SO(10) Cosmic Strings and SU(3)_{color} Cheshire Charge

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Abstract

Certain cosmic strings that occur in GUT models such as SO(10) can carry a magnetic flux which acts nontrivially on objects carrying SU(3)_{color} quantum numbers. We show that such strings are non-Abelian Alice strings carrying nonlocalizable colored “Cheshire” charge. We examine claims made in the literature that SO(10) strings can have a long-range, topological Aharonov-Bohm interaction that turns quarks into leptons, and observe that such a process is impossible. We also discuss flux-flux scattering using a multi-sheeted formalism.

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1. Introduction

Cosmic strings are vortex lines that occur as the result of spontaneous symmetry breaking in certain quantum field theories. Although there are none in the minimal $SU(5)$ GUT, such strings occur generically in many GUT models based on larger groups, such as $SO(10)^{[6]}$. Cosmic strings have been proposed as seeds for structure formation in the universe, and the approximate string tension required for the correct level of density perturbations nicely coincides with the GUT scale suggested by the unification of the coupling constants$^{[6]}$.

This paper is about the Aharonov-Bohm interactions of GUT cosmic strings. It has been noted that although the region surrounding the cosmic string is pure vacuum (as long as the region is simply-connected and sufficiently distant from the string core), the magnetic flux confined in the core of the string can give rise to an Aharonov-Bohm interaction$^{[7,8,9,10]}$. This long-range interaction is of a topological nature, and completely determined by the magnetic flux carried by the string, which we define in terms of the Aharonov-Bohm transformation that it generates

$$U(C, x_0) = P \exp \left[ i \int_{(C, x_0)} dx^i A_i \right]. \quad (1.1)$$

Here $x_0$ is an arbitrary basepoint and $C$ is a loop starting and ending at $x_0$ that encircles the string once in the counterclockwise direction$^{[11,12,26]}$. The flux $U(C, x_0)$ must lie in the unbroken symmetry group because the covariant derivative of the Higgs condensate must vanish along the path $C$. The Aharonov-Bohm interactions of matter fields with the string are most easily analysed using a basis in which $U(C, x_0)$ is diagonal, so that $U$ acting on the matter fields takes the form $\text{diag}[e^{i \xi_1}, e^{i \xi_2}, \ldots]$. Then the scattering cross section for each component is given by the classic result

$$\frac{d\sigma}{d\theta} = \frac{1}{2\pi} \frac{\sin^2(\xi/2)}{k \sin^2(\theta/2)}. \quad (1.2)$$

When the incident beam is a superposition of diagonal components, a gauge-dependent Aharonov-Bohm scattering amplitude must be used, and the relative phases between the components are relevant.

When the magnetic flux carried by the string $U(C, x_0)$ lies in the center of the unbroken symmetry group $H(x_0)$ (which we shall denote by $Z[H(x_0)]$), there is little
more to be said. However, when $U(C, x_0) \notin Z[H(x_0)]$, new physics arises. In this case, the cosmic string solution, considered from the point of view of classical field theory, is no longer invariant under the action of $H$. This fact has two important consequences: (1) The classical string solution has internal zero modes, which lead to a manifold of classical degenerate string solutions. The effect of these zero modes is most easily analyzed by considering a loop of string.† When the loop is quantized, this classical internal degeneracy is lifted, and a spectrum of charged string loop states emerges. This charge is “Cheshire” charge, discussed classically in refs. 11–13 and quantum mechanically in refs. 14 and 15. “Cheshire” charge is peculiar because it is nonlocalizable; it does not reside on the string, nor can it be attributed to a current source in the vicinity of the string. Rather, it is a sourceless charge due to the peculiar matching condition that arise owing to the magnetic flux carried by the loop. (2) The interaction of charged particles and the string becomes more complicated and is no longer simply described by equation (1.2). Even for particles that travel in a narrow beam near the string, with no interference between paths of different winding number around the loop, the loop distorts the electric field of the particle, thus creating a new interaction.

The nature of Cheshire charge is most easily understood in the context of $U(1)$ Alice strings. These exotic strings arise in a model with unbroken symmetry group $H = U(1)Q \times Z_2$, where the product is semidirect, and the generator $X$ of $Z_2$ conjugates $U(1)Q$ charge—that is, $XQX^{-1} = -Q$. Alice strings carry a magnetic flux that lies in the disconnected component $H_\phi = \{Xe^{i\phi}Q | 0 \leq \phi < 2\pi\}$. Since under a continuous gauge rotation $Q = e^{iQ}$, a magnetic flux $U = Xe^{i\phi}Q$ transforms into $\Omega U \Omega^{-1} = e^{iQ}[Xe^{i\phi}Q]e^{-iQ} = Xe^{i(\phi - 2\xi)}Q$, there is no preferred flux in $H_\phi$, so there is a zero mode leading to a manifold of degenerate classical solutions with the topology of $S^3$, which when quantized leads to a spectrum of charged states. One may study the electrodynamics of a string loop classically. Consider an electric charge $+q$ carried around a closed path passing through a loop of Alice string. Its charge seems to change from $+q$ to $-q$, creating the appearance that a charge $+2q$ has somehow disappeared. The resolution of the paradox is that the missing charge has been transferred to the string loop in the form of nonlocalizable Cheshire charge. At the

† To simply the discussion, we shall ignore the translational degrees of freedom (i.e., the ability of the loop to oscillate) and consider a static string loop of radius $R$. 

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classical level, Cheshire charge is possible because of the twist in $U(1)_Q$ as one passes through the loop. There exist vacuum solutions to Maxwell’s equations for which $(\nabla \cdot \mathbf{E}) = 0$ everywhere, while at the same time the electric field integrated over a surface surrounding the loop would lead one to infer using Gauss’s law that charge is enclosed by the surface. Alice strings—at least for $U(1)$ identified with ordinary electromagnetism—while interesting, are not very realistic because in models with Alice strings the $Z_2$ symmetry must be exact. One knows that in the real world charge reflection is not an exact symmetry. Therefore, the possibility of observing these strings can be ruled out.

However, $SU(3)$ color Cheshire charge, by contrast, does not require any exotic new symmetries that must be reconciled with experiment. In scenarios for grand unification, one typically has a pattern of symmetry breaking

$$\hat{G} \rightarrow \ldots \rightarrow U(1)_{EM} \times SU(3)_{color} \times D = H_{cont} \times D. \quad (1,3)$$

Here $D$ is a discrete group. To simplify the discussion of whether the theory has topologically-stable cosmic string solutions, we take the group $\hat{G}$ to be a simply-connected universal covering group, even when $\hat{G}$ does not act effectively on the fields of the theory. This choice of $\hat{G}$ has the consequence that the entire subgroup $D$ or a subgroup thereof may act trivially on the matter fields of the theory, thus avoiding the introduction of new physical discrete symmetries to be reconciled with experiment. Since $\pi_1[\hat{G}/H] = D$, to each nontrivial element of $D$ there corresponds a topologically-stable cosmic string solution. Without more details about the model, one cannot determine precisely what magnetic flux is carried by the string configuration of minimal energy in the topological sector described by $d \in D$. All that can be determined from the topology is that $U$ belongs to the coset $d[U(1)_{EM} \times SU(3)_{color}]$. We may write $U_d = g_{U(1)} \otimes g_{SU(3)} \otimes d$. If $g_{SU(3)} \notin Z_3$, then the string solution carries colored Cheshire charge.

In this paper, we discuss the quantization of strings with non-Abelian Cheshire charge, and also discuss the process by which Cheshire charge is transferred to string loops by passing charged objects through the loop. Before proceeding to a general discussion of non-Abelian Cheshire charge, we discuss in section 2 an $SO(10)$ GUT model with cosmic string solutions which support $SU(3)_{color}$ Cheshire charge. Although the existence of cosmic strings in models of $SO(10)$ grand unification had been
discussed long ago, only recently was it demonstrated by Ma that the magnetic flux that minimizes the string tension in the first $\text{Spin}(10) \to SU(5) \times Z_2$ phase transition takes a value that is not invariant under $SU(5)$.\footnote{Ma showed numerically that an ansatz with such an asymmetric flux leads to a lower energy per unit length than an Abrikosov-Nielsen-Olesen ansatz whose flux lies in the center of $SU(5) \times Z_2$. In section 2 we calculate which flux orientation for these strings is energetically preferred in the two subsequent phase transitions $SU(5) \times Z_2 \to SU(3) \times SU(2) \times U(1)_Y \times Z_2 \to SU(3) \times U(1)_Q \times Z_2$. We calculate the effect of the flux on the fermions through an Aharonov-Bohm scattering. Our conclusions regarding Aharonov-Bohm scattering differ from those of the authors of refs. 16 and 17. In particular, we note that a long-distance Aharonov-Bohm effect which does not involve core penetration cannot lead to processes forbidden by the unbroken symmetry group, such as the $B$ to $L$ processes claimed to exist in refs. 16 and 17. In section 3 we discuss colored Cheshire charge. In section 4 we discuss vortex-vortex scattering for the case where the unbroken symmetry group is discrete. In section 5 we discuss Alice vortex-vortex scattering.}

2. $\text{SO}(10)$ Strings and the Non-Abelian Aharonov-Bohm Effect

A potentially realistic $\text{SO}(10)$ model of grand unification can be constructed by the pattern of symmetry breaking

\[ \text{Spin}(10) \to SU(5) \times Z_2 \to SU(3) \times SU(2) \times U(1)_Y \times Z_2 \to SU(3) \times U(1)_Q \times Z_2, \]  

(2.1)

where

\[ \langle \phi_{126} \rangle = v_{126} \cdot \left[ (e_1 + ie_2) \wedge (e_3 + ie_4) \wedge (e_5 + ie_6) \wedge (e_7 + ie_8) \wedge (e_9 + ie_{10}) \right]; \]
\[ \langle \phi_{45} \rangle = v_{45} \cdot \text{diag}[2/3, 2/3, 2/3, -1, -1]; \]
\[ \langle \phi_{10} \rangle = v_{10} \cdot \left[ 0, 0, 0, 0, 0, 1, 0, 0, 0 \right]. \]

With this partial choice of gauge, the Cartan subalgebra for $SU(3) \times SU(2) \times U(1)_Y$

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in terms of the $SO(10)$ generators is

$$T_{3}^{SU(3)} = \frac{1}{2} [M_{1,2} - M_{3,4}],$$

$$T_{8}^{SU(3)} = \frac{1}{2\sqrt{3}} [-M_{1,2} - M_{3,4} + 2M_{5,6}],$$

$$T_{3}^{SU(2)_L} = \frac{1}{2} [M_{1,2} - M_{3,4}],$$

$$Y = \frac{2}{3} [M_{1,2} + M_{3,4} + M_{5,6}] - [M_{7,8} + M_{9,10}].$$

(2.3)

The $SU(5)$ generators may be written as $T_{ab}^{I} = [M_{(2a-1),(2b-1)} + M_{2a,2b}]$ for $a \leq b$

$T_{ab}^{R} = [M_{2a,(2b-1)} - M_{2a-1,2b}]$ for $a < b$, where $(a, b = 1, \ldots, 5)$. A possible basis

for those $SO(10)$ generators not included in $SU(5)$ consists of the two sets of ten

$$T_{ab}^{A} = [M_{(2a-1),(2b-1)} - M_{2a,2b}],$$

$$T_{ab}^{B} = [M_{2a,(2b-1)} + M_{2a-1,2b}]$$

(2.4)

where $a < b$, and the lone generator

$T_{ab}^{I} = [M_{1,2} + M_{3,4} + M_{5,6} + M_{7,8} + M_{9,10}]$, which comutes with all elements of $SU(5)$.

This GUT model has topologically-stable cosmic string solutions because of the $Z_2$ discrete symmetry generated by the $2\pi$ rotation in $SO(10)$, which we shall denote by $X$. The element $X$ acts on the matter fields of the theory in the following manner.

All fermions acquire a phase $-1$ under the action of $X$ because the fermions transform under a spinor representation of $Spin(10)$, and $X$ acts trivially on bosons because all bosons transform under tensor representations of $Spin(10)$. Therefore, in this model the $Z_2$ discrete symmetry does not forbid or even constrain any processes not already forbidden by the fermion/boson superselection rule. From topological arguments we know that there exists a stable cosmic string solution carrying a flux that lies in the coset $X[SU(5)]$. [For the moment, we consider only the first stage of symmetry breaking $Spin(10) \rightarrow SU(5) \times Z_2$.] Energetic considerations determine which flux (or set of fluxes) belonging to this coset minimizes the energy per unit length of the string.

To determine which flux is energetically preferred, one must examine competing possibilities. Aryal and Everett considered the ansatz

$$\phi_{126}(r, \theta) = e^{i\theta r} f(r) \phi_{126}$$

$$A(r, \theta) = \dot{\theta} g(r) \frac{1}{r} \cdot \tilde{r}$$

(2.5)
where \( \phi_{126} \) is given in eqn. (2.2), and \( f(0) = g(0) = 0 \), and \( f(r) \), \( g(r) \to 1 \) as \( r \to \infty \), and \( t^a = (1/5)T_{ab}^U \). The generator \( t^a \) is certainly the most symmetric choice since \( X = e^{i2\pi t} \), which commutes with all elements of \( SU(5) \). However, as pointed out by Ma, it is not the choice that minimizes the energy. Ma considered the ansatz, originally suggested by Aryal and Everett, \(^2\) [\( \phi_{126}(r, \theta) = e^{i \theta t^a} [f^{ext}(r) \phi^{ext}_{126} + f^{int}(r) \phi^{int}_{126}] \]

\[
A(r, \theta) = \frac{g(r)}{r} \cdot t^a
\]

where \( t^a = (1/2)T_{1,2}^A \), \( f^{ext}(0) = g(0) = 0 \) and \( f^{ext}(r), g(r) \to 1 \) and \( f^{int}(r) \to 0 \) as \( r \to \infty \). \( \phi^{ext}_{126} = \phi_{126} \) and

\[
\phi^{int}_{126} = v_{126} \cdot [(e_1 \land e_3 - e_2 \land e_4) \land (e_5 + ie_6) \land (e_7 + ie_8) \land (e_9 + ie_{10})].
\]

Ma found that for all potentials with at most quartic terms this ansatz leads to a lower energy per unit length than the ansatz (2.5).

From the point of view of magnetic energy, the flux direction \( t = t^s \) is preferred over \( t = t^a \) because \( \text{tr}[(t^s)^2] < \text{tr}[(t^a)^2] \). However, in ansatz (2.6) the Higgs field does not vanish in the core of the string, and \( \phi^{int}_{126} \) has the property that \( t^a \cdot \phi^{int}_{126} = 0 \). Therefore, the magnetic field in the core does not introduce any scalar gradient energy for that part of the scalar field pointing in the \( \phi^{int}_{126} \) direction. This means that if for some \( \lambda \neq 0 \), \( V[\lambda \phi^{int}_{126}] < V[0] = 0 \), then ansatz (2.6) can lead to a lower energy by having a larger core, which numerically has been shown to be the case.

We now consider how classical field configurations described by ansätze (2.5) and (2.6) transform under the action of the unbroken symmetry group \( SU(5) \). A configuration described by (2.5) is invariant under the action of \( SU(5) \), but the same is not true for (2.6). A global gauge transformation \( \Omega \in SU(5) \) acts on \( t^a \) according to the rule \( t^a \to \Omega t^a \Omega^{-1} \). By starting with \( t^a = t^a_0 = (1/2)T_{1,2}^A \) and acting on \( t^a \) with global gauge transformations, we can obtain the assignments \( t^a = (1/2) = T^A_{a,b} \) and \( t^a = (1/2) = T^B_{a,b} \) for arbitrary \( a, b \), as well as other flux assignments. If there were no subsequent stages of symmetry breaking, all such solutions would have degenerate energy at the classical level.

In the two subsequent stages of symmetry breaking—first to \( SU(3) \times SU(2) \times U(1) \times Z_2 \), and then to \( SU(3) \times U(1) \times Z_2 \)—the freedom to make global gauge transformation becomes restricted, and different orientations of \( t^a \) need not lead to solutions
of degenerate energy, because interactions with the fields $\phi_{45}$ and $\phi_{10}$ can, and generically do, break the degeneracy. Therefore, to determine the lowest energy solution, we must first determine the preferred orientation of $t^a$ within the conjugacy class $[t^a] = \{\Omega t^a \Omega^{-1} | \Omega \in SU(5)\}$ in light of the subsequent stages of symmetry breaking.

In general, a subsequent stage of symmetry breaking $G_1 \to G_2$ through the condensation of a field $\Phi$ can have two effects. Suppose that $U \in G_1$ and solutions with the fluxes in the conjugacy class $[U] = \{\Omega U \Omega^{-1} | \Omega \in G_1\}$ are degenerate when the condensation of the field $\Phi$ is not taken into account. If $[U] \cap G_2$ is empty, then all flux orientations would frustrate the Higgs field $\Phi$ at large distances (which would lead to an infinite energy per unit length), and consequently the cosmic string solution acquires an outer core with a $G_2$ screening flux to avoid large distance frustration of the field $\Phi$. Alternatively, suppose that the intersection $[U] \cap G_2$ is nonempty. Then these flux orientations are preferred because no $G_2$ screening flux is required. However, inside the core the field $\Phi$ may still be frustrated where there is a magnetic field, and those orientations that minimize this frustration are energetically preferred.

For the second stage of symmetry breaking, the three flux orientations $t^a_{(1)} = (1/2)T_{3,4}^A$, $t^a_{(2)} = (1/2)T_{3,4}^A$, and $t^a_{(3)} = (1/2)T_{3,4}^A$ are not related to each other by conjugation by $SU(3) \times SU(2) \times U(1) \times Z_2$. Although $\exp[i2\pi t^a_{(k)}] \notin SU(3) \times SU(2) \times U(1)$ for $(k = 1, 2, 3)$, $[t^a_{(k)}, \phi_{45}] = v_{45} [t^a_{(k)}, Y] \neq 0$. The generator that minimizes $\text{tr} \left( [t^a_{(k)}, Y]^2 \right)$ is energetically preferred. A simple calculation shows that of these three choices $t^a = (1/2)T_{3,4}^A$ gives the energetically preferred orientation; therefore, with respect to the second stage of symmetry breaking, the orientations $t_a = (1/2)\Omega T_{3,4}^A \Omega^{-1}$ where $\Omega \in SU(3) \times SU(2) \times U(1)$ are energetically preferred. Note that both $t_a = (1/2)T_{3,4}^A$ and $t_a = (1/2)T_{3,5}^A$ are included in this set. We now consider the final stage of symmetry breaking. The choice $t_a = (1/2)T_{3,4}^A$ is unacceptable because $\exp[i2\pi(1/2)T_{3,4}^A] \notin SU(3) \times U(1)Q \times Z_2$. However, $\exp[i2\pi(1/2)T_{3,5}^A] = \text{diag}[1, 1, -1, 1, -1]_{10} \in SU(3) \times U(1)Q \times Z_2$. Finally, we note that for $t_a = (1/2)T_{3,5}, \exp[i2\pi t_a]$ is not invariant under the action of $SU(3)_{\text{color}}$, and although $\exp[i2\pi t_a]$ is invariant under the action of $U(1)_{Q}$, $t_a$ is not invariant under the action of $U(1)_{Q}$. The consequences of this lack of invariance of the classical solution under unbroken continuous symmetries will be discussed in the next section. We now discuss the Aharonov-Bohm interaction of fermions with a classical string background.
A single fermion generation is introduced as an irreducible multiplet of the 16-dimensional spinor representation. With the following basis for the Clifford algebra

\[
\begin{align*}
\Gamma_1 &= \sigma_1 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3, \\
\Gamma_2 &= \sigma_2 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3, \\
\Gamma_3 &= 1 \otimes \sigma_1 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3, \\
\Gamma_4 &= 1 \otimes \sigma_2 \otimes \sigma_3 \otimes \sigma_3 \otimes \sigma_3, \\
\Gamma_5 &= 1 \otimes 1 \otimes \sigma_1 \otimes \sigma_3 \otimes \sigma_3, \\
\Gamma_6 &= 1 \otimes 1 \otimes \sigma_2 \otimes \sigma_3 \otimes \sigma_3, \\
\Gamma_7 &= 1 \otimes 1 \otimes 1 \otimes \sigma_1 \otimes \sigma_3, \\
\Gamma_8 &= 1 \otimes 1 \otimes 1 \otimes \sigma_2 \otimes \sigma_3, \\
\Gamma_9 &= 1 \otimes 1 \otimes 1 \otimes 1 \otimes \sigma_1, \\
\Gamma_{10} &= 1 \otimes 1 \otimes 1 \otimes 1 \otimes \sigma_2,
\end{align*}
\]

the particle content becomes

\[
\begin{align*}
|\psi\rangle &= |+, +, +, +, + \rangle \\
|\nu\rangle &= |-, -, -, +, - \rangle \\
|e\rangle &= |-, -, -, -, + \rangle \\
|e^c\rangle &= |+, +, +, -, - \rangle \\
|u_1\rangle &= |+, -, +, +, - \rangle \\
|u_2\rangle &= |-, +, +, +, - \rangle \\
|u_3\rangle &= |+, +, -, +, - \rangle \\
|d_1\rangle &= |+, -, +, -, + \rangle \\
|d_2\rangle &= |-, +, +, -, + \rangle \\
|d_3\rangle &= |+, +, -, -, + \rangle \\
|u_1^c\rangle &= |-, +, -, +, + \rangle \\
|u_2^c\rangle &= |+, -, -, +, + \rangle \\
|u_3^c\rangle &= |-, -, +, +, + \rangle \\
|d_1^c\rangle &= |-, +, -, -, - \rangle \\
|d_2^c\rangle &= |+, -, -, -, - \rangle.
\end{align*}
\]
\[ |d_5^i\rangle = |-, -, +, -\) \]

Since \(M_{ij} = (-i/4)[\Gamma_i, \Gamma_j]\),

\[(t^a)_{16} = \left(\frac{1}{2} T_3^A\right)_{16} = \left(\frac{1}{2} [M_{5,9} - M_{6,10}]\right)_{16} = \frac{1}{4}[1 \otimes 1 \otimes \sigma_1 \otimes \sigma_2 - 1 \otimes 1 \otimes \sigma_2 \otimes \sigma_1] \]

(2.7)

Thus \(t^a\) acts nontrivially on eight fermions as follows:

\[
\begin{align*}
    t^a |\nu^c\rangle &= +(i/2) |u_3\rangle, \\
    t^a |u_3\rangle &= -(i/2) |\nu^c\rangle, \\
    t^a |\nu\rangle &= +(i/2) |\bar{u}_3\rangle, \\
    t^a |\bar{u}_3\rangle &= -(i/2) |\nu\rangle, \\
    t^a |d_1^i\rangle &= +(i/2) |d_2\rangle, \\
    t^a |d_2\rangle &= -(i/2) |d_1^i\rangle, \\
    t^a |d_1^i\rangle &= +(i/2) |d_2\rangle, \\
    t^a |d_2\rangle &= -(i/2) |d_1^i\rangle.
\end{align*}
\]

(2.8)

The other eight fermions are annihilated by \(t^a\).

From the relations in equation (2.8) it may appear that the vector potential rotates certain quarks into leptons and vice versa. In fact, certain authors argue that strings of this sort can transform quarks into leptons by means of a long-range Aharonov-Bohm interaction that is completely of a topological nature, without any core penetration.\(^{[16,17]}\) However, this phenomenon does not occur because, while the fermion multiplet twists as one passes around the string, the Higgs fields also twist. Consequently, at the same time that the twisting described in equation (2.8) takes place, the definition of what is a quark and what is a lepton changes too, so that the purported process does not take place.

The crucial point is that away from the core of the string (except for exponentially decaying corrections) there is pure vacuum. The winding in the Higgs field and the non-vanishing gauge field has no local physical effect because the covariant derivative of the Higgs field and the curvature in the gauge field vanish. Away from the core of the string in a simply-connected region \(R\) one can make a unitary choice of gauge so that the Higgs field is constant and the gauge field \(A_\mu\) vanishes. However, such a
unitary choice of gauge is not possible in a multiply-connected region that encloses
the string because the winding of the Higgs field (and also the vector potential)
around the string poses a topological obstruction to a choice of gauge in which \( \phi(x) \)
is everywhere constant. It is this topological obstruction that can give rise to an
Aharonov-Bohm interaction between the string and the matter fields of the theory.

The Aharonov-Bohm interactions of a cosmic string are most easily analyzed in
a singular gauge, in which all of the winding in the Higgs field and the non-vanishing
gauge potential are concentrated on a singular sheet of zero thickness whose boundary
coincides with the string. In this singular gauge, there are nontrivial matching
conditions for the fields of the theory on opposite sides of the sheet, more specifically

\[
\Psi(r, \theta = 2\pi) = e^{i2\pi r^a} \Psi(r, \theta = 0).
\] (2.9)

Here \( \Psi \) is a 16 multiplet. This matching condition is derived by requiring that the
covariant derivative of \( \Psi \) be nonsingular on the sheet. This requirement is reasonable
because the position of the sheet is a gauge artifact and therefore of no physical
significance.

In the models described above, and the model considered in reference 16 as well,
the generator \( t^a \) can be put into a block diagonal form in which the diagonal consists
exclusively of zeros and \( 2 \times 2 \) matrices of the form

\[
t^a = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}
\] (2.10)

that act on the multiplets \((\nu^c, u_3), (u_3^c, \nu), (d_1^c, d_2), \) and \((d_2^c, d_1)\). One obtains the
matching conditions

\[
\begin{align*}
\nu(r, \theta = 2\pi, z) &= -\nu(r, \theta = 0, z) \\
\nu^c(r, \theta = 2\pi, z) &= -\nu^c(r, \theta = 0, z) \\
u_3(r, \theta = 2\pi, z) &= -u_3(r, \theta = 0, z) \\
u_3^c(r, \theta = 2\pi, z) &= -u_3^c(r, \theta = 0, z) \\
d_1(r, \theta = 2\pi, z) &= -d_1(r, \theta = 0, z) \\
d_1^c(r, \theta = 2\pi, z) &= -d_1^c(r, \theta = 0, z) \\
d_2(r, \theta = 2\pi, z) &= -d_2(r, \theta = 0, z) \\
d_2^c(r, \theta = 2\pi, z) &= -d_2^c(r, \theta = 0, z)
\end{align*}
\] (2.11)

The other fields have trivial matching conditions. These conditions imply that the
particles $\nu, \nu^c, u_3, u_3, d_1, d_2$, and $d_2$ scatter off the string with the cross section

$$\frac{d\sigma}{d\theta} = \frac{1}{2\pi k} \cdot \frac{1}{\sin^2(\theta/2)},$$

(2.12)
corresponding to an Aharonov-Bohm phase $\xi = \pi$. However, there is no mixing. In terms of flux, $U = U_{SU(3)} \otimes e^{i3\pi Q} \otimes X$ where $U_{SU(3)} = \text{diag}[-1, -1, +1]$.

We have shown that in the particular models considered in references 16 and 17 there is no $B$-violation of a long-range topological origin. Note that our analysis does not apply to short-range $B$-violation that takes place inside the core. For small momenta, core penetration is limited to the lowest partial wave. However, fractional angular momentum wave function enhancement can cause the core penetration cross section to be larger than the naive geometric cross section, actually growing as a fractional power of the wavelength. We now turn to the question of whether it is possible to construct other models in which there is a $B$-violating Aharonov-Bohm cross section of a topological nature. One might contemplate the possibility that even though the models in references 16 and 17 do not have Aharonov-Bohm $B$-violation, other realistic models which do have it could be constructed. We show that this is not possible if $U(1)Q$ is an exact symmetry which can be globally defined.

Since leptons (and antileptons) have electric charge $q = -1, 0, +1$, and quarks (and antiquarks) have electric charge $q = -2/3, -1/3, +1/3, +2/3$, a process that changes a quark into a lepton (or antilepton) cannot conserve electric charge. For processes that take place near the core of the string the missing charge can be transferred to the string core, where $Q$ is not necessarily a good symmetry. However, for Aharonov-Bohm scattering the $S$ matrix must leave all of the Higgs fields invariant. Since a transition from quark to lepton obviously fails to commute with the Higgs field which breaks $SU(5)$ to $SU(3) \times SU(2) \times U(1)$, such a process must be forbidden. In fact, even before we study the possibility that cosmic strings could be Alice strings, we already know that the associated delocalized Cheshire charge could not be ordinary electric charge: There is no operator which both changes electric charge and commutes with all the Higgs fields which must be present in any theory containing the standard model.
3. Colored Alice Strings

It is well known that when a topological defect solution—or soliton—in classical field theory is not invariant under the action of a continuous unbroken internal symmetry, there exist zero modes that lead to a manifold of classically degenerate solutions. In the quantum theory the classical degeneracy is broken, so that in addition to the $H$ invariant soliton, there exist dyonic states that carry nontrivial $H$ charge. The classic example, first discussed by Julia and Zee,[8] is the ’t Hooft-Polyakov monopole, whose core is not invariant under $U(1)_Q$ rotations. The result is the existence of electrically-charged dyonic excitations, which carry magnetic charge $g$ and electric charge $n\epsilon$, where $n$ is an integer. Another example is the Skyrmion[13].

The same phenomenon occurs for cosmic strings. When the flux $U \in H$ of a cosmic string does not lie in the center of $H$, loops of cosmic string can carry electric charge. The classic example is the Alice string, which can carry nonlocalizable Cheshire charge[11,12,13]. For Alice strings[10], the unbroken symmetry group is $H = U(1)_Q \times Z_2$, where the product is semidirect with the generator $X$ of $Z_2$ conjugating $U(1)_Q$ charge—that is, $XQX^{-1} = -Q$. Alice strings carry a magnetic flux which lies in the disconnected component $H_d = \{Xe^{i\phi}Q|0 \leq \phi < 2\pi\}$. Since under a continuous gauge rotation $\Omega = e^{iQ}$, a magnetic flux $U = Xe^{i\phi}Q$ transforms into $\Omega U \Omega^{-1} = e^{iQ}Xe^{i\phi}Qe^{-iQ} = Xe^{i(\phi-2\xi)Q}$, there is no preferred flux in $H_d$. At the classical level, there is a degeneracy, with each flux $U \in H_d$ corresponding to a distinct classical cosmic string solution of degenerate energy[14]. Quantum mechanically, this degeneracy is broken. The energy eigenstates are states of definite charge $2N$

$$|N> = \int_0^{2\pi} d\phi \ e^{i\phi N} |\phi>.$$  \hspace{1cm} (3.1)

Formally, quantizing the Alice string zero mode is precisely analogous to quantizing a rotor in two dimensions.

This discussion of Cheshire charge so far has been rather abstract, based on the transformation properties of magnetic flux. For Alice strings the existence—and also necessity—of Cheshire charge already is manifest at the level of classical field theory. As mentioned in the introduction, loops of string must be able to carry nonlocalizable Cheshire charge in order to resolve a paradox in which charge conservation would
appear to be violated. Consider a charge \( +q \) carried around a closed loop passing through a loop of Alice string. Its charge appears to change from \( +q \) to \( -q \), creating the appearance that a charge \( +2q \) has somehow disappeared. The resolution of the paradox is that the missing charge has been transferred to the string loop in the form of nonlocalizable Cheshire charge. At the classical level, Cheshire charge is possible because of the twist in \( U(1)Q \) as one passes through the loop. There exist vacuum solutions to Maxwell’s equations for which \( (\nabla \cdot E) = 0 \) everywhere, while at the same time the electric field integrated over a surface surrounding the loop would lead one to infer using Gauss’s law that there is charge enclosed.

Similar phenomena occur for the colored \( SO(10) \) strings discussed in the previous section. The flux of the strings discussed in the previous section is described by a matrix \( U^a_b \), which transforms as \( U \rightarrow \Omega U \Omega^{-1} \) under a gauge rotation \( \Omega \in SU(3) \). The matrix \( U \) after an appropriate gauge rotation has the form \( \dagger \)

\[
U = \text{diag}[1, 1, -1].
\] (3.2)

Physically, the orientation of \( U \) in internal color space is measurable; it indicates which color of quark experiences nontrivial Aharonov-Bohm scattering.

Classically, the cosmic string solution has zero modes. Since \( SU(3) \) acts nontrivially on the classical string solution, there exists a manifold of classically degenerate string solutions. The possible color orientations described by \( U \) lie in the coset space

\[
M \cong \frac{H}{H_{\text{inv}}} = \frac{SU(3)}{[SU(2) \times U(1)]/\mathbb{Z}_2}.
\] (3.3)

Here the subgroup \( H_{\text{inv}} \subset H \) is the part of the unbroken symmetry group that leaves \( U \) invariant.

Quantizing the color zero mode is analogous to the abelian Alice string, except that the representations are slightly more complicated. The Alice string loop zero mode was like that of a spinning top in two dimensions. The colored string loop is more like the symmetric rigid rotor in three dimensions, except that the symmetry group is \( SU(3) \) instead of \( SU(2) \), and also a quotient over a subgroup has been taken.

\( \dagger \) This is what \( U \) acting on the up quarks looks like. For the down quarks there is an additional overall factor of \( (-1) \), but this does not change the discussion of colored Cheshire charge.
Quantum mechanically, the state of the flux degrees of freedom of the string loop is described by a wave function $\Psi(m)$, whose domain is the manifold of classically degenerate vacua $M$, defined in (3.3). States of definite $SU(3)$ charge—that is, states which transform irreducibly under $SU(3)$—correspond to functions $Y^m_{\tilde{m}}(m)$, where the index $\tilde{m}$ labels the irreducible representations of $SU(3)$. Let $d^m$ be the dimensionality of the representation $\tilde{m}$. The index $i$ ranges from 1 to $d^m$. The index $a$ is included to account for multiple copies of $\tilde{m}$. Square-integrable functions on $M$ can be expanded as

$$
\psi(m) = \sum_{\tilde{m}, a, i} C^m_{a,i} Y^m_{\tilde{m}}(m). \quad (3.4)
$$

To determine what kinds of $SU(3)_{\text{color}}$ charge the loop may carry, it is convenient to represent the function $\psi(m)$ as a function whose domain is $SU(3)$ instead of the coset space $M$. We consider functions $\psi : SU(3) \rightarrow \mathcal{C}$ that are invariant under translation by $H_{\text{inv}} = [SU(2) \times U(1)]/Z_2$. The Peter-Weyl theorem implies that an arbitrary function on $SU(3)$ may be expanded in terms of the complete orthonormal set $\sqrt{d^m} D^m_{aa'}(g)$. Let $N^{\tilde{m}}$ be the number of times that the representation $\tilde{m}$ occurs in the expansion (3.4).

$$
N^{\tilde{m}} = \int_{g' \in [SU(2) \times U(1)]/Z_2} \int_{g \in SU(3)} \text{d}g' \text{d}g \; D^m_{aa'}(g') [D^m_{aa'}(g)]^* \quad \text{d}g',
$$

$$
= \int_{g' \in [SU(2) \times U(1)]/Z_2} \text{d}g' \; D^m_{aa'}(g') \int_{g \in SU(3)} \text{d}g \; D^m_{aa'}(g) [D^m_{aa'}(g)]^* \quad (3.5)
$$

$$
= \int_{g' \in [SU(2) \times U(1)]/Z_2} \text{d}g' \; \Gamma^m(g').
$$

The integral over $g'$ imposes invariance under translations by $H_{\text{inv}}$. Because of the orthonormality relation for characters, the last line indicates that $N^{\tilde{m}}$ is simply the number of times that the trivial representation occurs in the reducible representation of $[SU(2) \times U(1)]/Z_2$ induced by the irreducible representation of $SU(3)$ labeled by $\tilde{m}$. It is clear that representations of nontrivial triality cannot occur because the center $Z_3$ of $SU(3)$ is included in $[SU(2) \times U(1)]/Z_2$.

Charge can be transferred to the loop by passing objects with nontrivial $SU(3)_{\text{color}}$ quantum numbers through it. To see this, consider an initially uncharged loop, de-
scribed by the wave function

\[ |\psi_{\text{loop}}^{\text{initial}} \rangle = \int_{m \in M} \, dm \, |m \rangle, \]  

(3.6)

and a color singlet quark-antiquark pair, described by the wave function

\[ |\psi_{QQ}^{\text{initial}} \rangle = \frac{1}{\sqrt{3}} \psi_i^j |Q_j \rangle |\bar{Q}_i \rangle, \]  

(3.7)

so that the initial wave function for the color degrees of freedom of the entire system is

\[ |\psi^{\text{initial}} \rangle = |\psi_{QQ}^{\text{initial}} \rangle \otimes |\psi_{\text{loop}}^{\text{initial}} \rangle, \]  

(3.8)

a simple tensor product. Initially, there are no correlations between the two subsystems.

We now adiabatically transport the quark \( Q \) through the loop, while keeping the antiquark \( \bar{Q} \) in place, finally bringing the quark \( Q \) back to its original position, as indicated in fig. 8†

The wave function now is

\[ |\psi^{\text{final}} \rangle = \int_{m \in M} \, dm \, \frac{1}{\sqrt{3}} U_i^j(m) \, |Q_j \rangle |\bar{Q}_i \rangle \otimes |m \rangle, \]  

(3.9)

The state \( |\psi^{\text{final}} \rangle \) is no longer a simple tensor product; correlations have been established between the color degrees of freedom of the loop and of the quark-antiquark pair. These correlations are necessary to insure that the total color charge of the entire system remains trivial, as it must be owing to charge conservation.

Suppose that we now measure the \( SU(3)_{\text{color}} \) charge of the \( Q \bar{Q} \) pair in the final state. We first calculate the density matrix, by taking the trace over the string loop degrees of freedom. The result is

\[ \hat{\rho}_{QQ}^{\text{final}} = \frac{1}{3} \int_{m \in M} U_i^j(m) U_i^j(m) |Q_j \rangle |\bar{Q}_i \rangle <\bar{Q}_i | <Q_i | Q^n \rangle. \]  

(3.10)

To calculate the probability \( p_{\text{singlet}}^{\text{final}} \) of the final \( Q \bar{Q} \) pair being in a singlet state, we

† Actually, it is unnecessary to carry out the process adiabatically. As long as the quark propagates through the loop and the antiquark \( \bar{Q} \) does not, the wave function discussed above describes the \( SU(3)_{\text{color}} \) degrees of freedom exactly.
use the projection operator
\[ \hat{P}_{\text{singlet}} = \frac{1}{3} |Q^i > |\bar{Q}_i > < \bar{Q}_j | < Q^j | \] (3.11)
to obtain
\[ p_{\text{final}}^{\text{final}} = \text{tr}[\hat{P}_{\text{singlet}}^{\text{final}} P_{QQ}^{\text{final}}] = \frac{1}{9}, \]
\[ p_{\text{octet}}^{\text{final}} = 1 - p_{\text{singlet}}^{\text{final}} = \frac{8}{9}. \] (3.12)
These probabilities apply to the loop as well, because charge conservation mandates that the combined system remain in a color singlet state. The generalization to more complicated situations where the loop and the pair are initially charged or where objects with color charge belonging to a different representation are passed through the loop is straightforward.

So far we have ignored the effects of confinement and treated SU(3)\text{color} almost as if it were a global symmetry. This approximation is justified for a large range of length scales, because the string thickness is of order \(10^{16} GeV^{-1}\), while the effects of confinement become relevant for length scales greater than approximately \(\Lambda_{QCD}^{-1} \approx [10^{-1} GeV]^{-1}\). Of course, for loops larger than \(\Lambda_{QCD}^{-1}\), confinement is relevant. On scales much larger than \(\Lambda_{QCD}^{-1}\), there are no colored objects; particles are color singlets. However, the possibility of diffractive scattering of hadrons by the string, with cross sections of order \(\Lambda_{QCD}^{-1}\), exists. Consider a hadron that propagates very close to the string in such a way that some of the partons pass around one side of the string while the rest of the partons pass around the other side. After this happens, a hadron, which previously was a color singlet, ceases to remain a color singlet. Cheshire charge has been transferred from the hadron to the string loop, but since confinement does not allow isolated color charge, either the hadron bounces back elastically or becomes excited to form a resonance, or if there is enough energy a string of electric color flux forms, which eventually fragments to form a jet of hadrons. Since cosmic strings in an astrophysical context typically travel at a fraction of the speed of light, and near cusps very close to the speed of light, such interactions are plausible.

In the above we have considered the transformation properties of the magnetic flux \(U = \exp[2\pi i t^a]\) under \(SU(3) \times U(1)\), but we have not considered the transformation properties of Lie algebra element \(t^a\) generating \(U\) under \(SU(3) \times U(1)\). These transformation properties are not necessarily the same because for a given
there may be several choices of $t$ that give $U = \exp[i2\pi t]$. For example, for the $Spin(10)$ strings presented in section 2, $\exp \left[ i(2\pi)\frac{1}{2} \left( \cos \xi \, T^A_{3,5} + \sin \xi \, T^B_{3,5} \right) \right]$ is independent of $\xi$, so there is a continuum of such choices with the topology of $S^1$. Consider the action of $Q$ on $T^A_{3,5}$. Recall that since $Q_{10} = \text{diag} [+1/3, +1/3, +1/3, 0, -1]$ and $U_{10} = \text{diag} [+1, +1, -1, +1, -1]$, the flux $U$ as seen away from the core of the string is invariant under $U(1)_Q$. However, $[Q, T^A_{3,5}] = \frac{2}{3} T^B_{3,5}$, and similarly $[Q, T^B_{3,5}] = -\frac{2}{3} T^A_{3,5}$. Therefore, the fields excited inside the string core—the vector field, and the scalar field too—are not invariant under the action of $U(1)_Q$. This fact implies that the strings are superconducting. The $SO(10)$ strings considered here exhibit both a ‘traditional’ charged scalar condensate in the core $^{[5]}$ and a charge carrying condensate of vector fields in the core $^{[B2]}$. Here both fields are charged, because the direction $\phi^{int}$ also is not invariant under $U(1)_Q$.

One peculiar feature of these superconducting strings is the quantization condition for the electric charge on a string loop. The commutation relations $[Q, T^A_{3,5}] = \frac{2}{3} T^B_{3,5}$ and $[Q, T^B_{3,5}] = -\frac{2}{3} T^A_{3,5}$ imply that the charge $q$ on a loop satisfies $q = \frac{2}{3} en$ where $n$ is an integer and $e$ is the magnitude of the charge of an electron.

Because of the monopoles potentially present in the theory, a loop state with fractional electromagnetic charge and but trivial triality would violate the Dirac quantization condition. Recall that our loop states always carry trivial triality. The resolution of this paradox is the following. In our discussion of Cheshire charge we considered only the transformation properties of the long range flux under $SU(3)$. We did not consider the transformation properties of the generator $t^a$ under $SU(3)$. It turns out that the subgroup $[SU(2) \times U(1)]/\mathbb{Z}_2 \subset SU(3)$ that leaves $U = \exp[i2\pi t^a]$ invariant does not leave $t^a$ invariant. This feature allows loop states with fractional electromagnetic charge to carry color charge corresponding to nontrivial triality localized on the string itself, so that the Dirac quantization condition is not violated. It should be stressed that the electric charge and the color charge of nontrivial triality is not Cheshire charge, because it is localized on the string core. In contrast to Cheshire charge, it is not sourceless; there is a measurable, localizable charge on the string.
4. Non-Abelian Vortex-Vortex Scattering

In its classic form, the Aharonov-Bohm effect involves the scattering of charged particles by localized magnetic flux where the charged particle is forbidden to penetrate the region where the gauge field has curvature (that is, where \( F_{\mu\nu} \neq 0 \)), or where such penetration can be neglected. For Abelian gauge fields, magnetic flux is uncharged; however, for non-Abelian gauge fields, this generally is not the case, since in general \( \Omega F_{\mu\nu} \Omega^{-1} \neq F_{\mu\nu} \). Thus it becomes possible to consider Aharonov-Bohm experiments in which two straight, parallel flux tubes pass by each other, moving only in the plane normal to the tube axes. Because this is such a strong restriction on the allowed tube motions, the resulting problem of vortex motion in two space dimensions may be considered artificial, but perhaps still interesting because a great deal can be found about its solution.

The possibility for nontrivial Aharonov-Bohm scattering of non-Abelian magnetic flux by other non-Abelian magnetic flux pointing in a different direction in internal symmetry space is best illustrated by considering vortex-vortex scattering in a theory where a gauge symmetry \( G \) is broken to a non-Abelian discrete subgroup \( H \) by the formation of a Higgs condensate\(^{[25-32]}\). We shall take \( G \) to be the simply-connected covering group, so that to each nonunit element of \( H \) there corresponds a topologically stable vortex excitation. Using a discrete little group \( H \) alleviates certain complications, which shall be discussed later.

Before considering the quantum theory, it is useful to consider the classical description of a system with such vortices. It is the manner in which the magnetic flux carried by the vortices evolves classically as the vortices are moved that gives rise to nontrivial Aharonov-Bohm scattering in the quantized system. A classical state of a system with \( N \) vortices at a time \( t \) is specified by giving the positions of the vortices \( x_1, x_2, \ldots, x_N \) and their magnetic fluxes as well. If the unbroken group \( H \) is Abelian, the magnetic fluxes can be specified by enclosing each vortex once in the counterclockwise direction by a set of closed paths \( C_1, C_2, \ldots, C_N \) and evaluating the fluxes

\[
h_j = P \exp \left[ i \oint_{C_j} dx \cdot A(x) \right] \in H, \tag{4.1}\]

Here \( P \) indicates that the exponential is path ordered. The flux \( h_j \) is an element of
the unbroken symmetry group and indicates the phase that results from adiabatically transporting a particle around the vortex.

When $H$ is non-Abelian, a more elaborate formalism is required because definition (4.1) becomes ambiguous. $h_j$ is no longer well defined because it too changes upon being adiabatically transported around another vortex. Upon parallel transport around a vortex with flux $h_i$, one has $h_j \rightarrow h_i h_j h_i^{-1}$. To avoid this ambiguity, one must use paths that start and end at a particular basepoint $x_0$, as indicated in fig. 1. Expression (4.1) is modified to become

$$h(C, x_0) = P \exp \left[ i \int_{(C,x_0)} dx \cdot A(x) \right] \in H(x_0). \quad (4.2)$$

Because the curvature of the gauge field vanishes everywhere except at the vortex cores, the flux $h(C, x_0)$ is invariant under continuous deformations of $C$ that avoid the vortex cores. Therefore, the expression (4.2) can be thought of as defining a mapping from the fundamental group of the punctured plane $M = \mathcal{R} - \{x_1, x_2, \ldots, x_N\}$ to the little group $H(x_0)$ based at $x_0$. This mapping is a group homomorphism. As shown in fig. 2, it is apparent that in the non-Abelian case not only the winding number of a path around a vortex is relevant in determining the flux $h(C, x_0)$, but also its threading around the neighboring vortices. Paths $\alpha$ and $\alpha'$ both have unit winding number around the vortex $A$ and vanishing winding number around $B$; however, they differ in how they thread around the vortex $B$. Since $\alpha' = \beta \alpha \beta^{-1}$, $h(\alpha') = h(\beta)h(\alpha)h(\beta)^{-1}$. Therefore, $h(\alpha') = h(\alpha)$ if and only if the fluxes $h(\alpha)$ and $h(\beta)$ commute.

The formalism just outlined allows one to describe the magnetic fluxes of a set of vortices at fixed time. A classical vortex configuration at time $t_a$ is completely specified by the positions of the vortices $x_1(t_a), x_2(t_a), \ldots, x_N(t_a)$ and the mapping $h : \pi_1(M(t_a), x_0) \rightarrow H(x_0)$ where $M(t_a) = \mathcal{R}^2 - \{x_1(t_a), x_2(t_a), \ldots, x_N(t_a)\}$. We now consider what happens when the vortices move. At a later time $t_b$ the state of the system is described by their new positions $x_1(t_b), x_2(t_b), \ldots, x_N(t_b)$ and a new mapping $h_{t_b} : \pi_1(M(t_b), x_0) \rightarrow H(x_0)$. The new mapping is completely determined by the trajectory $X(t) = (x_1(t), x_2(t), \ldots, x_N(t))$. The mapping $h_{t_b}$ is obtained from the mapping $h_{t_a}$ by taking paths in $\pi_1(M(t_b), x_0)$ and dragging them back to $\pi_1(M(t_a), x_0)$ using the trajectory $X(t)$ to deform $M(t)$.

\[\dagger\] For simplicity we shall treat the vortices as point particles that never overlap.
Now that we have succeeded in describing classical vortex evolution, we turn to describing the quantized system. The classical states introduced above serve as a basis of orthogonal states for the Hilbert space of physical states, analogous to the position eigenstates \( |x> \) in ordinary non-relativistic quantum mechanics. To calculate the matrix element

\[
\langle x_1(t_b), x_2(t_b), \ldots, x_N(t_b); h_{t_b} \mid x_1(t_a), x_2(t_a), \ldots, x_N(t_a); h_{t_a} \rangle
\]

(4.3)

for the \( N \)-vortex sector using the path integral formalism, we sum over paths \( X(t) = (x_1(t), x_2(t), \ldots, x_N(t)) \) that in addition to the usual requirement that \( X(t) = X_a(t_a) \) for the initial state and \( X(t_b) = X_b(t_b) \) for the final state also satisfy the requirement that \( X(t) \) causes \( h_{t_a} \) to deform into \( h_{t_a} \). Paths satisfying the first requirement can be classified into braid classes—that is, homotopy classes. The completely-colored braid group describes the relation between different such homotopy classes. The last requirement restricts the sum to a subset of the possible braid classes—namely those which give the proper final flux \( h_{t_b} \). For a system with two vortices, \( A \) and \( B \) which initially at \( t \) are at positions \( x_a \) and \( x_b \) and propagate to positions \( x'_a \) and \( x'_b \) at a later time \( t' \), paths \( X(t) \) connecting the initial and final states may be classified by a winding number \( \tilde{N} \). Unless the initial and final positions coincide, the choice of homotopy class corresponding to \( \tilde{N} = 0 \) is arbitrary. Increasing \( \tilde{N} \) by one unit involves allowing vortex \( A \) to wind around vortex \( B \) by one more unit.

Vortex-vortex scattering for \( N = 2 \) is illustrated in fig. 3. For simplicity vortex \( B \) is held fixed. As indicated in fig. 3(b), vortex \( A \) can propagate quantum mechanically to its new position \( A' \) via many homotopically inequivalent paths, two of which are indicated in the figure. Figs. 3(a) and 3(c) indicate alternative flux bases for the initial state, and fig. 3(d) indicates a flux basis for the final state. If vortex \( A \) propagates from \( A \) to \( A' \) along path 1, then \( h(\alpha_f) = h(\alpha_1) \) and \( h(\beta_f) = h(\beta) \). Alternatively, if vortex \( A \) propagates from \( A \) to \( A' \) along path 2, then \( h(\alpha_f) = h(\alpha_2) \) and \( h(\beta_f) = h(\alpha_2 \alpha^{-1}) \). Therefore, it is clear that the amplitudes do not add, for they correspond to classically distinguishable final states. Paths 1 and 2 are only two of infinitely many homotopically inequivalent paths. To consider other homotopy classes, let us consider the effect of increasing the relative winding number \( \tilde{N} \) by one unit, by the process indicated in fig. 4(b), which we shall call \( \mathcal{R}^2 \) for reasons that
shall become apparent later. One has
\[
\alpha_{n+1} = [\beta(n)\alpha(n)]\alpha(n)[\beta(n)]^{-1} = [\beta(0)\alpha(0)]\alpha(n)[\beta(0)]^{-1},
\]
(4.4)
\[
\beta_{n+1} = [\beta(n)\alpha(n)]\beta(n)[\beta(n)]^{-1} = [\beta(0)\alpha(0)]\beta(n)[\beta(0)\alpha(0)]^{-1}.
\]
The quantity \(\beta_{(j)}\alpha_{(j)}\) is independent of \(j\) because it is the path enclosing both vortices, and the combined flux is a conserved quantity. There exists a smallest \(M \geq 1\) so that
\[
h(\alpha_{(M)}) = h(\alpha(0))\) and \(h(\beta_{(M)}) = h(\beta(0)) — \)in other words, \(M\) indicates the number of windings required to restore the system to its original state.

We have shown that the classical configuration space of the system may be thought of as an \(M\)-sheeted Riemannian surface \(R^2(M)\), described in polar coordinates by the metric \(ds^2 = dr^2 + r^2 d\theta^2\) where \(|\theta| \leq M\pi\). There is a natural \(M\)-to-1 projection \(\pi : R^2(M) \rightarrow R^2\) into physical space. For \(m \in R^2\), the \(M\) points \(\pi^{-1}(m)\) in \(R^2(M)\) correspond to the \(M\) values of the magnetic flux that are possible at each possible position of the vortex \(A\). The multiply-sheeted formalism is a natural way to keep track of magnetic flux without introducing unphysical cuts that are merely gauge artifacts. Solving vortex-vortex scattering in the multiply-sheeted formalism reduces to solving the Schroedinger equation for a free particle in the multiply-sheeted spacetime, a mathematical problem that was solved long ago in a different context, that of rigorous diffraction theory. A. Sommerfeld and H. Carslaw\([24,35,36]\) solved the Helmholtz equation \([\nabla^2 + k^2]\psi = 0\) for an incident plane wave in an attempt to understand rigorously diffraction by a semi-infinite absorbing screen.

To define ‘scattering,’ one must first specify what kind of propagation would constitute the absence of scattering. For \(M = 1\) this is completely obvious; a plane wave signifies the absence of scattering. For \(M > 1\) the plane wave
\[
\psi_{pw}(r, \phi) = \begin{cases} 
e^{-ikr \cos \phi}, & \text{for } |\phi| < \pi, \\ 0, & \text{for } \pi < |\phi| < M\pi \\ \end{cases}
\]
(4.5)
does not satisfy the equations of motion, even in the absence of a nontrivial potential. This is because in the multiply-sheeted space the plane wave (4.5) has sharp edges at \(\phi = \pm \pi\), which would correspond to an infinitely sharp shadow, as predicted by geometric optics but contrary to the nature of wave propagation. We define scattering by the asymptotic form
\[
\psi(r, \phi) = \psi_{pw}(r, \phi) + \frac{f(\phi)e^{ikr}}{\sqrt{r}},
\]
(4.6)
do so that \(\sigma(\phi) = |f(\phi)|^2\). Contrary to the usual convention, here \(\phi = 0\) points in the
backward direction, while $\phi = \pm \pi$ points in the forward direction. This convention allows one to distinguish between forward propagation to the left the scattering center and forward propagation to the right of the scattering center. For fixed $r$—irrespective of how large $r$ is—the asymptotic form (4.6) breaks down for $|\phi|$ sufficiently close to $\pi$, because the asymptotic form (4.6) represents “far-field,” or Fraunhofer, diffraction, and therefore is valid only for $|\phi| - \pi \geq \sqrt{(\lambda/r)}$.

Following Carslaw, we calculate $\psi(r, \phi; M)$ exactly \([53]\). We re-express the plane wave (i.e., the $M = 1$ solution) with the contour integral

$$\psi(r, \phi - \phi'; M = 1) = e^{-ikr \cos[\phi - \phi']} = \frac{1}{2\pi} \int_C \frac{d\alpha}{e^{i\alpha} - e^{i\phi}} e^{-ikr \cos[\alpha - \phi']}$$

(4.7)

where the contour $C = C_{top} + C_{bottom}$ is sketched in fig. 5. The shaded regions indicate where the integrand is vanishes as one approaches infinity. Note that for equation (4.7) the contributions from $C_{left}$ and $C_{right}$ cancel because of periodicity. (Later this will no longer be true.) The equality of the contour integral and the plane wave is demonstrated by adding the contours $C_{left}$ and $C_{right}$ to the original contour $C$, so that the contour becomes closed and Cauchy’s theorem can be applied.

We now modify (4.7) by altering to integrand to change the periodicity. Equation (4.7) is modified to become

$$\psi(r, \phi - \phi'; M) = \frac{1}{2\pi M} \int_C \frac{d\alpha}{e^{i(\alpha/M)} - e^{i(\phi/M)}} e^{-ikr \cos[\alpha - \phi']}.$$  

(4.8)

For $M = \infty$, we define

$$\psi(r, \phi - \phi'; \infty) = \frac{1}{2\pi i} \int_C \frac{d\alpha}{(\alpha - \phi)} e^{-ikr \cos[\alpha - \phi']}.$$  

(4.9)

Deforming $M$ away from unity spreads out the simple poles of unit strength in the integrand, thus changing the periodicity of $\psi$, so that $\psi(r, \phi + 2\pi M; M) = \psi(r, \phi; M)$.\(^\dagger\)

The variable $\phi'$, which hereafter shall be set to zero, allows one to show that the Helmholtz equation is satisfied. Since $\psi$ depends only on the difference $(\phi - \phi')$, $\partial_{\phi}^2 \psi = \partial_{\phi'}^2 \psi$, so consequently $[\nabla^2 + k^2] \psi = 0$. It now only remains to be shown that $\psi(r, \phi; M)$ has the required asymptotic form.

\(^\dagger\) None of the arguments here rely on $M$ being an integer. The situation where $M$ is not an integer is of interest for studying wave propagation in the presence of conical defects, which arise from point sources in $2 + 1$ dimensional gravity and in the vicinity of cosmic strings in $3 + 1$ dimensional gravity.
The asymptotic form for equation (4.8), defined in equation (4.6), is derived using the saddle point approximation, which to leading order in \( kr \) gives

\[
\int d\alpha \ F(\alpha) \exp \left[ -(kr)G(\alpha) \right] = \sqrt{\frac{\pi}{kr}} \sum_{l} \frac{(\pm)^{l}}{\sqrt{G''(\alpha_l)}} F(\alpha_l) \exp \left[ -(kr)G(\alpha_l) \right]
\]  

(4.10)

where the index \( l \) labels the relevant saddle points, which for this problem, with \( G(\alpha) = i \cos \alpha \), are located at \( \alpha = -\pi, \ 0, \ +\pi \). As indicated in fig. 6(a), we deform the contours in fig. 5 to pass through the saddle points. In doing so, for \( |\phi| < \pi \) it is necessary to push either \( C_{\text{upper}} \) or \( C_{\text{lower}} \) to over a pole on the real axis in the interval \(-\pi < \alpha < +\pi \). The deformation around the pole adds an additional closed contour, shown in fig. 6(b), giving a contribution equal to \( \psi_{pw} \) — which is precisely the “unscattered” wave. We now calculate the contributions from the saddle point.

The contributions from \( \alpha = 0 \) cancel, leaving

\[
\frac{1}{2\pi M} \left( \frac{1 - i}{\sqrt{2}} \right) \sqrt{\frac{2\pi}{kr}} e^{ikr} \left[ e^{i(\pi/M)} - e^{i(\phi/M)} - e^{-i(\phi/M)} \right] 
\]

\[
= \left( \frac{1 + i}{\sqrt{2}} \right) \frac{1}{\sqrt{2\pi k M}} e^{ikr} \left[ \frac{i \sin(\pi/M)}{\cos(\pi/M) - \cos(\phi/M)} \right],
\]

so that the scattering amplitude in equation (4.6) is

\[
f_{M}(\phi) = \left( \frac{1 + i}{\sqrt{2}} \right) \frac{1}{\sqrt{2\pi k M}} \left[ \frac{\sin(\pi/M)}{\cos(\pi/M) - \cos(\phi/M)} \right].
\]  

(4.11)

The infinitely-sheeted scattering amplitude has the particularly simple form

\[
f_{\infty}(\phi) = \left( \frac{1 + i}{\sqrt{2}} \right) \sqrt{\frac{2\pi}{k}} \cdot \frac{1}{\phi^2 - \pi^2}.
\]  

(4.12)

Equation (4.12) has also been derived by direct summation of the Feynman path integral in polar coordinates with the summation restricted to a particular homotopy class.\cite{46} Equation (4.11) is the amplitude for vortex-vortex scattering where the initial and final states are both flux eigenstates. Here \( M \) indicates the minimal number of windings to recover the original fluxes. As expected, the scattering amplitudes (4.11) and (4.12) have two forward peaks at \( \phi = \pm \pi \) where the cross section diverges, and away from these peaks the amplitude decreases. When \( |\phi| \to \pi \), a pole approaches one of the two relevant saddle points, eventually invalidating the saddle point approximation made above for fixed \( kr \). This happens because for \( |\phi| \) close to \( \pi \) at fixed \( r \) sufficiently close to the forward direction one sees a Fresnel diffraction pattern rather than a small-angle divergence in the wave function.
It is interesting to note that with the infinitely-sheeted scattering amplitude one can calculate the usual Aharonov-Bohm scattering amplitude in a very simple way. Consider Aharonov-Bohm scattering of a particle of charge \( q \) by an infinitely thin flux tube carrying a flux \( \Phi = (\alpha/2\pi)\Phi_0 \) where \( \Phi_0 = (\hbar c/q) \) is the quantum of flux with respect to the charge \( q \). It follows that

\[
f_{AB}(\phi; \alpha) = \sum_{n=-\infty}^{+\infty} e^{i\alpha} f_\infty(\phi + 2\pi n)
\]

\[
= \left( \frac{1 + i}{\sqrt{2}} \right) \sqrt{\frac{2\pi}{k}} \sum_{n=-\infty}^{+\infty} e^{i\alpha} \frac{1}{(\phi + 2\pi n)^2 - \pi^2}.
\]  

(4.13)

In other words, the Aharonov-Bohm scattering amplitude is the superposition of contributions with all possible winding numbers, added together with a relative phase factor of \( e^{i\alpha} \) for \( \Delta N = +1 \). This sum is evaluated by considering contour integral

\[
\frac{1}{2\pi} \int_C \frac{dz}{e^z - 1} \frac{1}{(z + \phi)^2 - \pi^2}
\]

(4.14)

where \( C \) is a closed curve enclosing the entire real axis. For \( 0 < \alpha < 2\pi \), the integral (4.14) vanishes, so that the sum of the residues vanishes. In other words,

\[
\sum_{n=-\infty}^{+\infty} e^{i\alpha} \frac{1}{(\phi + 2\pi n)^2 - \pi^2} + \frac{1}{2\pi} \sin(\alpha/2) e^{-i(\alpha/2\pi)} = 0.
\]

(4.15)

Because of periodicity, the result extends to all values of \( \alpha \). Therefore we get

\[
f_{AB}(\phi; \alpha) = \left( \frac{1 + i}{\sqrt{2}} \right) \sqrt{\frac{2\pi}{k}} \frac{1}{\cos(\phi/2)} e^{-i(\alpha/2\pi)}.
\]

(4.16)

the classic Aharonov-Bohm result. [Because of our peculiar convention for defining the scattering angle, there is a cosine rather than a sine in the denominator.] So far in our discussion of vortex-vortex scattering we have not included the possibility of “exchange” effects, whose relevance was first pointed out by Lo and Preskill\(^\text{[17]}\). One might not expect exchange effects to be relevant for the non-Abelian vortex-vortex interaction discussed here because vortices carrying the same flux do not interact—at least not by means of the purely topological effects we consider here. However, an exchange interaction does sometimes arise for vortices carrying different fluxes. In our discussion of two-vortex scattering we considered only paths that take \( A \) to \( A' \).
and \( B \) to \( B' \). We did not consider paths that take \( A \) to \( B' \) and \( B \) to \( A' \). However, when \( h(\alpha) \) and \( h(\beta) \) lie in the same conjugacy class, it is possible that amplitudes from the two types of paths must be added together.

To consider this possibility, it is useful to consider the braid operation \( \mathcal{R} \), whose square was considered in fig. 4. \( \mathcal{R} \) exchanges two vortices in a counter-clockwise sense, as indicated in fig. 7(a). With the flux basis indicated in fig. 7(b), one has

\[
\mathcal{R} : \alpha \rightarrow \beta, \\
\mathcal{R} : \beta \rightarrow \beta \alpha \beta^{-1}.
\]  

(4.17)

Let us write \( \alpha_{(n+\frac{1}{2})} = \mathcal{R} \alpha_{(n)} \) and \( \beta_{(n+\frac{1}{2})} = \mathcal{R} \beta_{(n)} \). Then, as before, define \( \tilde{M} \) to be the smallest integer or half-integer such that \( \alpha_{(\tilde{M})} = \alpha_{(0)} \) and \( \beta_{(\tilde{M})} = \beta_{(0)} \). If \( \tilde{M} \) is a half-integer, then exchange effects are relevant. As a simple example, consider the symmetric group \( S^3 \). Suppose that \( \alpha_{(0)} = (12) \) and \( \beta_{(0)} = (23) \). A simple calculation shows that \( \tilde{M} = 3/2 \). In the multiply-sheeted formalism, the exchange contribution is taken into account very simply. One just inserts the proper half-integral value of \( M \) into the amplitude (4.11).

5. Abelian Alice Vortex-Vortex Scattering

Let us consider vortex-vortex scattering of Abelian Alice strings, for the moment ignoring the possibility of Cheshire charge, which gives rise to a classical force, in addition to the purely quantum-mechanical flux holonomy effect.

We start with two Alice vortices in flux eigenstates, so that

\[
\begin{align*}
  h(\alpha_{(0)}) &= X, \\
  h(\beta_{(0)}) &= Xe^{iQ}.
\end{align*}
\]  

(5.1)

We have set the phase of \( h(\alpha_{(0)}) \) to zero by an overall global gauge rotation. Recall

\[
\begin{align*}
  \alpha_{(j+\frac{1}{2})} &= \alpha_{(j)}\beta_{(j)}^{-1} \alpha_{(j)}^{-1}, \\
  \beta_{(j+\frac{1}{2})} &= \alpha_{(j)}.
\end{align*}
\]  

(5.2)
so that one has

\[
\begin{align*}
h(\alpha(0)) &= X, & h(\beta(0)) &= Xe^{i\xi Q}, \\
h(\alpha(\frac{1}{2})) &= Xe^{-i\xi Q}, & h(\beta(\frac{1}{2})) &= X, \\
h(\alpha(1)) &= Xe^{-2i\xi Q}, & h(\beta(1)) &= Xe^{-i\xi Q}, \\
h(\alpha(\frac{3}{2})) &= Xe^{-3i\xi Q}, & h(\beta(\frac{3}{2})) &= Xe^{-2i\xi Q}, \\
\end{align*}
\]

(5.3)

…

Note that \( h(\alpha(j))h(\beta(j)) \) is independent of \( j \). This is because of the superselection rule for the combined flux. The path \( \alpha\beta \) surrounds both vortices and can be thought of as measuring the flux at spatial infinity.

When \( (\xi/2\pi) \) is irrational, the sequence never repeats itself. In this case, the classical configuration space is simply the previously discussed Riemannian surface with \( M = \infty \). When \( (\xi/2\pi) \) is rational, there is a distinction between fractions with even denominators and fractions with odd denominators. When \( (\xi/2\pi) \) has an even denominator, an even number of exchanges (or braid operations) restores the original flux state. Thus paths related by an odd number of exchanges will always produce a different flux state, and should not be included in the path integral sum. The path integral for a process involves only “direct” paths—that is, paths that take \( A \) to \( A' \) and \( B \) to \( B' \)—and does not involve any “exchange” paths—that is, paths that take \( A \) to \( B' \) and \( B \) to \( A' \). By contrast, when \( (\xi/2\pi) \) has an odd denominator, an odd number of exchanges restores the original flux state. Thus the path integral includes both “direct” and “exchange” contributions. For \( (\xi/2\pi) \) irrational, the path integral includes only one homotopy class of paths, because all homotopically inequivalent paths result in different final flux states. However, for certain large winding numbers the distinction becomes arbitrarily hard to discern.

The discussion so far has been simplified because it has ignored the zero mode and its coupling to the massless degrees of freedom that exist because there is a continuous unbroken symmetry. A flux state with \( h(\alpha) = Xe^{i\theta Q} \) and \( h(\beta) = Xe^{i(\theta + \xi)Q} \), which we shall denote as

\[
|\theta > \otimes |(\theta + \xi) >
\]

(5.4)

is a superposition of total charge/flux eigenstates \( |Q, \xi > \), with every charge occurring
in the superposition with the same amplitude. The bases are related by the expression

$$|Q, \xi> = \int_0^{2\pi} \frac{d\theta}{\sqrt{2\pi}} e^{i\theta Q} |\theta > \otimes |\theta + \xi >.$$  \hspace{1cm}(5.5)$$

Let $R$ be the vortex-vortex separation. For total charge/flux eigenstates, there is a classical potential of the form $-Q^2 \ln[R]$. The repulsion is due to the fact that a pair with Cheshire charge lowers its self-energy by separating, and thus further delocalizing the Cheshire charge.

A realistic discussion of vortex-vortex scattering should take into account both effects. In the weak coupling limit, the ‘Coulomb’ interaction diminishes, while the strength of the purely quantum-mechanical holonomy effect remains constant.

6. Conclusion

This paper has explored several of the Aharonov-Bohm phenomena which would occur if the simplest realistic cosmic strings exist. Among possibilities we find would not occur is AB catalysis of quark-lepton transitions for trajectories passing outside the core of a string. Nevertheless, there would be interesting and observable color interactions which would produce hadronic excitations and perhaps characteristic hadronic jets resulting from hadron-string collisions. These interactions represent perhaps the first realistic example and consequence of Alice strings, and they turn out to be non-Abelian Alice strings rather than the Abelian ones which introduced the concept.

The general formalism for AB scattering has been applied to a beautiful formal problem, the scattering of straight parallel strings on each other, so that the motion is restricted to the two space dimensions perpendicular to the string axes, and thus is really a problem in 2+1 dimensional quantum mechanics. Because of its entirely topological nature, the flux-flux scattering discussed in section 4 can be interpreted as a statistical interaction of a rather exotic sort. It is well known that in two spatial dimensions (as contrasted with three or more spatial dimensions) there are more possible types of statistics because the homotopy classes for ways of interchanging $n$ particles are described by the infinite braid group $A_n$ rather than the
much smaller finite symmetric group $S^n$. “Anyons” have exotic statistics described by one-dimensional unitary representations of $A_n$. Anyonic statistics are Abelian in the sense that the non-Abelian structure of $A_n$ is not probed—that is, $g_1g_2g_1^{-1}g_2^{-1}$ is always mapped into a trivial phase. This is no longer the case for the non-Abelian vortices discussed here. It would be very interesting to find examples of systems in a two-dimensional condensed matter context with elementary excitations which obey similar non-Abelian statistics.

An intriguing subject which still presents some open questions is the issue of string excitations corresponding to localization of charge on the string. Depending on the energy cost of such excitations the transition from quark to lepton for fermions which penetrate the string, a process which can have large cross section growing as a fractional power of the de Broglie wavelength, could be accomplished by deposit of the lost charges on the string core. Estimates of the charge deposit energy would be a worthwhile subject for future study.

Cosmic strings in themselves are an example of the Aharonov-Bohm effect, in the sense that the continuity and covariant constancy of the Higgs fields surrounding the string constrains the allowed fluxes carried by the string. Thus the two subjects are intimately connected, and very likely there remain new links between them to be explored in the future.

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FIGURE CAPTIONS

Figure 1. Describing a Classical Vortex Configuration. Four vortices $V_1$, $V_2$, $V_3$, $V_4$ are represented in a constant timeslice. A basis of paths $C_1$, $C_2$, $C_3$, $C_4$ starting and ending around $x_0$ is indicated. The paths are used to define the fluxes carried by the vortices. Each path $C_j$ winds around the vortex $V_j$ exactly once in the counterclockwise direction without winding around any of the other vortices.

Figure 2. Braid Dependence. In this figure it is shown that the winding number of a path does not determine the flux completely. The two paths $\alpha$ and $\alpha'$ have the same winding numbers around the vortices $A$ and $B$. However, when the vortices $A$ and $B$ carry non-Abelian fluxes that do not commute, the fluxes measured with respect to $\alpha$ and $\alpha'$ differ. Since $\alpha' = \beta \alpha \beta^{-1}$, $h(\alpha') = h(\beta) h(\alpha) h(\beta)^{-1}$. Thus $h(\alpha') = h(\alpha)$ if and only if $[h(\alpha), h(\beta)] = h(\alpha) h(\beta) h(\alpha)^{-1} h(\beta)^{-1} = e$.

Figure 3. Vortex-Vortex Scattering. In (a) is indicated a basis for two vortices positioned at $A$ and $B$. For simplicity, we imagine that vortex $B$ is held
fixed while vortex $A$ is allowed to propagate quantum mechanically. We consider the propagation of vortex $A$ to its new position $A'$, as indicated in (b). In (b) are shown two homotopically inequivalent paths, path 1 and path 2 from $A$ to $A'$. (Actually, there is an infinite number of homotopy classes of paths from $A$ to $A'$.) In (d) is indicated a basis for the final state. If path 1 is taken $\alpha_f = \alpha_1$. If path 2 is taken $\alpha_f = \alpha_2$.

**Figure 4.** Effect of Moving One Vortex Around Another Vortex.

**Figure 5.** Contour for Calculating Exact Solution.

**Figure 6.** Saddle Points and Deformations of Contours.

**Figure 7.** The Braid Operation. The braid operation $\mathcal{R}$ is illustrated in (a). Fig. (b) indicates a basis. Fig. (c) indicates what happens to basis path defined in (b) when the $\mathcal{R}$ operation is reversed, dragging the paths back. $\alpha(0)$ is dragged back to $\alpha(1/2)$ and $\beta(0)$ is dragged back to $\beta(1/2)$ as the $\mathcal{R}$ operation is reversed.

**Figure 8.** Moving a quark through a loop of colored Alice string. Initially, the colored Alice string is uncharged and the $Q\bar{Q}$ pair is in a singlet state. The quark is taken through the loop and then back to its original position, while the antiquark remains fixed. Because of the transfer of Cheshire charge, the final state is a superposition of a state for which no charge transfer has taken place, with both the loop and the $Q\bar{Q}$ pair in color singlet states, and a state with the both the $Q\bar{Q}$ pair and the loop in color octet state, indicating the transfer of Cheshire charge.