Gamma_R}{2\pi}} \frac{1}{\omega - (E_R - i \Gamma_R/2)^2} \right).$$

(15)

We have two alternatives for the boundaries of the integration. If we stick with the rules of standard quantum mechanics we have to choose the boundaries of integration according to (12) to be $0 \leq \omega < \infty$. If we want the Fourier transform to be the exponential, so that $\psi^G$ has an exponential time evolution then we have to choose the range to cover $-\infty < \omega < \infty$.

For values of $\Gamma_R/E_R \sim 10^{-1}$ and less, there is not much numerical difference between these two choices. For the choice $0 \leq \omega < \infty$, which we call the vector $\psi^G_{\text{appr.}}$, the scalar product can be calculated as

$$\langle \psi^G_{\text{appr.}}, \psi^G_{\text{appr.}} \rangle = \int_0^\infty d\omega \ \langle \psi^G_{\text{appr.}}, |\omega\rangle \langle \omega | \psi^G_{\text{appr.}} \rangle = \int_0^\infty d\omega \ \frac{\Gamma_R}{2\pi} \frac{1}{(\omega - E_R)^2 + (\Gamma_R/2)^2} = \frac{1}{\pi} \int_{-2E_R/\Gamma_R}^{2E_R/\Gamma_R} dx \frac{1}{x^2 + 1} = 1 - \frac{1}{\pi} \left( \frac{\Gamma_R}{2E_R} - \frac{1}{3} \left( \frac{\Gamma_R}{2E_R} \right)^3 + \cdots \right).$$

(16a)

The vector $\psi^G_{\text{appr.}}$ is a vector in $\mathcal{H}$, but it is not in the domain of the Hamiltonian $H$.

For the choice $-\infty < \omega < \infty$, which we call $\psi^G$, we obtain

$$\langle \psi^G, \psi^G \rangle = \int_{-\infty}^{\infty} d\omega \ \frac{\Gamma_R}{2\pi} \frac{1}{(\omega - E_R)^2 + (\Gamma_R/2)^2} = 1.$$

(16b)
Although the numerical difference between (16a) and (16b) is small, the mathematical difference is enormous. Although the quantity $(\psi^G, \psi^G)$ is a scalar product in $L^2(\mathbb{R})$, it is no longer a scalar product in the Hilbert space $\mathcal{H}$ represented by $L^2(\mathbb{R}^+)$, since in (14) the scalar product in that space is defined by integration only over the range $0 \leq E < \infty$.

The difference between the energy distributions is
\[
|\langle E|\psi_{\text{appr.}}^G \rangle|^2 = \begin{cases} 
\frac{\Gamma_R}{2\pi} \frac{1}{(E-E_R)^2 + (\Gamma_R/2)^2} & \text{for } 0 \leq E < \infty \\
0 & \text{for } -\infty < E < 0
\end{cases}
\] (17a)
and
\[
|\langle E|\psi^G \rangle|^2 = \frac{\Gamma_R}{2\pi} \frac{1}{(E-E_R)^2 + (\Gamma_R/2)^2} \quad \text{for } -\infty < E < \infty.
\] (17b)

For small values of $\Gamma_R/E_R$ the numerical difference between (17a) and (17b) for values $E < 0$ is negligible. This is why Fermi may have thought he could neglect this difference and make his approximation by extending the lower limit of integration to $-\infty$ without qualitatively affecting the results. But this can be considered a mistake, because there is an important difference between $\psi_{\text{appr.}}^G$ and $\psi^G$. While the vector $\psi_{\text{appr.}}^G$ is a vector in the Hilbert space $(\langle E|\psi_{\text{appr.}}^G \rangle \in L^2(\mathbb{R}^+))$, it is not an eigenvector of the Hamiltonian $H$ and it does not have exponential time evolution [17]. If one wants an eigenvector of a self-adjoint $H$ with exponential time evolution, then the lower limit of the integration must be extended to $-\infty$. Therefore, we follow Fermi and extend the boundary of integration in (15) to define $\psi^G$:
\[
\psi^G = \int_{-\infty}^{\infty} \omega \langle -\omega|\psi^G \rangle = \int_{-\infty}^{\infty} \omega \langle -\omega|\psi^G \rangle \left( i \sqrt{\frac{\Gamma_R}{2\pi}} \frac{1}{\omega - (E_R - i\Gamma_R/2)} \right).
\] (18a)

This vector $\psi^G$ is now no more an element of the physical Hilbert space $\mathcal{H} \leftrightarrow L^2(\mathbb{R}^+)$, but $\psi^G$ is a generalized vector with ideal Breit-Wigner energy distribution that extends over $(-\infty, \infty)$ [18]:
\[
\langle -E|\psi^G \rangle = i \sqrt{\frac{\Gamma_R}{2\pi}} \frac{1}{E - (E_R - i\Gamma_R/2)} \quad \text{for } -\infty < E < \infty.
\] (18b)

Unlike the ordinary Dirac kets $|\omega\rangle$ in (12), we have denoted the kets in (18) by $|\omega^-\rangle$. This notation has been chosen in conformity with the notation for the solutions of the Lippmann-Schwinger equation with $-i\epsilon$ in the denominator (the out-going plane wave solutions). We will give their exact definition after further developing the notion of a generalized vector.

The Gamow vector $\psi^G$ is generalized in the sense that it is not an element of the physical Hilbert space $\mathcal{H}$, but, like Dirac kets, it has meaning as a continuous antilinear functional
\[ \langle \psi | F \rangle = F(\psi) \] on a dense subspace of the Hilbert space, i.e. for \( \psi \in \Phi \subset \mathcal{H} \). Generalizing the scalar product \( \langle \psi, \phi \rangle = \phi(\psi) \) to the bra-ket \( \langle \psi | F \rangle = F(\psi) \) introduces a larger class of vectors \( F \in \Phi^* \supset \mathcal{H}^* \) than the continuous antilinear functionals defined by the Hilbert space scalar product \( \langle \psi, \phi \rangle = \phi(\psi) \). This results in a triplet of spaces, a Rigged Hilbert Space, \( \Phi \subset \mathcal{H} \subset \Phi^* \); the kinds of generalized vectors \( |F\rangle \) in the dual space \( \Phi^* \) are determined by the choice of \( \Phi \).

For the Hilbert space vectors \( \phi \), the scalar product with all \( \psi \in \mathcal{H} \) is a functional \( \phi(\psi) = \langle \psi | \phi \rangle = (\psi, \phi) \) on all \( \psi \in \mathcal{H} \). It is defined in (14) (using Lebesgue integration) and makes mathematical sense for every \( \psi \in \mathcal{H} \). However, for the Dirac ket \( |E\rangle \in \Phi^* \),

\[ |E\rangle = \int_0^\infty d\omega \ |\omega\rangle \delta(\omega - E), \tag{19a} \]

the functional \( \langle \psi | E \rangle \) on some vector \( \psi \) makes mathematical sense only if \( \psi \) is from a smaller space \( \Phi \). For such \( \psi \in \Phi \subset \mathcal{H} \), the wave function \( \psi(E) \)

\[ \overline{\psi}(E) = \langle \psi | E \rangle = \int_0^\infty d\omega \ \langle \psi | \omega \rangle \delta(\omega - E) \tag{19b} \]

is a smooth function in the Schwartz space, \( \langle E | \psi \rangle = \overline{\psi}(E) \in \mathcal{S}(\mathbb{R}^+) \). Then, the meaning of Dirac’s eigenket equation (13) is that of a generalized eigenvector equation

\[ \langle H \psi | E \rangle = \langle \psi | H^* | E \rangle = E \langle \psi | E \rangle \] for all \( \psi \in \Phi \),

where \( H^* \) is the unique extension of the operator \( H^\dagger = H \) to the set of functionals \( F \in \Phi^* \).

The vector \( \psi^G \) defined by (18a) is still more generalized than the Dirac ket \( |E\rangle \) and the generalization of the scalar product, the bra-ket \( \langle \psi^- | \psi^G \rangle \), makes sense only for every \( \psi^- \) in a still smaller subspace \( \Phi_+ \subset \Phi \subset \mathcal{H} \). The space \( \Phi_+ \) will be chosen such that the energy wave functions \( \langle -E | \psi^- \rangle \) will be smooth Hardy functions in the upper half complex energy plane, i.e. \( \langle -E | \psi^- \rangle \in \mathcal{H}_+^2 \cap \mathcal{S}(\mathbb{R}^+) \). Hardy functions are those smooth functions \( \langle -E | \psi^- \rangle \) which can be analytically continued into the upper complex plane, and \( \langle \psi^- | E^- \rangle = \langle -E | \psi^- \rangle \) can be continued into the lower half complex plane and they vanish at the infinite semicircle sufficiently fast [19]. Only for these \( \psi^- \in \Phi_+ \) does the bra-ket \( \langle \psi^- | \psi^G \rangle \) make sense; i.e. we must have \( \psi^G \in \Phi^*_+ \), and then the value of the functional \( \psi^G \) at \( \psi^- \) is

\[ \langle \psi^- | \psi^G \rangle = \frac{i}{2\pi} \int_{-\infty}^{\infty} d\omega \ \langle \psi^- | \omega^- \rangle \frac{\sqrt{2\pi \Gamma R}}{\omega - (E_R - i\Gamma_R/2)} = \frac{1}{2\pi} \int_C d\omega \ \frac{\sqrt{2\pi \Gamma R} \langle \psi^- | \omega^- \rangle}{\omega - (E_R - i\Gamma_R/2)}. \tag{21} \]

In the second integral the closed contour \( C \) is from \( +\infty \) to \( -\infty \) and then along the infinite semicircle; the equality follows because \( \langle \psi^- | \omega^- \rangle \) vanishes on the infinite semicircle if \( \psi^- \in \Phi_+ \).
In distinction to the standard Dirac kets $|\omega\rangle \in \Phi^\times$, the Dirac kets $|\omega^-\rangle$ used in (21) (and before in (18)) are functionals over the Hardy space $\Phi_+$, i.e. $|\omega^-\rangle \in \Phi_+^\times$. This we take as the mathematical definition of the out-going plane wave solutions of the Lippmann-Schwinger equation. From $\Phi_+ \subset \Phi$, it follows that $\Phi_+^\times \supset \Phi^\times$ and so the set of $|\omega^-\rangle$ is larger than the set of standard Dirac kets; the functionals $\langle \omega^- | \psi \rangle$ with $\psi \in \Phi_+$ are defined for any complex number $\omega$ of the lower half complex plane. There are also Hardy spaces $\Phi_-$ for which the kets $|\omega^+\rangle$ are defined for any complex number $\omega$ of the upper half complex plane; the role of these spaces will be discussed in the final section.

Evaluating formula (21) using the Cauchy formula [19], we obtain

$$\langle \psi^- | \psi^G \rangle = \frac{i}{2\pi} \int_{-\infty}^{\infty} d\omega \left( \langle \psi^- | \omega^- \rangle \sqrt{2\pi\Gamma_R} \right) \frac{\sqrt{2\pi\Gamma_R}}{\omega - (E_R - i\Gamma_R/2)} = \langle \psi^- | E_R - i\Gamma_R/2^- \rangle \sqrt{2\pi\Gamma_R}. \quad (22)$$

This means that $\psi^G = \sqrt{2\pi\Gamma_R} |E_R - i\Gamma_R/2^-\rangle \in \Phi_+^\times$ is something like a Dirac ket extended to the complex value $E_R - i\Gamma_R/2$. Also, comparing this with equation (19b) shows that the Breit-Wigner distribution $i/2\pi(E - (E_R - i\Gamma_R/2))^{-1}$ is something like a Dirac $\delta$-function when applied to the well-behaved Hardy functions, i.e. just as integrating over $\delta(\omega - E)$ maps every function $\langle \psi | \omega \rangle \in S$ to its value at $E$, integrating over $i/2\pi(\omega - (E_R - i\Gamma_R/2))^{-1}$ maps every function $\langle \psi^- | \omega^- \rangle \in H_2^\times \cap S_{\mathbb{R}^+}$ to its value at $E_R - i\Gamma_R/2$ with $\Gamma_R \geq 0$.

The properties of well-behaved Hardy class vectors ensure that if $\psi^- \in \Phi_+$, then also $H\psi^- \in \Phi_+$. Then replacing $\psi^-$ in (22) with $H\psi^-$ (or equivalently replacing $\langle \psi^- | \omega^- \rangle$ with $\langle H\psi^- | \omega^- \rangle = \langle \psi^- | H^\times | \omega^- \rangle = \omega \langle \psi^- | \omega^- \rangle$) one proves

$$\langle H\psi^- | \psi^G \rangle = \langle \psi^- | H^\times | \psi^G \rangle = (E_R - i\Gamma_R/2) \langle \psi^- | \psi^G \rangle \text{ for all } \psi^- \in \Phi_+. \quad (23)$$

This proves that the Gamow vector $\psi^G$ is a generalized eigenvector of the self-adjoint and semi-bounded Hamiltonian $H$ with a complex eigenvalue (note that this cannot be proved for $\psi^-_{\text{appr.}}$; it is not an eigenket of $H$). Omitting the arbitrary $\psi^- \in \Phi_+$ in (23), the eigenvector property is written in the Dirac notation as

$$H^\times |\psi^G\rangle \equiv H^\times |E_R - i\Gamma_R/2^-\rangle \sqrt{2\pi\Gamma_R} = (E_R - i\Gamma_R/2) |\psi^G\rangle. \quad (24)$$

Such eigenvectors (eigenkets) of a self-adjoint Hamiltonian with complex eigenvalue do not exist in the Hilbert space or even in $\Phi^\times$, the dual to the Schwartz space, but they are in $\Phi_+^\times \supset \Phi_+$, the dual to the well-behaved Hardy class space $\Phi_+$ of the upper half complex plane.
5 Exponential Decay Law from Breit-Wigner Distribution

As shown above, the Gamow vectors are eigenvectors of a self-adjoint Hamiltonian with complex eigenvalues, thus justifying the heuristic equation (6b). However this is not enough to justify the exponential time evolution (6a); for that we return to the Cauchy formula.

Note that if \( \langle \psi^- | \omega^- \rangle \in \mathcal{H}_2^[-] \cap \mathcal{S}_{[R^+]}, \) then \( \exp(-i\omega t)\langle \psi^- | \omega^- \rangle \) will also be in \( \mathcal{H}_2^[-] \cap \mathcal{S}_{[R^+]}, \) though only for \( t \geq 0. \) For \( t < 0, \) the exponential factor blows up as \( \omega \) approaches the infinite semicircle. With this fact, the time evolution of the Gamow vector is calculated for all \( \psi^- \in \Phi_+ \) as follows:

\[
\langle e^{iHt} \psi^- | E_R - i\Gamma_R/2 \rangle = \langle \psi^- | e^{-iHt} | E_R - i\Gamma_R/2^- \rangle \\
= \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega \frac{\langle \psi^- | e^{-iHt} | \omega^- \rangle}{\omega - (E_R - i\Gamma_R/2)} \\
= \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega \frac{e^{-i\omega t} \langle \psi^- | \omega^- \rangle}{\omega - (E_R - i\Gamma_R/2)} \\
= e^{-iE_R t} e^{-\Gamma_R t/2} \langle \psi^- | E_R - i\Gamma_R/2^- \rangle
\]

for all \( \psi^- \in \Phi_+ \) and for \( t \geq 0 \) only.

The mathematical result (25) is the generalization of the formula for the Fourier transform of the Lorentzian (10b), which one observes if one multiplies the Lorentzian in (10b) by a Hardy class function of the lower complex energy plane, i.e. by a function \( \langle \psi^- | \omega^- \rangle \in \mathcal{H}_2^[-] \cap \mathcal{S}_{[R^+]}: \)

\[
\frac{i}{2\pi} \int_{-\infty}^{+\infty} d\omega \frac{e^{-i\omega t} \langle \psi^- | \omega^- \rangle}{\omega - (E_R - i\Gamma_R/2)} = \theta(t)e^{-i(E_R - i\Gamma_R/2)t} \langle \psi^- | E_R - i\Gamma_R/2^- \rangle
\]

for any \( \langle \psi^- | \omega^- \rangle \in \mathcal{H}_2^[-] \cap \mathcal{S}_{[R^+]}: \)

The mathematical result (25) for the Gamow ket \( \psi^G = | E_R - i\Gamma_R/2^- \rangle \sqrt{2\pi \Gamma_R} \) can also be written in the Dirac notation if one omits the arbitrary \( \psi^- \in \Phi_+: \)

\[
e^{-iH^*t} | E_R - i\Gamma_R/2^\prime \rangle = \int_{-\infty}^{+\infty} d\omega \frac{i}{2\pi} \frac{e^{-i\omega t}}{\omega - (E_R - i\Gamma_R/2')} | \omega^- \rangle
\]

\[
= e^{-iE_R t} e^{-\Gamma_R t/2} | E_R - i\Gamma_R/2^\prime \rangle \quad \text{for } t \geq 0 \text{ only.}
\]

The equality in the second line of (27) is the vector analog of the Fourier transformation formula (10b), and (26) is the wave function analog of (10b). It expresses the exponential law for the time evolution of the vector \( \psi^G(t) \) with idealized Breit-Wigner energy distribution.
Thus the Gamow vector is needed to accomplish the task posed: derive the exponential law from the Lorentzian energy distribution. The well-known Fourier transformation formula (10) suggests that we should follow Fermi’s “mistake” and extend the energy to $-\infty$. As can be expected from the $\theta(t)$ in (10), time asymmetry about the time $t = 0$ (which can be any time $t_0 \neq -\infty$) results.

Such a privileged moment of time $t = 0$ cannot exist in standard quantum mechanics. The time evolution in the Hilbert space is necessarily given by the reversible, unitary group of operators $U(t) = \exp(-iHt)$ [8]; with every $U(t)$ there exists also an inverse $(U(t))^{-1} = U(-t)$. However, in the space $\Phi^+$, the time evolution operator $U^\times(t) = \exp(-iH^\times t)$ (27) is the uniquely defined extension of the unitary operator $U(t)$ for $t \geq 0$ only; $(U^\times(t))^{-1}$ is not defined on $\Phi^+$ and $U^\times(t)$ is a semigroup. The mathematical necessity of this time asymmetry and special time $t = 0$ has a physical consequence: the semigroup time transformation properties of the Gamow vector (27) have the notion of time’s arrow (or irreversibility) and a privileged time $t_0$ built in.

6 Conclusion

In summary, the derivation in introductory textbooks which relates the Breit-Wigner energy distribution to the exponential time evolution, and which makes no sense in standard mathematical theory of quantum mechanics in the Hilbert space, can be given a mathematical meaning. To do so leads to the Gamow vectors and their irreversible semigroup time evolution. How one has to construct the Gamow vector is indicated by the mathematical relations (10) for the Fourier transform between Lorentzian and exponential. The relation (10) already contains the first appearance of time asymmetry and forebodes the inadequacy of the unitary time evolution of standard quantum mechanics. The time evolution of the Gamow vectors (27) can be viewed as just the generalized vector version of the relation (10) between generalized functions.

To incorporate generalized vectors requires a partial revision of the mathematical theory of quantum physics. The Hilbert space is a vector space with topology (i.e. the notion of convergence for infinite sequences of vectors) given by norm convergence. While the appropriateness of a linear space of states is experimentally verified by the glorious success of the superposition principle, the appropriateness of the norm topology cannot be directly tested due to the limited resolution of experimental equipment and the infinite number of measurements required to make a topological distinction. It is the algebraic properties of a linear scalar product space, and not the topological properties, to which most physicists refer when discussing the Hilbert space. The Hilbert space results when this algebraic structure is completed with the norm topology, and then the mathematical theory used in [5, 6, 8] is
obtained.

The first revision of this mathematical theory of quantum mechanics was made so that elements like the Dirac ket could be incorporated; it is now well-accepted and uses the Rigged Hilbert space

\[ \Phi \subset \mathcal{H} \subset \Phi^\times \]  

where \( \Phi \) is usually chosen to be the Schwartz space (which has a stronger topology than \( \mathcal{H} \)) and then \( \Phi^\times \), the space of continuous antilinear functionals on \( \Phi \), is the space of tempered distributions [20].

The scalar product \( |(ψ, φ)|^2 \) in the Hilbert space is interpreted as the probability to detect the observable \( |ψ⟩⟨ψ| \) in the state \( φ \). For the generalized “scalar product” with a Dirac ket, the physical interpretation of \( |⟨E|ψ⟩|^2 \) is as the probability density in energy using an apparatus described by \( ψ \). Since the macroscopic apparatus will have a smooth energy resolution, the amplitude \( ⟨E|ψ⟩ \) is best described by a smooth function, \( ⟨E|ψ⟩ \in S(\mathbb{R}^+) \) or equivalently \( ψ \in \Phi \). Thus the mathematic necessity of using the space \( \Phi \) to define the Dirac kets has a physical justification.

The second revision is to incorporate the Gamow vector. In order to include generalized vectors with complex (and also negative) energies, the energy wave functions \( ⟨−E|ψ⟩ \) cannot be just smooth functions but must also be analytically continuatable. Instead of using the Hilbert space axiom

\[ \{\text{space of prepared states}\} = \{\text{space of detected observables}\} = \mathcal{H} \]  

or the slightly more general revision

\[ \{\text{space of prepared states}\} = \{\text{space of detected observables}\} = \Phi \subset \mathcal{H}, \]  

we distinguish mathematically between states and observables [21] and make the new hypothesis [22]:

The prepared states are described by:

\[ \{ϕ^+\} = \Phi_- \subset \mathcal{H} \subset \Phi_\times^− \]

and the registered observables by:

\[ \{ψ^−\} = \Phi_+ \subset \mathcal{H} \subset \Phi_\times^+. \]

Here we use a pair of Rigged Hilbert spaces of Hardy type, where the energy wave functions of the vectors \( ϕ^+ \in \Phi_- \) and \( ψ^− \in \Phi_+ \) are well-behaved Hardy functions in the lower and upper half complex planes, respectively [23]. The Gamow vectors, together with the out-plane wave solutions of the Lippmann-Schwinger equations, are elements of the space \( \Phi_\times^+ \).

The states prepared by the preparation apparatus are, by hypothesis (31), not generalized vectors like the Gamow vectors (18); the states \( ϕ^+ \in \Phi_- \subset \mathcal{H} \) are represented by the Dirac
basis vector expansion (12) with wave functions \( \phi^+(\omega) = \langle ^+\omega | \phi^+ \rangle \) that are well-behaved Hardy functions in the lower half plane. From (31) one can prove (see the reference in citation [19]) that an alternate “complex basis vector” expansion holds for every \( \phi^+ \in \Phi_- : \)

\[
\phi^+ = \sum_{i=1}^{N} |\psi^G_i\rangle \langle \psi^G_i| \phi^+ \rangle + \int_{-\infty}^{+\infty} d\omega \ |\omega^+\rangle b(\omega).
\]

Here, the \( \psi^G_i \) represent \( N \) Gamow vectors with eigenvalues \( z_{R_i} = E_{R_i} - i\Gamma_i/2 \) which are associated to \( N \) first order resonance poles at the pole positions \( z_{R_i} \) of the analytically continued S-matrix. However, in addition to the sum over the resonance states \( \psi^G_i \), there appears an integral over the continuous basis vectors with weight function \( b(\omega) \). Whereas each Gamow vector in (32) corresponds to a Breit-Wigner resonance amplitude, the integral corresponds to a slowly varying background in the scattering amplitude.

Considering just one resonance \( (N = 1) \), then the first term of (32) represents the state of the resonance with its characteristic exponential time behavior. However, there is always also a background term whose time evolution is not exponential and whose energy-dependence \( b(\omega) \) depends on the way in which the state was prepared. This background term is a theoretical necessity but in a particular experiment may be unimportant. The influence of this background can explain observed deviations from the exponential decay law [24], which had been derived mathematically a long time ago [6] for Hilbert space vectors, and thus also for \( \phi^+ \in \Phi_- \subset \mathcal{H} \). But the Gamow vectors are elements of \( \Phi^\times_- \) and are not in \( \mathcal{H} \), and therefore their time evolution can follow the time-honored exponential law.

In the spaces \( \Phi_- \) and \( \Phi^\times_- \) and in the space \( \Phi_+ \) and \( \Phi^\times_+ \) (but not in \( \mathcal{H} \)) only semigroup time evolution is defined. The semigroup time evolution is the crucial difference between standard time symmetric quantum mechanics in the Hilbert space or in a triplet such as (28) and a quantum theory that includes time asymmetry. By changing the boundary conditions, i.e. the space of allowed solutions of the dynamical equations (Schrödinger or Heisenberg) from (29 or 30) to (31), we obtain semigroup evolution. Time asymmetric boundary conditions of time symmetric dynamical (differential) equations are nothing new in physics. The cosmological arrow of time and the radiation arrow of time are consequences of such a theory. The idea of a fundamental arrow of time in the laws of quantum mechanics is also nothing new, c.f. [25] where it had to be affixed artificially on top of the time symmetric solutions because the Hilbert space boundary condition for the quantum mechanical Cauchy problem excluded time asymmetry. With the hypothesis (31), asymmetric time evolution is a consequence of the mathematical property of the Hardy space functions [4] (specifically the Paley-Wiener theorem for their Fourier transform [19]) in the same way as reversible time evolution is a mathematical consequence of the properties of Hilbert space functions [8].

The Gamow vectors \( \psi^G \) describe resonance states (without background). They are the
only mathematical entity which can exactly combine the properties of Lorentzian energy
distribution and exponential time evolution. The relation between the lifetime \( \tau \) (from the
counting rate (4)) and the Lorentzian width \( \Gamma \) (from the cross section (3)) is precisely and
exactly \( \tau = \frac{\hbar}{\Gamma} \) and this lifetime-width relation holds exactly only for the Gamow vectors.

The Gamow vectors also predict causal probabilities. As a consequence of their semigroup
time evolution (25,27) the concept of a finite time \( t_0 \geq \infty \) (represented by the semigroup
time \( t = 0 \)) is introduced for a quantum system described by \( \psi^G \), and before that time
the quantity \( \langle e^{iHt}\psi^-|\psi^G \rangle = \langle \psi^-|e^{-iHt}\psi^G \rangle = \langle \psi^-|\psi^G(t) \rangle \) does not exist. Extending the
usual probability interpretation of \( |(\psi,\phi)|^2 \) to the Gamow vector, the generalized scalar
product \( |\langle \psi^-|\psi^G(t) \rangle|^2 \) describes the probability to detect the decaying quantum state \( \psi^G = |E_R-i\Gamma_R/2\rangle \sqrt{2\pi\Gamma_R} \) at the time \( t \) with an apparatus described by \( \psi^- \). Lets put this into a
specific physical context for the purpose of example by considering the process where a \( K^0 \)
is created by \( \pi^-p \rightarrow K^0\Lambda \) and then decays via the channel \( K^0 \rightarrow \pi^+\pi^- \). The Gamow vector
\( \psi^G \) represents the decaying state \( K^0 \) and \( \psi^- \) represents the the decay products \( \pi^+\pi^- \) which
the detector registers. Since the decay products cannot be detected before the decaying state
\( \psi^G \) has been created (or prepared) at an arbitrary but finite time \( t_0 = 0 \), the probability
to detect the decay product \( \psi^- \), \( |\langle \psi^-|\psi^G(t) \rangle|^2 = |\langle e^{iHt}\psi^-|\psi^G \rangle|^2 \), makes physical sense only
for \( t \geq 0 \). This means that for the decaying kaon system, we should expect a non-zero
counting rate for the decay products \( \pi^+\pi^- \) only for times after the \( K^0 \) has been created in
the reaction \( \pi^-p \rightarrow K^0\Lambda \) and leaves the proton target. Therefore the probability for this
decay \( |\langle \psi^-|\psi^G(t) \rangle|^2 \) must be different from zero only for \( t > t_0 = 0 \) and this is precisely the
time asymmetry predicted by (25). Thus the semigroup time \( t = 0 \) is interpreted as the time
\( t_0 \) at which the creation of the decaying state is completed and the registration of the decay
products can begin. For the \( K^0 \) system this time is very accurately measurable because the
\( K^0 \) is created by the strong interaction with a time scale of \( 10^{-23} \) s and it decays weakly with
a time scale of \( 10^{-10} \) s.

The result (25) predicts that the probability \( |\langle \psi^-|U^\times(t)|\psi^G(t) \rangle|^2 \) to
detect decay products \( \psi^- \) from a state \( \psi^G \) which has been prepared at \( t_0 \) is zero for \( t < t_0 \).
Therefore the probability of precursor events is zero, in contrast to the Hilbert space result
obtained in [5] which does predict precursor events. This feature is what we mean by
causality.

This prediction applies when the detector \( \langle \psi^-|\psi^- \rangle \) is placed right at the position of the
decaying particle \( \psi^G \) and is in the rest frame of the decaying state. If the decaying state and
detector are displaced or have relative motion, then time translation alone cannot describe
the situation and Poincaré transformations must be considered. To represent decaying states
that undergo relativistic transformations, relativistic Gamow vectors were introduced [26].
Unlike the relativistic stable particle states which furnish a unitary (Wigner) representation
of the Poincaré group [27], relativistic Gamow vectors furnish only a semigroup representation of the Poincaré transformations into the forward light cone

\[ x^2 = t^2 - x^2 \geq 0, \quad t \geq 0, \]  

(33)

(c.f. Appendix and references thereof). One therefore predicts in this relativistic theory space-time translated probability amplitudes

\[ \langle \psi^- | U^\times (1, (t, x)) | \psi^G \rangle = \langle \psi^- | e^{-i(H^\times t - P \cdot x)} | \psi^G \rangle \]  

(34)

only for those space-time translations \((t, x)\) which fulfill (33).

This is the relativistic analogue of the time asymmetry \(t \geq 0\) in (25). Therefore, decay events of the space-time translated state \(U^\times (1, (t, x))|\psi^G\rangle\) are only predicted to occur for \(t \geq 0\) and \(t^2 \geq x^2 \equiv r^2/c^2\) or for \(r/t \leq c\). This means the probability for decay events cannot propagate faster than the speed of light and Einstein causality is obeyed by these semigroup representations of the Poincaré transformations. This confinement to semigroup transformations into the forward light cone follows from the Hardy space hypothesis (31). Unitary representations of the Poincaré group in the Hilbert space are not restricted to the forward light cone (33) and therefore do not fulfill Einstein causality. This can be deduced from the theorem in reference [5] which is based on the Hilbert space axiom (29). The Hardy space hypothesis predicts probabilities only for semigroup time evolution (for non-relativistic) or Poincaré semigroup evolution in the forward light cone (for relativistic) and as a consequence both the causality conditions “no registration of an observable in a state before that state has been prepared” and the Einstein causality condition “no propagation of probabilities with a speed faster than light” are fulfilled.

Fermi’s fortuitous (but at the time unjustified) approximation, which was made so that causality was maintained, pointed the way to the Gamow vectors, and the mathematical relation of the Fourier transform (10) foretold the time asymmetry.

**Appendix: Poincaré Transformations of Relativistic Resonance States**

To address Einstein causality for the representation of resonances and decaying states, we must consider Poincaré transformations of a relativistic Gamow vector. We will therefore briefly record some definitions and results. Since this subject exceeds the scope of the present paper we shall just review the results here and refer to [26] for details.

The non-relativistic Gamow vector was in some sense a generalization of the Dirac ket to complex energies and so we begin by summarizing the representations of relativistic stable
states. Stable particles are described by irreducible unitary representation spaces of the Poincaré group characterized by the invariant mass squared \(m^2\) and by spin \(j\) [27]. The basis vectors \([j, m^2]|\mathbf{p}j_3\rangle\) of an irreducible representation space are usually labelled by a component of the spin \(j_3\) and the spatial components \(\mathbf{p}\) of the 4-momentum \(p = (p_0, \mathbf{p})\). Instead of \(\mathbf{p}\) one could equivalently use the spatial components \(\hat{\mathbf{p}}\) of the 4-velocity \(\hat{\mathbf{p}} = p/m = (\gamma, \hat{\mathbf{p}}) = (\gamma, \gamma \mathbf{v})\) and then use the 4-velocity eigenkets \([j, m^2]|\hat{\mathbf{p}}j_3\rangle\) as basis vectors.

The relativistic Gamow vector \([j, s_R]|\hat{\mathbf{p}}j_3\rangle\) is defined from the resonance pole \(s_R = (M - i\Gamma/2)^2\) of the S-matrix for a resonance scattering process \(1 + 2 \to R \to 3 + 4\) by

\[
[j, s_R]|\hat{\mathbf{p}}j_3\rangle = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{ds}{s - s_R} [j, s]|\hat{\mathbf{p}}j_3\rangle. \tag{A1}
\]

Here \(s = (p_1 + p_2)^2\) is the invariant mass squared and the integration extends along the lower edge of the real \(s\)-axis in the second sheet (the sheet that contains the pole at \(s_R\)). The definition (A1) is the relativistic analogue of (18a). The kets \([j, s]|\hat{\mathbf{p}}j_3\rangle\) are the out-plane wave solutions of the Lippmann-Schwinger equation and are functionals over a Hardy space \(\Phi_+\) (which is slightly different than the space \(\Phi_+\) chosen for the non-relativistic case but has the same analyticity properties), whereas the usual Wigner basis kets for unitary representations are functionals over the Schwartz space.

The transformation properties of the relativistic Gamow vectors under Poincaré transformations \((\Lambda, x)\) (where \(x = (t, \mathbf{x})\)) are given by

\[
U^x(\Lambda, x)[j, s_R]|\hat{\mathbf{p}}j_3\rangle = e^{-i\gamma \sqrt{s_R}(t-\mathbf{x} \cdot \mathbf{v})} \sum_{j_3'} D^j_{j_3 j_3'} (W(\Lambda^{-1}, \hat{\mathbf{p}})) [j, s_R]|\Lambda^{-1} \hat{\mathbf{p}}j_3'\rangle \tag{A2a}
\]

only for

\[
t \geq 0 \text{ and } t^2 \geq \mathbf{x}^2. \tag{A2b}
\]

The matrix \(D^j_{j_3 j_3'}\) is the \((2j+1)\)-dimensional representation of the Wigner rotation \(W(\Lambda^{-1}, \hat{\mathbf{p}}) = L^{-1}(\Lambda \hat{\mathbf{p}})AL(\hat{\mathbf{p}})\) and \(L(\hat{\mathbf{p}})\) is the standard boost, which depends only on \(\hat{\mathbf{p}}\) and not the momentum \(p = \sqrt{s_R} \hat{\mathbf{p}}\). It is this property that allows construction of the representation \([j, s_R]\) by analytic continuation of the Lippmann-Schwinger kets to the Gamow kets,

\[
[j, s]|\hat{\mathbf{p}}j_3\rangle \to [j, s_R]|\hat{\mathbf{p}}j_3\rangle \tag{A3}
\]

in such a way that \(\hat{\mathbf{p}}\) remains unaffected and always real. These representations \([j, s_R]\) are the “minimally complex” representations.
In the limit as the complex invariant mass squared $s_R$ becomes real, $s_R = (M - i\Gamma/2)^2 \rightarrow m^2$, the transformation formula (A2a) looks exactly like the well-known formula for Wigner’s unitary representations $[j, m^2]$:

$$U^x(\Lambda, x) |[j, s]p_j3^-\rangle = e^{-ip \cdot x} \sum_{j'_3} D_{j3j'_3}^j (W(\Lambda^{-1}, \hat{p})) |[j, s]\Lambda^{-1}\hat{p}j'_3^-\rangle,$$  \hspace{1cm} (A4)

where $\exp(-ip \cdot x) = \exp(-i\gamma m(t - v \cdot x))$. However, the important difference between (A4) and Wigner’s transformation formula for unitary representations is that Wigner’s transformation formula holds for the whole Poincaré group

$$\mathcal{P} = \{(\Lambda, x) : \Lambda \in S0(1, 3), x \in \mathbb{R}^4 \}$$  \hspace{1cm} (A5)

whereas the transformation formulas (A4) for the Lippmann-Schwinger kets and (A2a) for the Gamow kets hold only for the orthochronous Lorentz transformations and space-time translations into the forward light cone:

$$\mathcal{P}_+ = \{(\Lambda, x) : \Lambda \in S0(1, 3), x \in \mathbb{R}^4 | x^2 = t^2 - x^2 \geq 0, t \geq 0 \}.$$  \hspace{1cm} (A6)

The formulas (A2a,A2b) are the relativistic generalizations of the transformation formula (27) so as to include all allowed space-time translations, which, as for the time translations in (27), form only a semigroup $\mathcal{P}_+$.

Specializing (A2a) to space-time translations $(1, x)$, we obtain from (A2a) the space-time translated probability amplitude in analogy to (25):

$$\langle \psi^- (x) | [j, s_R]p_j3^- \rangle = \langle U(1, x)\psi^- | [j, s_R]p_j3^- \rangle = e^{-i\gamma (M - i\Gamma/2)(t - x \cdot v)} \langle \psi^- | [j, s_R]p_j3^- \rangle \text{ for all } \psi^- \in \Phi^+ \hspace{1cm} (A7)$$

but only for $t \geq 0$ and $t^2 \geq x^2$ because the Hardy space $\Phi_+$ does not contain vectors $U(1, x)|\psi^-\rangle$ for which (A2b) is not fulfilled. Introducing the speed of light, the condition for the predicted probability (A2b) becomes $t \geq 0$ and $c^2 t^2 \geq x^2$ or $|x|/t \leq c$. This means the probability for decay events is only predicted for the forward light cone and cannot propagate faster than the speed of light. Thus Einstein causality is not violated.

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References


[16] Gamow was the first to use eigenfunctions of the Hamiltonian with complex energy for a heuristic description of unstable states in G. Gamow, Z. Phys. 51, 204 (1928). At that time in mathematics, these functions were considered even more pathological than the Dirac delta function, and so they did not achieve the popularity of the Dirac kets.


[18] The mathematics is a little subtle here. The wave function (18b) is an element of $\mathcal{H}_2^+ (\mathbb{R})$. In $L^2 (\mathbb{R}) = \mathcal{H}_2^- \oplus \mathcal{H}_2^+$, the Hamiltonian is represented by the multiplication operator. This multiplication operator has deficiency indices $(0, 1)$ in $\mathcal{H}_2^-$ and $(1, 0)$ in $\mathcal{H}_2^+$. On the other hand, the multiplication operator has deficiency indices $(0, 0)$ in $\mathcal{H}_2^\pm \cap \mathcal{S}_{|\mathbb{R}^+}$ and therefore $H$ is essentially self-adjoint in the spaces $\Phi^\pm$. See O. Civitarese, M. Gadella, R. Id Betan, Nucl. Phys. A 660, 255 (1999).

[19] Equation (22) and several subsequent formulas (23,25,26) follow also from theorems for Hardy functions. Hardy functions were first suggested for this purpose by H. Baumgartel, private communication. For a summary of properties of Hardy class functions needed here, see Appendix A by M. Gadella in A. Bohm, S. Maxson, M. Loewe, M. Gadella, Physica A 236, 485 (1997).


[21] The concept of the time $t_0$ is already contained in the historical paper of Feynman, R.P. Feynman, Rev. Mod. Phys. 20, 367 (1948), in particular p. 372 and 379, who also distinguishes between the state at times $t' < t_0$ which is defined by the preparation (our prepared states $\{\phi^+\}$) and what he calls the “state characteristic of the experiment” at time $t'' > t_0$ (our registered observables $\{\psi^-\}$). The possibility that $\{\phi^+\} \neq \{\psi^-\}$
he mentions in Footnote 14, attributing it to H. Snyder, but does not consider it. We implement this possibility by the two Hardy spaces $\Phi_-$ and $\Phi_+$ in (31) and arrive at a consistent description of resonances and decay which obeys the principle of causality. This is not possible in the Hilbert space theory.


