An $O(3)$ Global Anomaly in 0+1 Dimension

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Abstract

We present a simple exactly solvable quantum mechanical example of the global anomaly in an $O(3)$ model with an odd number of fermionic triplets coupled to a gauge field on a circle. Because the fundamental group is non-trivial, $\pi_1(O(3)) = \mathbb{Z}_2$, fermionic level crossing - circling occurs in the eigenvalue spectrum of the 1-dim. Dirac operator under continuous external field transformations. They are shown to be related to the presence of an odd number of normalizable zero modes in the spectrum of an appropriate 2-dim. Dirac operator. We argue that fermionic degrees of freedom in the presence of an infinitely large external field violate perturbative decoupling.

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The phenomenon of fermion level crossing in field theory and its relation to the existence of normalizable fermionic zero modes of the Dirac operator in the presence of topologically non-trivial gauge fields is well known [1].

The $SU(2)$ global anomaly [2] in particular is related to the phenomenon of level crossing in the eigenvalue spectrum of the 4-dim. ($D = 4$) Dirac operator. This eigenvalue flow corresponds to the existence of a zero mode for an appropriately defined 5-dim. ($D = 5$) Dirac operator in the presence of an external topologically non-trivial gauge field. As a result the fermionic path integral is not gauge invariant and the theory is not self-consistent. Recently a generalization was presented [3] for the case of an $SU(2)$ theory with an odd number of Weyl doublets and arbitrary Yukawa couplings and quark masses.

As a result of the antisymmetric character of the $D = 4$ Dirac operator its spectrum of eigenvalues is organized in paires $\lambda, -\lambda$ on the complex plane. Under topologically non-trivial gauge transformation a fermionic level circling effect was shown to occur for an odd number of positive eigenvalues. This in turn was related to the existence of an odd number of normalizable zero modes in the spectrum of the $D = 5$ Dirac operator consistently with the Atiyah-Singer index theorem. While the general picture appears to be unambiguous some subtle question still remain with regard to the precise nature of fermionic eigenvalue rearrangement on the complex plane, consistently with the index theorem.

Of equal interest is also the issue of pertubative decoupling of heavy fermions from the light sector of a theory in the presence of Yukawa interactions and global anomalies [4]. More specifically it was argued, in the context of effective field theories with $U(1)$ global anomalies that the low energy physics is in fact sensitive to the presence of otherwise integrated out heavy fermions through their zero modes(nonperurbative non-decoupling). We would like to investigate this possibility for the case of the $SU(2)$ global anomaly as well.

In this paper we examine a simple quantum mechanical model which captures, we believe, the essential properties of the $D = 4$ $SU(2)$ gauge field theory with odd number of fermionic Weyl doublets that posseses an anomaly[2, 3]. This is an $O(3)$ gauge model with odd number of real fermion $\psi$ in the fundamental representation of the gauge group. With no loss of generality we first consider the case of a single real fermionic triplet. A generalization to the case of an odd number of fermionic triplets is of course straightforward. The lagrangian reads as

$$L = i\dot{\psi}\psi + \psi A\psi,$$

where $A$ is an $O(3)$ gauge field and $\dot{\psi}$ means the derivative in time variable $t$. We shall consider the compactified version of the theory where the 'fields' live on a circle, so that $0 \leq t \leq \beta$ where $\beta$ is a positive parameter. We also introduce appropriate boundary conditions for fields

$$\psi(\beta) = -\psi(0), \quad A(\beta) = A(0).$$

This model has a gauge invariance under transformations

$$\psi \rightarrow e^{i\alpha}\psi, \quad A \rightarrow e^{i\alpha}i\partial_t e^{-i\alpha}.$$
where $\alpha$ is a periodic function taking values in the $O(3)$ algebra. It is clear that the
gauge field $A$ has almost no physical degrees of freedom. Indeed by an appropriate
gauge transformation it may take the form of a constant matrix belonging to the
$O(3)$ algebra. Subsequently by a constant $O(3)$ rotation it can be transformed to
\begin{equation}
A = aT_2, \tag{4}
\end{equation}
where $a$ is a real number and $T_2$ is a generator of $O(3)$ group. If $a$ is not a multiple
of $2\pi/\beta$ then this gauge field can not be removed by a gauge transformation with
a parameter periodic in time. In turn if $a = 2\pi n/\beta$, $n \in \mathbb{Z}$, then this gauge field is
gauge equivalent to $a = 0$. It is also clear that the spectrum of the ’Dirac’ operator
\begin{equation}
iD = i\partial_t + A \tag{5}
\end{equation}
does not change under a shift $a \to a + 2\pi/\beta$.

However because of the homotopic theorem [5]
\begin{equation}
\pi_1(O(3)) = \mathbb{Z}_2 \tag{6}
\end{equation}
and in analogy with Witten’s $SU(2)$ anomaly we expect our quantum mechanical
model be equally inconsistent. This is indeed the case, as we will see below.

First let us note that the ’Dirac’ operator $D$ is antisymmetric and antihermitian
\begin{equation}
D^T = -D, \quad D^+ = -D. \tag{7}
\end{equation}
The eigenvalues of this operator are purely imaginary. Moreover the non-zero eigen-
values $i\lambda$ are paired as $(i\lambda, -i\lambda)$. We shall soon see that for a generic gauge field
this operator has no zero modes.

The partition function of our model is proportional to the Pfaffian of the operator
$D$. Let us define this Pfaffian as a square root of the determinant of the ’Dirac’
operator $D$
\begin{equation}
PfD = (\det D)^{1/2}. \tag{8}
\end{equation}
For a generic gauge field this square root can be defined as a product of all positive
eigenvalues.

Now let us find the spectrum of the ’Dirac’ operator $D$ in an external gauge field
$A = aT_2$ for an arbitrary real constant $a$. The equation for the eigenfunction that
corresponds to an eigenvalue $i\lambda$ reads as
\begin{equation}
D\psi = i\lambda\psi. \tag{9}
\end{equation}
The solution is given by
\begin{equation}
\psi = e^{it\lambda}e^{-itaT_2}\psi_0, \tag{10}
\end{equation}
where $\psi_0$ is a constant 3-component vector. The spectrum is determined by imposing
the antisymmetric boundary condition to the fermionic field
\begin{equation}
e^{i\beta\lambda}e^{-i\beta aT_2}\psi_0 = -\psi_0. \tag{11}
\end{equation}
The existence of a non-zero solution $\psi_0$ to the latter equation is equivalent to
\[
\det \left( e^{i\beta \lambda} e^{-i\beta aT_2} \psi_0 + 1 \right) = 0.
\] (12)
From this equation we find three types of eigenvalues $(+, 0, -)$ given by
\[
\lambda_k^{(0)} = (\pi + 2\pi k)/\beta, \quad \lambda_k^{(+)} = a + (\pi + 2\pi k)/\beta;
\lambda_k^{(-)} = -a + (\pi + 2\pi k)/\beta, \quad k \in \mathbb{Z}.
\] (13)
The spectrum of the Dirac operator is organized in pairs $(\lambda, -\lambda)$ of eigenvalues with opposite sign and type. As it is also depicted in fig.2 when we adiabatically change $a$ into $a + 2\pi/\beta$ there exists exactly two eigenvalues $(\lambda_3^+, -\lambda_3^-)$ that cross the zero of the real eigenvalue axis and into the $(\lambda_1^+, -\lambda_1^-)$ levels respectively preserving their type. All the other eigenvalue pairs of the positive and negative eigenvalue type rearrange themselves accordingly into distinctly different available levels of the same type and sign character. The eigenvalues of the zero type don’t feel the external field and thus they intact.

It is easy to calculate the value of the Pfaffian. We find it to be
\[
PfD = \prod_{k=0}^{\infty} \left(a + \frac{\pi + 2\pi k}{\beta}\right) \left(-a + \frac{\pi + 2\pi k}{\beta}\right) \left(\frac{\pi + 2\pi k}{\beta}\right) = \text{const} \prod_{k=0}^{\infty} \left(1 - \frac{a \beta}{\pi + 2\pi k}\right)^2 \sim \cos \frac{a \beta}{2}.
\] (14)
It is thus obvious that when $a$ changes adiabatically to $a + 2\pi/\beta$ the Pfaffian changes sign.

At this point we remark that in order for our model to make sense as a quantum gauge theory we have to integrate over all gauge field configurations. To that end we choose a gauge for $A = aT_2$. Because the global anomaly breaks the gauge invariance under “big” gauge rotation $a \rightarrow a + 2\pi/\beta$ we must integrate over the whole real line. We may observe that the partition function
\[
Z = \int_{-\infty}^{\infty} da \cos \frac{a \beta}{2} = 0
\] vanishes identically, a manifestation of the global anomaly.

It is interesting now to look at the case when $a$ is complex. In this case the ‘Dirac’ operator becomes non-hermitian and hence its eigenvalues are complex (Fig. 1 & 3). Eqs. (13) suggest that as the parameter $a$ changes adiabatically to $a = 2\pi/\beta$ the pair of eigenvalues $(\lambda_3^+, \lambda_3^-)$ that previously crossed the zero real axis in this case bypass each other at a distance (“level circling”) which is proportional to twice the imaginary part of the external field $a$.

We can find a corresponding normalizable zero mode of an appropriate euclidean $D = 2$ Dirac operator. The natural candidate reads as
\[
\sigma_1 \partial_t + \sigma_2 \partial_x - i\sigma_1 A,
\] (15)
where $x$ is a second coordinate parametrizing a continuous change of the gauge field (the $D = 2$ field is taken in the 'Hamiltonian' gauge $A_x = 0, A_t = A$). Multiplying this equation by $i\sigma_1$ we get the following equivalent form

$$\hat{D} = i\partial_t - \sigma_3 \partial_x + A.$$  \hspace{1cm} (16)

This equation splits into two equations for the components of a spinor $\psi = (\psi_1, \psi_2)$. However it is clear that we should take only one of these equations to describe the process of level crossing. Thus we choose

$$(\partial_z - A)\psi_1 = 0,$$ \hspace{1cm} (17)

where $z = x + it$. It is clear that the operator $\partial_z - A$ is antisymmetric and hence its non-zero eigenvalues are paired. Hence the number of zero modes of the operator $\partial_z - A$ is invariant under deformations of the gauge field. Let us take

$$a(x) = a + \frac{2\pi}{\beta}(\text{th} x + 1).$$ \hspace{1cm} (18)

Then the field $A$ interpolates between values $a$ at $x \to -\infty$ and $a + 2\pi/\beta$ at $x \to +\infty$ and hence it plays the role of an instanton. It is easy to check that there is a unique solution to the above equation which is normalizable and periodic in the "time" variable $t$. For $0 \leq a < \pi/\beta$ we have

$$\psi = (\cosh x)^{-\pi}e^{-ax - it\pi/\beta}\psi_0,$$ \hspace{1cm} (19)

where $\psi_0$ is the $O(3)$ triplet given by (up to a normalization factor)

$$\psi_0 = (1, -i, 0).$$ \hspace{1cm} (20)

In the case $\pi/\beta < a \leq 2\pi/\beta$ we get

$$\psi = (\cosh x)^{-\pi}e^{(2\pi/\beta - a)x - it\pi/\beta}\psi_0.$$ \hspace{1cm} (21)

Finally when $a = \pi/\beta$ there is no anti-periodic in time $t$ normalizable zero mode. This case corresponds to the absence of level crossing. Indeed there is a double degeneracy of levels $\lambda^\pm = 2\pi k/\beta$, $k \in \mathbb{Z}$, in the spectrum of the $D = 1$ Dirac operator. As a result its determinant vanishes at this value of the gauge field due to the contribution $k = 0$. Hence its sign does not change under a "big" gauge rotation. In turn in an analogy with Witten’s anomaly the existence of the normalizable zero mode for $a \neq \pi/\beta$ is related to the level crossing (circling) of the $D = 1$ 'Dirac' operator.

This completes our demonstration for the existence of a zero mode for the $D = 2$ Dirac operator. As stated before in an $O(3)$ model with an odd number of fermionic triplets we would expect to find an odd number of zero modes for the $D = 2$ Dirac operator. This would be related to a level crossing/circling of an odd number of eigenvalues in the spectrum of the $D = 1$ Dirac operator.
We proceed now to examine a healthy version of our model with an even number, say two, of fermionic triplets. We introduce a “scalar” field $\Phi$ which transforms under gauge rotations parametrized by $\alpha$ (an element of the $O(3)$ algebra)

$$\Phi \rightarrow e^{i\alpha} \Phi e^{-i\alpha}.$$  \hspace{1cm} (22)

Our lagrangian is given by

$$\mathcal{L} = i\bar{\psi} \dot{\psi} + \bar{\psi} A\psi + i \chi \dot{\chi} + \chi (A + \Phi) \chi.$$  \hspace{1cm} (23)

In this theory because the gauge invariance is intact we must integrate in the partition function over the gauge field. It is trivially seen that the model is indeed healthy as

$$Z = \int_0^{2\pi/\beta} da \cos^2 \frac{a\beta}{2} = \frac{\pi}{\beta}.$$  \hspace{1cm}

Let us now explore the limit where $\Phi \rightarrow \infty$ and a perturbative decoupling of $\chi$ field is expected. Due to gauge invariance we take $A + \Phi \sim hT_2$ ($T_2$ is an $O(3)$ generator and $h$ is a constant). If we let $\Phi \rightarrow \infty$ then $h \rightarrow \infty$ two components of the $\chi$ fermionic triplet become “heavy”. From eq.(13) we observe that the third $\chi$ component remains light and free.

By integrating out the $\chi$ field in the functional integral we get a factor $\sim \cos h\beta/2$. When we let $\Phi \rightarrow \infty$ ($h \rightarrow \infty$) this limit does not exist. At this point we may draw two independent conclusions:

Firstly our initially healthy model of two fermionic triplets in the “decoupling” limit where one of them ($\chi$) becomes infinitely massive remains healthy. This is because level crossing/circling in the spectrum of the $D = 1$ Dirac operator for the perturbatively decoupled $\chi$ still occurs. In other words we observe a non-perturbative non-decoupling of the heavy $\chi$ field.

Secondly, the non-existence of the limit $h \rightarrow \infty$ also implies that there is no well defined low energy effective lagrangian for the $\psi$ field alone. In the realistic case of the standard electro-weak theory with an infinitely heavy top-quark doublet we may infer that there is no well defined low-energy theory.

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References


Figure Captions

Fig.1. The spectra of the Dirac operator in the external gauge field $A = aT_2$ in two cases: 1) $a =$ real and 2) $a =$ complex. The square root of the determinant is defined respectively as a product of eigenvalues which 1) are positive and 2) have positive real part.

Fig.2. The flow of real eigenvalues ($\equiv$ level crossing) as the gauge field varies from $A = aT_2$ to $A^U = (a + 2\pi/\beta)T_2$ ($2\pi/\beta \equiv$ radius of the circle). Solid lines indicate the eigenvalues that define the determinant.

Fig.3. The flow of complex eigenvalues is depicted on the complex $\lambda$-plane. The complex gauge field $A$ varies from $aT_2$ to $(a + 2\pi/\beta)T_2$. Solid lines indicate eigenvalues that define the determinant. Level circling occurs between levels $\lambda_3$ and $-\lambda_3$. 