A Loop Representation for the Quantum Maxwell Field

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Quantization of the free Maxwell field in Minkowski space is carried out using a loop representation and shown to be equivalent to the standard Fock quantization. Because it is based on coherent state methods, this framework may be useful in quantum optics. It is also well-suited for the discussion of issues related to flux quantization in condensed matter physics. Our own motivation, however, came from a non-perturbative approach to quantum gravity. The concrete results obtained in this paper for the Maxwell field provide independent support for that approach. In addition, they offer some insight into the physical interpretation of the mathematical structures that play, within this approach, an essential role in the description of the quantum geometry at Planck scale.
1 Introduction.

This paper discusses only the free quantum Maxwell field. However, the motivation for the work comes from certain issues that arise in quantum gravity.

Over the last decade, it has become increasingly clear that perturbative methods, which have been so successful in the treatment of non-gravitational interactions, cannot be used to construct a quantum theory of gravity. One must face this problem non-perturbatively. The canonical approach is well-suited for this task. Indeed, already in the sixties, through the work initiated by Bergmann, Dirac, Arnowitt, Deser, Misner and others, a concrete, non-perturbative quantization program was formulated within this approach. The idea was to first represent quantum states as functionals of 3-metrics and then select physical states by demanding that they be annihilated by the quantum constraint operators. Unfortunately, the task of solving the constraint equations turned out to be difficult in the quantum theory and not a single physical state could be obtained. More recently, it was realized that the problem simplifies considerably if, prior to quantization, one performs a transform and works with new canonical variables which are analogous to those normally used in the Yang-Mills theory [1]. Thus, in this framework, the emphasis is shifted from 3-metrics to certain connections on the spatial 3-manifold. Consequently, concepts such as the trace of holonomies—the Wilson loops—now play an important role in quantum gravity. In fact, one can introduce [2] a new representation for quantum states—called loop representation—where states are functionals of closed loops. This representation has turned out to be particularly useful in solving the quantum constraints; an infinite dimensional space of solutions is now known [2, 3]. We thus have a large class of physical states of exact quantum gravity which can be used to analyze the micro-structure of space-time in the Planck regime. What we lack, however, is sufficient physical intuition for these states and for the natural mathematical operators which operate on them. There does exist a formal transform, called the loop transform, which enables one to relate objects on the loop space to those on the space of connections which are generally easy to interpret geometrically and hence also physically. However, the transform is only formal: it involves an integration on the infinite dimensional space of connections and very little is known about the existence of the required measure.
In order to gain physical insight into the loop representation, it seems natural to apply these general ideas to simpler systems. Perhaps the simplest toy-model is general relativity in 2+1-dimensions. This model has been investigated in some detail [5] and has provided insights into several “geometric” aspects of the loop representation, particularly on the role of the diffeomorphism invariance of the theory in the quantum description. However, in this model there are only a finite number of degrees of freedom, whence issues related to the presence of the infinite dimensional reduced phase-space of 3 + 1 general relativity have remained unexplored. To gain insight into these issues, it is natural to use the loop representation for quantization of Maxwell and Yang-Mills theories. The goal is three-folds. First, one wants to check if one can indeed reproduce the known, “correct” quantum physics via loop quantization. A positive result would give considerable confidence in the loop approach since nowhere in its development was it required that it should yield, e.g., the Poincaré invariant Fock space of photons. Second, one hopes that the results obtained in these model systems may provide the much needed physical intuition for loop states, as well as technical tools that may be useful in solving the open mathematical problems in the gravitational case. Third, within the models themselves, one may be able to apply the loop space methods to obtain new results. For example, one can argue that the loop representation is especially well-suited to the computation of higher excited states and their energy levels in lattice QCD, to couple the theory to dynamical fermions, to provide a firmer relation between string like structures (a la Nambu and Goto) and QCD, to analyze properties of coherent states in QED and in the discussion of issues related to flux quantization in condensed matter physics.

The purpose of this paper is to provide a loop quantization of the free Maxwell field. In particular we will construct, entirely within the loop picture, the analog of the Fock space of photons and represent on it physically interesting operators, including the Hamiltonian. Furthermore, we will provide a direct interpretation of the loop states themselves. Finally, the mathematical techniques we introduce to obtain the inner-product between physical states are likely to be useful in quantum gravity.

\[\text{\footnotesize 1}\text{The Gelfand spectral theory offers a promising approach to making the transform well-defined [4].}\]
Perhaps the most intriguing aspect of the loop representation is that
the quantum states are now functionals of loops. In field theory, one often
uses a functional representation. However, in these descriptions, the states
are functionals of the \textit{dynamical variables} of the theory. For example, in
the case of the Maxwell field, one often use a basis \( |A(\vec{x})\rangle \), on which the
field operator \( \hat{A}_a(\vec{x}) \) acts as a multiplication operator, and represents states
as functionals of the vector potential \( A(\vec{x}) \). In a loop representation, by
contrast, the states are functionals, \( \Psi[\gamma] \), of closed loops \( \gamma \) on a 3-manifold.
Now, closed loops themselves have no \textit{dynamical} significance whatsoever in
the classical Maxwell theory. How can such a description then be viable and
why should it have any relation at all to the Fock space, represented, e.g., as
the space of functionals of vector potentials?

Let us first discuss a simple example to see that there is no a priori
conflict. Consider, within the standard non-relativistic quantum mechanics,
a Hydrogen atom. Normally, its states are represented as functions, \( \Psi(\vec{x}) \), of
the configuration variable \( \vec{x} \). However, one can also go to the basis in which
the Hamiltonian, \( \hat{H} \), the total angular momentum, \( \hat{L}^2 \), and the \( z \)-component
of the angular momentum, \( \hat{L}_z \), are diagonal, and represent states as functions,
\( \Psi[n,l,m] \) of three integers, \( n, l, m \). The sets of these three integers have no
physical significance at all in the classical dynamics of the hydrogen atom.
And yet the representation of states as functions \( \Psi[n,l,m] \) is both viable
and extremely useful. If some one hands us just the space of (normalized)
functions \( \Psi[n,l,m] \), it would be difficult to see they have anything to do what
so ever with the Coulomb problem. However, if in addition we are given the
physical interpretation of a complete set of operators \( \hat{H}, \hat{L}^2, \hat{L}_z \) which act
by multiplication on \( \Psi[n,l,m] \) –i.e., \( \hat{H} \cdot \Psi[n,l,m] := (-13.6 \text{ ev}/n^2) \Psi[n,l,m] \),
etc– then it follows immediately that the functions \( \Psi[n,l,m] \) of 3 integers
represent the quantum states of the hydrogen atom. Furthermore, we can
then express any other physical observable –such as \( \hat{X} \)– as an operator on
these wave functions.

The situation is similar in loop representations. In the one constructed in
this paper, it is the (positive frequency) electric fields –which form a complete
commuting set– that act as multiplication operators on loop states. The
fact that the loops are closed ensures consistency with the constraint that
electric fields must be divergence-free. Armed with this interpretation, not
only can one relate the loop states with the familiar Fock space but one can
also express physical observables \textit{directly} in the loop representation. This
detailed analysis removes much of the unease that one experiences when one first encounters the loop representation in canonical gravity.

Since our motivation comes from quantum gravity, the paper is primarily addressed to gravitational theorists. Therefore, we begin in section 2 by recalling the Bargmann representation for a quantum Maxwell field which may be unfamiliar to our intended audience. In section 3, we exhibit the transform from the Bargmann to the loop representation, show that it exists rigorously and discuss its properties. In particular, we present the loop states which correspond to the n-photon states, normally used as the basis in the Fock description, as well as the images in the loop representation, of the familiar operators on the Fock space. The Bargmann states (section 2) have been used widely, especially in quantum optics, and the general ideas behind the loop transform (section 3) are well-known in the literature on lattice gauge theories. The new element here is the synthesis of the two ideas. Indeed, the loop transform in the continuum is well-defined precisely because we depart from the practice of using wave functions $\Psi(A)$ of real connections and use instead the Bargmann states which are holomorphic functionals of the positive frequency connections. In section 4, we construct the loop representation ab initio, without any reference to the Bargmann representation or the loop transform. Strictly, from the viewpoint of the Maxwell theory alone, this step is unnecessary. Its motivation comes, rather, from quantum gravity where, as noted above, the transform exists only as a formal device. In section 5, we point out that the quantization procedure used in section 4 is precisely the one that was first introduced in the gravitational case and comment on the relation between our framework to other similar treatments.

The idea of formulating quantum theory of gauge fields in terms of loops and holonomies has in fact a long tradition [6]. In particular, a somewhat different treatment of the quantum Maxwell theory in terms of loops was given by Gambini and Trias [7]. The present paper itself belongs to a series. The second paper [8] in this series provides a general framework for lattice QCD with fermions, the third [9] uses this framework to carry out computations of ground and excited state energies in 2+1-dimensional QCD without fermions. Continuum QCD without fermions is discussed in [10]. Finally, the analogous loop representation of linearized gravity (i.e. gravitons) is given in [11].
2 Bargmann representation

Our purpose here is to recall the Bargmann representation [12] of free photons, using, however, a canonical framework. In this representation, quantum states arise as holomorphic functionals of complex connections. The choice of this representation as the point of departure is a key element of our analysis: we will see in the next section that it is this strategy that enables us to pass to the loop picture in a straightforward fashion, without encountering any divergences. In addition, the general procedure used is closely related to the one proposed [2] in gravity where it is the canonical quantization method and the holomorphic connection representation that are naturally available.

Let us begin with a summary of the phase space description of the Maxwell field. Let $\Sigma$ denote a $t=$const. slice in Minkowski space. $\Sigma$ is topologically $\mathbb{R}^3$ and equipped with a flat, positive definite metric $q_{ab}$. The configuration variable for the Maxwell field is the $U(1)$--connection 1-form $A_a(\vec{x})$ on $\Sigma$, the vector potential for the magnetic field $B^a(\vec{x})$. Its canonically conjugate momentum is the electric field $E^a(\vec{x})$. The fundamental Poisson bracket is

$$\{A_a(\vec{x}), E^b(\vec{y})\} = \delta_a^b \delta^3(\vec{x}, \vec{y}). \quad (1)$$

The system has one first class constraint, $\partial_a E^a = 0$. One can therefore pass to the reduced phase space $\hat{\Gamma}$, by, for example, fixing the Coulomb gauge. Let us do so. The true dynamical degrees of freedom are then contained in the pair $(A^T_a(\vec{x}), E^a_T(\vec{x}))$ of transverse (i.e. divergence free) fields on $\Sigma$. The only non vanishing Poisson bracket now is

$$\{A^T_a(\vec{x}), E^b_T(\vec{y})\} = \delta_a^b \delta^3(\vec{x}, \vec{y}) - \Delta^{-1} \partial_a \partial^b \delta^3(\vec{x}, \vec{y}), \quad (2)$$

where $\Delta$ is the negative of the Laplacian of $q_{ab}$. (Thus $\Delta$ is a non-negative operator.)

It is easier –although, by no means essential– to work with momentum space variables. Let us therefore express $A^T_a(\vec{x})$ and $E^a_T(\vec{x})$ in terms of their Fourier components:

$$A^T_a(\vec{x}) = \frac{1}{(2\pi)^{3/2}} \int d^3k \ e^{i\vec{k} \cdot \vec{x}} \ [g_1(\vec{k})m_a(\vec{k}) + g_2(\vec{k})\bar{m}_a(\vec{k})],$$

$$E^a_T(\vec{x}) = -\frac{1}{(2\pi)^{3/2}} \int d^3k \ e^{i\vec{k} \cdot \vec{x}} \ [p_1(\vec{k})\bar{m}^a(\vec{k}) + p_2(\vec{k})m^a(\vec{k})], \quad (3)$$
where \(m_a(\vec{k})\) is a complex vector field in the momentum space which is transverse, \(m_a(\vec{k})k^a = 0\), and normalized such that \(m^a(\vec{k})m_a(\vec{k}) = 0\), \(m^a(\vec{k})\bar{m}_a(\vec{k}) = 1\). (The negative sign in front of the r.h.s of the second equation is inserted only to simplify subsequent equations.) The \(q_j(\vec{k})\) and \(p_j(\vec{k})\), with \(j = 1, 2\), capture the true degrees of freedom in the phase space of the Maxwell field; they describe the two radiative modes of the Maxwell field corresponding to the two helicities. The Poisson bracket relations (2) are equivalent to:

\[
\{q_i(-\vec{k}), p_j(\vec{k}')\} = \delta_{ij} \delta^3(\vec{k}, \vec{k}').
\] (4)

In the Bargmann representation, one works with positive and negative frequency fields. Given a pair \((A^T, E_T)\) in the reduced phase space, one can evolve it to obtain a (real) solution to Maxwell’s source-free equations, decompose it into positive and negative frequency parts and consider the data \((\pm A, \pm E)\) induced on the initial slice by the two parts. It is easy to verify that the positive frequency connection \(+A\) is given by

\[
\begin{align*}
+ A_a(x) & = \frac{1}{\sqrt{2}} \left( A^T_a(x) + i \Delta^{-1/2}(E_T)_a(x) \right) \\
& \equiv \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{|k|} e^{i\vec{k}\cdot \vec{x}} \left[ \zeta_1(\vec{k})m_a(\vec{k}) + \zeta_2(\vec{k})\bar{m}_a(\vec{k}) \right],
\end{align*}
\] (5)

with

\[
\zeta_j(\vec{k}) = \frac{1}{\sqrt{2}} \left( |\vec{k}| q_j(\vec{k}) - i p_j(\vec{k}) \right).
\] (6)

It is conjugate to the negative frequency electric field, \(-E\) given by

\[
- E^a(x) = \frac{1}{\sqrt{2}} \left( E^a_T(x) + i \Delta^{1/2}(A^T)^a(x) \right)
\equiv \frac{-i}{(2\pi)^{3/2}} \int d^3\vec{k} e^{i\vec{k}\cdot \vec{x}} \left[ \bar{\zeta}_1(-\vec{k})m_a(\vec{k}) + \bar{\zeta}_2(-\vec{k})\bar{m}_a(\vec{k}) \right],
\] (7)

The true degrees of freedom are now coded in the two complex fields \(\zeta_j(\vec{k})\) in momentum space: the Fourier coefficients of the positive frequency, transverse connection \(+A_a(x)\). These \(\zeta_j(\vec{k})\) provide us with a complex chart on the reduced phase space of the real Maxwell field. The only non-vanishing Poisson brackets among \(\zeta_j(\vec{k})\) and \(\bar{\zeta}_j(\vec{k})\) are:

\[
\{\zeta_i(\vec{k}), \bar{\zeta}_j(\vec{k}')\} = i|k| \delta_{ij} \delta^3(\vec{k}, \vec{k}').
\] (8)
The $\zeta_j(\vec{k})$ play the same role as the complex coordinates, $z = \frac{1}{\sqrt{2}}(\omega q - ip)$ on the phase space of a harmonic oscillator. Thus, the Maxwell field is now represented as an assembly of harmonic oscillators, two for each 3-momentum $\vec{k}$.

To quantize this system, let us begin by letting the $\zeta_j(\vec{k})$ and $\bar{\zeta}_j(\vec{k})$ be the “elementary classical variables” which are to have unambiguous quantum analogs, $\hat{\zeta}_j(\vec{k})$ and $\hat{\bar{\zeta}}_j(\vec{k})$. To begin with, we consider them as abstract operators. Let $\mathcal{A}$ denote the associative algebra they generate, subject to the canonical commutation relations (CCRs):

$$\left[\hat{\zeta}_i(\vec{k}), \hat{\bar{\zeta}}_j(\vec{k}')\right] = -\hbar |\vec{k}| \delta_{ij} \delta^3(\vec{k}, \vec{k}').$$

(9)

On this algebra, we define an involution operation, $*$, via

$$\left(\hat{\zeta}_i(\vec{k})\right)^* = \hat{\bar{\zeta}}_j(\vec{k}).$$

(10)

Denote the resulting $*$-algebra by $\mathcal{A}^{(*)}$. In the quantum theory, the abstract operators that form $\mathcal{A}^{(*)}$ are realized as linear operators on a Hilbert space of states. Thus, we now construct the quantum theory as a linear representation of the $*$-algebra $\mathcal{A}^{(*)}$ on a Hilbert space.

In the Bargmann method, this representation is constructed by using the “anti-holomorphic polarization” on the reduced phase space. Thus, one begins with the vector space $V$ of functionals $\Psi(\zeta_j)$ which are polynomials in $\zeta_j(\vec{k})$. (Since they are independent of $\bar{\zeta}_j(\vec{k})$, they are entire, holomorphic.) On this $V$, the operators $\hat{\zeta}_j(\vec{k})$ are introduced in the obvious fashion

$$\hat{\zeta}_j(\vec{k}) \circ \Psi(\zeta_i) = \zeta_j(\vec{k})\Psi(\zeta_i).$$

(11)

To ensure that the CCRs (9) are satisfied, it is then natural to represent $\hat{\bar{\zeta}}_j(\vec{k})$ via

$$\hat{\bar{\zeta}}_j(\vec{k}) \circ \Psi(\zeta_i) = h|\vec{k}| \frac{\delta \Psi(\zeta_i)}{\delta \bar{\zeta}_j(\vec{k})}.$$  

(12)

The final task is to equip $V$ with an inner-product. The idea [13] is to use the $*$-relations (10) to constraint the inner product. Thus, we seek an inner product $<,>$ with respect to which the operator appearing on the right
side of (12) is the Hermitian adjoint of that appearing on the right side of (11), i.e. such that

$$< \Psi, \zeta_j(\vec{k}) \Phi > = < \bar{h} | \vec{k} | \frac{\delta \Psi}{\delta \zeta_j(\vec{k})}, \Phi >,$$

(13)

for all $\Psi$ and $\Phi$ in $V$. Since the $*$-relations (10) capture, in the quantum theory, the fact that $(A^T, E_T)$ are real classical fields, (13) is equivalent to demanding that the quantum field operators $(\hat{A}^T, \hat{E}_T)$ be self-adjoint. Hence the requirement (13) on the permissible inner products will be called the quantum reality condition.

To find an inner product that satisfies this condition, we utilize a formal procedure; the resulting inner product, however, will be well defined. We begin with the ansatz

$$< \Psi, \Phi > = \int \prod_j \mathcal{d} \zeta_j \wedge \mathcal{d} \bar{\zeta}_j \mu(\zeta_j, \bar{\zeta}_j) \overline{\Psi(\bar{\zeta}_j)} \Phi(\zeta_j),$$

(14)

where $\prod_j \mathcal{d} \zeta_j \wedge \mathcal{d} \bar{\zeta}_j$ is the “translation invariant volume element” on the reduced phase space $\hat{\Gamma}$ and regard (13) as a condition on the “measure”, namely on $\mu(\zeta_j, \bar{\zeta}_j)$. Using the definitions (11, 12) of $\zeta_j(\vec{k})$ and $\bar{\zeta}_j(\vec{k})$, it is straightforward to verify that (13) determines $\mu(\zeta_j, \bar{\zeta}_j)$ uniquely (except for a trivial multiplicative constant) as

$$\mu(\zeta_j, \bar{\zeta}_j) = \exp - \int \frac{d^3 \vec{k}}{\hbar |\vec{k}|} (|\zeta_1(\vec{k})|^2 + |\zeta_2(\vec{k})|^2).$$

(15)

Thus, $\prod_j \mathcal{d} \zeta_j \wedge \mathcal{d} \bar{\zeta}_j \mu(\zeta_j, \bar{\zeta}_j)$ turns out to be a (well-defined) Gaussian measure. The Hilbert space $\mathcal{H}$ of all quantum states is obtained by the Cauchy completion of $V$ with respect to this inner product. Note that the quantum reality conditions have led us directly to the standard Gaussian measure; we did not have to invoke Poincaré invariance of the inner product or of the vacuum. One may therefore expect this procedure to select the inner product to succeed also beyond Minkowskian field theories. This expectation is borne out in, e.g., 2+1-dimensional gravity. [5]

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2In this paper we ignore the issues of domains of various (densely defined) operators. In particular, we do not distinguish between self-adjoint and symmetric operators.
We conclude this section by exhibiting a few states and operators in the Bargmann representation. The Fock vacuum $|0\rangle$ is represented by the functional $\Psi(\zeta_j) = 1$. The 1-photon state, $|\vec{k}_0, \epsilon = 1\rangle$, with momentum $\vec{k}_0$ and helicity +1, is represented by the linear functional $\Psi(\zeta_j) = \zeta_1(\vec{k}_0)$ which assigns to each $\zeta_j(\vec{k})$ the value of $\zeta_1$ at $\vec{k}_0$. A general 1-photon state, say with helicity +1, is a superposition,

$$\int d^3\vec{k} \bar{h}|\vec{k}|^2 \sum_{j=1}^2 \zeta_j(\vec{k}) \delta \delta \zeta_j(\vec{k}),$$

where the repeated indices $j_i$ are summed over $j_i = 1, 2$. The operators $\hat{\zeta}_j(\vec{k})$ are the creators; $\hat{\bar{\zeta}}_j(\vec{k})$ are the annihilators. Finally, the normal ordered Hamiltonian is given by

$$: \hat{H} : = \int d^3\vec{k} \ h|\vec{k}| \sum_{j=1}^2 \zeta_j(\vec{k}) \frac{\delta}{\delta \zeta_j(\vec{k})}.$$

This completes the summary of the Bargmann representation of the free Maxwell field.

3 A loop transform

The idea now is to pass to a loop representation by performing a transform from functionals $\Psi(\pm A) \sim \Psi(\zeta_j)$ of transverse, positive frequency connections $\pm A_a(\vec{x})$ on $\Sigma$ via equation (5) ) to functionals $\psi(\gamma)$ of closed loops $\gamma$ on $\Sigma$. (Recall that $\pm A_a(\vec{x}) \equiv (\zeta_1(\vec{k}), \zeta_2(\vec{k})$.) The transform is analogous to the Fourier transform

$$\psi(\vec{p}) = \frac{1}{(2\pi)^{3/2}} \int d^3\vec{x} \ e^{i\vec{p} \cdot \vec{x}} \Psi(\vec{x}),$$

which enables one to pass from the position to the momentum representation in non-relativistic quantum mechanics. The idea underlying this loop
transform was first introduced by Rovelli and Smolin [2] in the context of canonical gravity, where they used the expression

$$\psi(\gamma) = \int d\mu(A) \, T[\gamma, A] \, \Psi(A), \tag{19}$$

as a formal tool to pass from the connection representation, in which states $\Psi(A)$ arise as holomorphic functionals of (self dual) connections, to the loop representation, in which states are certain functionals on the loop space. Here, the trace of the holonomy $T[\gamma, A]$ of the gravitational connection $A_a(x)$ around the closed loop $\gamma$ is analogous to the Kernel $e^{ik \cdot \vec{x}}$ of the Fourier transform and $d\mu(A)$ is to be a suitable measure on the space of connections. However, as pointed out already in [2], equation (19) is only a formal expression because we do not yet know an appropriate measure on the space of gravitational connections $A_a(x)$. [4] We will see in this section that the transform can, however, be made rigorous in the case of the Maxwell field: not only does $\psi(\gamma)$ exist for all Bargmann states $\Psi(\zeta)$, but it can also be evaluated explicitly!

This section is divided into three parts. In the first, we introduce some notions associated with closed loops on $\Sigma$, in the second we show that the transform exists and in the third we show that it is faithful.

### 3.1 Loops

Let us begin with some definitions. By a loop we shall mean a continuous and piecewise smooth mapping $\gamma(s)$ from $S^1$ to $\Sigma$, where $s \in [0, 2\pi]$ (the end points 0 and $2\pi$ being identified). Two loops $\gamma$ and $\beta$ will be said to be holonomically equivalent if, for every smooth connection $A_a$, we have

$$\oint_\gamma A_a \, ds^a = \oint_\beta A_a \, ds^a.$$  

Thus, if holonomically equivalent, $\gamma$ and $\beta$ can differ from each other only through

i) reparametrization, $\gamma(s) = \beta(s')$ for some reparametrization $s \to s'$ of the curve $\beta(s)$. (Note that the reparametrization does not have to be continuous at the points where $\beta$ intersects itself);

ii) retracing identity, $\gamma = l \circ \beta \circ l^{-1}$, where $l$ is a line segment and $\circ$ indicates the obvious composition of segments;

or, any combination of retracings and reparametrizations.

Each equivalence class will be referred to as a holonomic loop and the set of all these equivalence classes will be called the holonomic loop space and
denoted by $\mathcal{HL}$.

Given a loop $\gamma$, we define its form factor, $F^a(\gamma, \vec{x})$, to be a distributional vector density of weight one via:

$$\int F^a(\gamma, \vec{x}) \omega_a(\vec{x}) \, d^3\vec{x} = \oint_\gamma \omega_a \, ds^a. \tag{20}$$

Thus, $F^a(\gamma, \vec{x})$ may be more directly expressed as

$$F^a(\gamma, \vec{x}) = \oint_\gamma ds \, \dot{\gamma}^a(s) \, \delta^3(\vec{x}, \vec{\gamma}(s)), \tag{21}$$

where $\vec{\gamma}(s)$ is the point on the loop $\gamma$ at parameter value $s$ and $\dot{\gamma}^a(s)$ the tangent vector to $\gamma$ at $\vec{\gamma}(s)$. It follows immediately from the definition (20) that

$$\int F^a(\gamma, \vec{x}) \, \partial_a \omega(\vec{x}) \, d^3x = 0 \tag{22}$$

for all $\omega(\vec{x})$, whence $F^a(\gamma, \vec{x})$ is divergence-free:

$$\partial_a F^a(\gamma, \vec{x}) = 0. \tag{23}$$

Note, incidentally, that neither the notion of the form factor nor its properties refer to the metric or even the topology of $\Sigma$. Therefore the notion is useful well beyond Maxwell theory. [14] For the purpose of Maxwell theory, however, it is convenient to use the flat metric $q_{ab}$ and perform a Fourier transform to obtain the momentum space representation of $F^a(\gamma, \vec{x})$. We have:

$$F^a(\gamma, \vec{k}) = \frac{1}{(2\pi)^{3/2}} \int d^3x \, e^{-i\vec{k} \cdot \vec{x}} \, F^a(\gamma, \vec{x}) = \frac{1}{(2\pi)^{3/2}} \oint_\gamma ds \, \dot{\gamma}^a(s) \, e^{-i\vec{k} \cdot \vec{\gamma}(s)}. \tag{24}$$

Let us note a few properties of these form factors. First, it follows from the very definition (20) that two loops $\gamma$ and $\delta$ have the same form factors if and only if they are holonomic. Thus, $F^a(\gamma, \vec{x})$—or $F^a(\gamma, \vec{k})$—can be used to characterize holonomic loops, whence the name “form factors”. (Note, however, that they do not coordinatize $\mathcal{HL}$ in the sense normally used in differential geometry because, given a loop $\gamma$ with the form factor $F^a(\gamma, \vec{k})$, there is in general no loop $\delta$ such that its form factor satisfies $F^a(\delta, \vec{k}) = \ldots$
\( \lambda F^a(\gamma, \vec{k}) \) for a general real number \( \lambda \). Hence, attractive as it may first seem, the strategy of using form factors to induce a manifold structure on \( \mathcal{H} \mathcal{L} \) fails to work.) Next, since \( F^a(\gamma, \vec{x}) \) is divergence-free, \( F^a(\gamma, \vec{k}) \) is transverse: 
\[ k_a F^a(\gamma, \vec{k}) = 0. \]
Hence it has only two independent components. Let us label them by \( F^j(\gamma, \vec{k}) \):

\[ F_1(\gamma, \vec{k}) = \frac{\hbar}{(2\pi)^{3/2}} \oint_{\gamma} ds \, \dot{\gamma}^a(s) \, \bar{m}_a(\vec{k}) \, e^{-i\vec{k} \cdot \vec{\gamma}(s)}, \]  
(25)

\[ F_2(\gamma, \vec{k}) = \frac{\hbar}{(2\pi)^{3/2}} \oint_{\gamma} ds \, \dot{\gamma}^a(s) \, m_a(\vec{k}) \, e^{-i\vec{k} \cdot \vec{\gamma}(s)}, \]  
(26)
so that

\[ F^a(\gamma, \vec{k}) = \frac{1}{\hbar} \left( F_1(\gamma, \vec{k})m^a(\vec{k}) + F_2(\gamma, \vec{k})\bar{m}^a(\vec{k}) \right). \]  
(27)

Here, we have chosen normalization and other conventions that will simplify later calculations. This transversality of form factors will play an important role in the loop quantization because it captures in a natural way the gauge invariance of the theory, i.e., the transversality of the photon. The next property follows from the fact that \( F^a(\gamma, \vec{x}) \) is real; its Fourier transform \( F^a(\gamma, \vec{k}) \) satisfies the “reality condition” \( \bar{F}_j(\gamma, \vec{k}) = -F_j(\gamma, -\vec{k}) \). (The fields \( q_j(\vec{k}) \) and \( p_j(\vec{k}) \) introduced in section 2 satisfy the same “reality conditions”: \( \bar{q}_j(\vec{k}) = -q_j(\vec{k}) \) and \( \bar{p}_j(\vec{k}) = -p_j(\vec{k}) \). The fields \( \zeta_j(\vec{k}) \), on the other hand, are Fourier coefficients of a complex field, \( +A^a_T(\vec{x}) \), and are therefore not subject to these restrictions.

Finally, given two holonomic loops \( \gamma \) and \( \delta \), we define a new holonomic loop, \( \gamma \# \delta \), called the eye-glass loop as follows: \( \gamma \# \delta \equiv l \circ \gamma \circ l^{-1} \circ \delta \) where \( l \) is any line segment joining a point on \( \gamma \) to a point on \( \delta \). \(^3\)

Thus, if \( p \) and \( q \) are points on loops \( \gamma \) and \( \delta \) respectively, and \( l \) joins \( p \) to \( q \), then \( \gamma \# \delta \) is the holonomic loop obtained by first going around \( \gamma \) starting and ending at \( p \), then going along \( l \) from \( p \) to \( q \), then around \( \delta \), and then back along \( l \) to the point \( p \) on \( \gamma \). Although the specific loop so obtained depends on the choices of \( p, q \) and \( l \), the resulting holonomic loop is the same. Using

\(^3\)Note that the operation on the loop space defined here by \( \# \) is different from the one defined in the non-Abelian case. However, the difference arises only because of the trace identities for \( U(1) \) and \( SL(2, \mathbb{C}) \) are different. There is a well-defined sense in which the operation defined here is the Abelian analog of that defined in [2].
the definition of form factors, we now have:

\[ F_j(\gamma \# \delta, \vec{k}) = F_j(\gamma, \vec{k}) + F_j(\delta, \vec{k}). \]  

(28)

In particular, if \( \gamma = \delta \), we have \( F_j(\gamma \# \gamma, \vec{k}) = 2F_j(\gamma, \vec{k}). \) Thus, the space of form factors — i.e., the space of the fields \( F^a(\vec{k}) \) such that there is a loop \( \gamma \) for which \( F^a(\vec{k}) = F^a(\gamma, \vec{k}) \) — does have the structure of a vector space over, however, the ring of integers rather than the ring of reals or complexes.

### 3.2 A transform

We are now ready to define a loop transform. As remarked at the beginning of this section, to define the analog of (19), we need, for Maxwell fields, the analog of the trace of the holonomy and a measure on the space of positive frequency connections \( +A_a(\vec{x}) \). The analog of \( T[\gamma, A] \) is simply \( T[\gamma, +A] = \exp \oint A_a ds^a \). (We have set the electric change equal to 1 and omitted the conventional factor of \( i \) in front of the integrand because now the connection \( +A_a \) is itself complex). For the measure, we do have a satisfactory candidate: the Poincaré invariant measure (15) that defines the inner product in the Bargmann representation. Let us therefore set

\[ \psi(\gamma) = \int \prod_j d\zeta_j \wedge d\bar{\zeta}_j e^{-\frac{1}{2} \int \frac{d^3\vec{k}}{|k|} |\zeta_j(\vec{k})|^2} e^{\oint +A_a ds^a} \Psi(\zeta_j). \]  

(29)

The question is if the integral exists for all Bargmann states \( \Psi(\zeta_j) \) and, if so, whether we can evaluate it explicitly. We shall show that the answer to both questions is in the affirmative.

Let us begin by re-expressing the holonomy in terms of \( \zeta_j \). Using the expression (5) of \( +A(\vec{x}) \) in terms of its Fourier components, we have:

\[ \oint +A_a ds^a = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3\vec{k}}{|k|} \left[ \zeta_1(\vec{k}) \oint e^{i\vec{k} \cdot \vec{\gamma}} \bar{\zeta}^a(s) m_a(\vec{k}) ds + \zeta_2(\vec{k}) \oint e^{i\vec{k} \cdot \vec{\gamma}} \bar{\zeta}^a(s) \bar{m}_a(\vec{k}) ds \right] = \int \frac{d^3\vec{k}}{|k|} \sum_j \zeta_j(\vec{k}) F_j(\gamma, \vec{k}). \]  

(30)
Thus, using the form factors, we have re-expressed the holonomy as an integral in momentum space, where the integrand splits up into two factors: the first, $\zeta_j(\vec{k})$, depending only on the connection $+A(\vec{x})$ and the second, $F_j(\gamma, \vec{k})$, depending only on the the holonomic loop $\gamma$. Let us substitute (30) into (29). The question then reduces to that of the existence (and of the value) of the Gaussian integral

$$\psi(\gamma) = \int \prod_j d\zeta_j \wedge d\zeta_j \exp \left( -\frac{\hbar}{4\pi} \int d^3\vec{k} \bar{\zeta}_j(\vec{k}) F_j(\gamma, \vec{k}) \right) \Psi(\zeta_j).$$

(31)

Let us begin by focussing on just one of the doubly infinite modes contained in $\zeta_j(\vec{k})$. If we denote this mode simply by $\zeta$, the analog of (31) is

$$\psi(F) = \int \frac{d\zeta \wedge d\bar{\zeta}}{4\pi i\hbar} \exp \left( -\frac{\hbar}{4\pi} \zeta \bar{\zeta} \right) e^{-F\bar{\zeta}} \Psi(\zeta).$$

(32)

where $\Psi(\zeta)$ is a Bargmann state for a single harmonic oscillator representing the mode $\zeta$. Let us therefore ask if the integral on the right side of (32) exists for all elements $\Psi(\zeta)$ of the Bargmann Hilbert space. Set

$$C_F(\zeta) = \exp F\zeta.$$ 

(33)

Then the integral in question is precisely the Bargmann inner product between the states $C_F(\zeta)$ and $\Psi(\zeta)$ :

$$\psi(F) = \langle C_F, \Psi \rangle.$$ 

(34)

Note that $C_F(\zeta)$ is precisely a coherent state in the Bargmann Hilbert space. (It is normalizable, with norm $\langle C_F, C_F \rangle = \exp F\bar{F}$.) Thus, not only does the inner product (34) exist for all Bargmann states $\Psi(\zeta)$, but the result, $\psi(F)$, can be interpreted as giving the components of $\Psi$ along the (overcomplete) coherent state basis. Furthermore, the explicit expression of $\psi(\zeta)$ is easy to evaluate. Let $\Psi(\zeta) = \sum a_n \zeta^n$. Expanding the coherent state $C_F(\zeta)$ in a power series and using the fact that, being eigenstates of the number operator, the functions $\sqrt{\frac{\lambda}{\pi}} \zeta^n$ are orthonormal, we find $\psi(F) = \sum a_n F^n$. Since the polynomials are dense in the Bargmann Hilbert space, it follows that the image $\psi(F)$ of the Bargmann state $\Psi(\zeta)$ under the transform (32) is simply

$$\psi(F) = \Psi(F).$$ 

(35)
The transform is astonishingly simple precisely because of the use of the coherent state basis, i.e. of the Bargmann representation.

Let us now return to the doubly infinite modes, $\zeta_j(\vec{k})$ of the Maxwell field. The analogs of the polynomials $\sum a_n\zeta^n$ are the cylindrical functionals $\Psi(\zeta_j) = \sum \int \frac{d^3\vec{k}}{\hbar|\vec{k}|} \cdots \frac{d^3\vec{k}}{\hbar|\vec{k}|} \ a_{j_1 \ldots j_n}(\vec{k}_1, \ldots, \vec{k}_n) \ \zeta_{j_1}(\vec{k}_1) \zeta_{j_2}(\vec{k}_2) \cdots$, where the coefficients $a_{j_1 \ldots j_n}(\vec{k}_1, \ldots, \vec{k}_n)$ are such that the integral $\int \frac{d^3\vec{k}}{\hbar|\vec{k}|} \cdots \frac{d^3\vec{k}}{\hbar|\vec{k}|} |a_{j_1 \ldots j_n}(\vec{k}_1, \ldots, \vec{k}_n)|^2$ converges. These cylindrical functionals are well-defined on the 1-photon Hilbert space $\mathcal{H}_1$ spanned by the $\zeta_j(\vec{k})$ for which $\int \frac{d^3\vec{k}}{\hbar|\vec{k}|} (|\zeta_1(\vec{k})|^2 + |\zeta_2(\vec{k})|^2)$ converges. This is precisely the Hilbert space on which the integral in the loop transform (29) is being defined. From the definition of Gaussian integrals on Hilbert spaces and the result discussed above for a harmonic oscillator, it now follows that the loop transform exists for all Bargmann photon states $\Psi(\zeta_j)$. Furthermore if we denote the transform by $T$, \[ T \circ \Psi(\zeta_j) = \psi(\gamma), \] where $\psi(\gamma)$ is given by (29), we have the simple result: \[ T \circ \Psi(\zeta_j) = \Psi(F_j(\gamma)), \] (37)

Thus, the transform does map Bargmann states to well defined functionals on the holonomic loop space $\mathcal{HL}$ which depend on holonomic loop $\gamma$ only through their form factors $F_j(\gamma, \vec{k})$.

Let us consider a few examples. Since the Fock vacuum $|0\rangle$ is represented by the functional $\Psi_0(\zeta_j) = 1$ in the Bargmann representation, it is represented by the unit functional, $\psi_0(\gamma) = 1$ also in the loop representation. A one photon state, $|\vec{k}_0, \epsilon = 1\rangle$, with momentum $\vec{k}_0$ and helicity $+1$, is represented by $\Psi_{k_0,+1}(\zeta_j) = \zeta_1(\vec{k}_0)$ in the Bargmann representation and hence by the functional

\[
\psi_{k_0,+1}(\gamma) = F_1(\gamma, \vec{k}_0) = \frac{\hbar}{(2\pi)^{3/2}} \int_\gamma ds \ \zeta_1(s)e^{-i\vec{k}_0 \cdot \vec{\gamma}(s)} \bar{m}_a(\vec{k}_0)
\]

\[ = \hbar \int_\gamma ds^a A_a^{\vec{k}_0,+1} \] (38)

in the loop representation, where $A_a^{\vec{k}_0,+1} = \frac{1}{(2\pi)^{3/2}} e^{i\vec{k}_0 \cdot \vec{\gamma}(s)} m_a(\vec{k}_0)$ is the plane wave connection with momentum $\hbar \vec{k}_0$ and helicity $+1$. A general 1-photon
state is given by the superposition $\int \frac{d^3k}{\hbar|k|} f_j(k) \left| k, j >\right.$ In the loop representation it is simply

$$\psi_f(\gamma) = \int \frac{d^3k}{\hbar|k|} f_j(k) F_j(\gamma, \vec{k})$$

$$= \oint_{\gamma} ds^a A^{(f)}_a,$$  \hspace{1cm} (39)

where

$$A^{(f)}_a = \frac{1}{(2\pi)^{d/2}} \int \frac{d^3k}{\hbar|k|} e^{i\vec{k} \cdot \vec{\gamma}(s)} (f_1(k)m_a(k) + f_2(k)m_a(k))$$  \hspace{1cm} (40)

is the connection on $\Sigma$ defined by the given 1-photon state. Thus, in the loop representation, 1-photon states arise simply as the line integral of the (negative frequency) connection along the loop! More precisely, we have the following: a normalizable, transverse, positive frequency connection $^+A_a(\vec{x})$ gives rise to a one photon state in the Fock space which, in the loop picture, is represented by the line-integral of the complex conjugate of the given connection.

The description of a $n$-photon state is straightforward. Recall that in the Bargmann representation such a state is given by an $n$-th order functional, $\Psi_n(\zeta_j)$, on the 1-photon Hilbert space $H_1$ (see equation (16) ). Hence, its image under the transform $\mathcal{T}$ is simply

$$\psi_n(\gamma) = \oint_{\gamma} ds^{a_1} ... \oint_{\gamma} ds^{a_n} A^{(f)}_{a_1,..,a_n},$$  \hspace{1cm} (41)

where we have used the standard (differential geometry) notation

$$\oint_{\gamma_1} ds^{a_1} ... \oint_{\gamma_n} ds^{a_n} \omega_{a_1...a_n} =$$

$$\oint_{\gamma_1} ds_1 \gamma^{a_1}(s_1) ... \oint_{\gamma_n} ds_n \gamma^{a_n}(s_n) \omega_{a_1...a_n}(\gamma_1(s_1), ..., \gamma_n(s_n)).$$  \hspace{1cm} (42)

In equation (41), $A^{(f)}_{a_1,..,a_n}$ is the totally symmetric field (with index $a_j$ in the tangent space of the point $\vec{x}_j$ ), divergence-free in each index, given by:

$$A^{(f)}_{a_1,..,a_n}(\vec{x}_1, ..., \vec{x}_n) =$$

17
\[
\frac{1}{(2\pi)^{3n/2}} \int \frac{d^3 k_1}{\hbar |k_1|} \cdots \int \frac{d^3 k_n}{\hbar |k_n|} e^{i \vec{k}_1 \cdot \vec{x}_1} p_{a_1}^{(j_1)}(\vec{k}_1) \cdots e^{i \vec{k}_n \cdot \vec{x}_n} p_{a_n}^{(j_n)}(\vec{k}_n) \]
\]
\[f_{j_1 \ldots j_n}(\vec{k}_1, \ldots, \vec{k}_n), \quad (43)\]

where the polarization vectors \(p_a^{(j)}(\vec{k})\) are given by \(p_a^{(1)}(\vec{k}) = m_a(\vec{k})\) and \(p_a^{(2)}(\vec{k}) = \bar{m}_a(\vec{k})\). Thus, an n-photon state \(\psi_n(\gamma)\) is given simply by a sum of products (41) of holonomies of (negative frequency) connections around the loop \(\gamma\).

Alternatively, since \(\psi_n(\gamma) = T \circ \Psi_n(\zeta_j) = \Psi_n(F_j(\gamma))\), equation (43) can be regarded as an n-nomial in the form factors \(F_j(\gamma, \vec{k})\) of the loop \(\gamma\). Although these two descriptions of loop states are clearly equivalent, their emphasis is different. The first description will be at forefront when we construct, in section 4, the loop representation directly, without any reference to the transform. The second, on the other hand, is useful in translating notions and formulae between the loop and the Bargmann pictures.

We conclude this sub-section by pointing out a subtle difference between the transform (32) for the harmonic oscillator and the transform (29) for the Maxwell field. As pointed out above, the integral on the right side of (32) can be regarded simply as the scalar product \(\langle C_F, \Psi \rangle\) between the coherent state \(C_F(\zeta) := \exp \int \frac{d^3 \vec{k}}{\hbar |\vec{k}|} F_j(\gamma, \vec{k}) \zeta_j(\vec{k})\) and the given state \(\Psi(\zeta_j)\). Similarly, in the case of the Maxwell field, the integral in (29) can be considered as the scalar product between the “coherent state”
\[C_{F \gamma}(\zeta_j) := \exp \int \frac{d^3 \vec{k}}{\hbar |\vec{k}|} F_j(\gamma, \vec{k}) \zeta_j(\vec{k})\]
\[\] and the Bargmann state \(\Psi(\zeta_j)\). However, now, this interpretation is somewhat formal because the “coherent state” \(C_{F \gamma}(\zeta_j)\) is not normalizable, i.e., does not belong to the Bargmann Hilbert space. The failure, however, is mild. To see this, recall first that there is a 1-1 correspondence between (normalizable) 1-photon states and coherent states, the latter being simply the exponentials of the former. The states \(C_{F_j}(\zeta_j)\) can thus be regarded as the coherent states associated with the Bargmann 1-photon states \(\Psi_{\gamma}(\zeta_j) = \int \frac{d^3 \vec{k}}{\hbar |\vec{k}|} F_j(\gamma, \vec{k}) \zeta_j(\vec{k})\). In the physical space picture, this is the state defined by the positive frequency connection whose restriction to \(\Sigma\) is given by \(+ A_a(\vec{x}) = F_a(\gamma, \vec{x})\). Since \(F_a(\gamma, \vec{x})\) is distributional (see section 3.1), the 1-photon state \(\Psi_{\gamma}\) fails to be normalizable. However, its “inner
product" with every normalizable, 1-photon state $\tilde{\Psi}(\zeta_j)$ is finite, being the holonomy of the connection defined by $\tilde{\Psi}(\zeta_j)$ around the loop $\gamma$. Furthermore, $<\Psi_\gamma, \tilde{\Psi}> = 0$ for all $\gamma$ if and only if the normalizable 1-photon state $\tilde{\Psi}$ itself vanishes. In this sense $\Psi_\gamma$ are (over)complete in the 1-photon Hilbert space. Thus, the states $\Psi_\gamma(\zeta_j)$ are analogous to the plane-wave states $\Psi(\vec{x}) = \exp i \vec{k} \cdot \vec{x}$ which fail to be normalizable but have finite "inner product" with every square integrable function and form a complete basis. In this sense the non-normalizability of the states $\Psi_\gamma(\zeta_j)$ is "mild". Therefore, we can still regard $C_{F(\gamma)}$ defined in (44) as a coherent state which, however, is the exponential of the generalized 1-particle state $\Psi_\gamma(\zeta_j)$. (The states $C_{F(\gamma)}$ are over-complete in the Fock space.) In this sense, we can regard the the integral in the Maxwell loop transform (29) as a scalar product $<C_{F(\gamma)}, \Psi>$ between generalized coherent states $C_{F(\gamma)}$ and the given quantum state $\Psi$.

In the Dirac bra-ket notation, then, we can denote the coherent state $C_{F(\gamma)}$ simply as $|\gamma\rangle$ and the loop transform (29) simply by

$$\psi(\gamma) = <\gamma|\Psi>.$$ (45)

Finally, note that, both for the harmonic oscillator and the Maxwell field, there are technical differences between the use of an overcomplete coherent state basis and of an orthonormal basis. For the harmonic oscillator, for example, the basis $|x\rangle$ is orthonormal. It is an eigenbasis of the (Hermitian) position operator $\hat{X}$ which acts by multiplication on wave functions $\Psi(x) := <x|\Psi>$. The basis $|z\rangle$ (or, $|F\rangle$), on the other hand is overcomplete and therefore fails to be orthonormal. It is an eigenbasis of the (non-Hermitian) annihilation operator $\hat{\tilde{z}} \equiv d/dz$, but it is the operator $\hat{z}$ that acts by multiplication on wave functions $\Psi(z) := <z|\Psi>$.

### 3.3 Properties

In this subsection we first show that the loop transform $T$ is faithful and then use this property to pull-back physically interesting operators from the Bargmann to the loop representation.

We saw in the last subsection that the action of $T$ is rather simple: $\psi(\gamma) \equiv T \circ \Psi(\zeta_j)$ is given by $\psi(\gamma) = \Psi(F_j(\gamma))$. Since each holonomic loop is completely characterized by its form factor, the map $T$ from the Bargmann states to functionals on the loop space is clearly well-defined and linear. However, it is not obvious that it is 1-1. In the case of the harmonic oscillator,
the transform is clearly 1-1 since it maps the holomorphic function \( \Psi(\zeta) \) of a complex variable \( z \) to the holomorphic function \( \Psi(F) \) of a complex variable \( F \). In the case of the Maxwell field, on the other hand, the argument of the Bargmann states is two complex functions \( \zeta_j(\vec{k}) \) of three real variables while the argument of the loop states is (equivalence classes at) only three real functions \( \gamma^a(s) \) of a single real variable. Put differently, while \( \zeta_j(\vec{k}) \) form a complex vector space, the form factors \( F_j(\gamma, \vec{k}) \) which characterize holonomic loops constitute only a “sparse subset” of this vector space.\(^4\) Therefore, there does exist a large class of functionals of \( \zeta_j(\vec{k}) \) whose restriction to the subset of form factors vanishes identically. We must show that none of the Bargmann states belong to this class. Roughly speaking, the image \( \psi(\gamma) = \Psi(F_j(\gamma)) \) of a Bargmann state \( \Psi(\zeta_j) \) “samples” the values of \( \Psi \) only on a sparse subset and we have to show that the \( \Psi \) have a specific functional form that enables one to reconstruct them uniquely from their values at these “sample points”. We will proceed in two steps. First we show that there is no \( n \)-photon state \( \Psi_n(\zeta_j) \) for which the image \( \psi_n(\gamma) \equiv T \circ \Psi_n(\zeta_j) \) vanishes on the loop space. In the second step we will show that the same result holds for a general Bargmann state.

Let us begin with a general 1-photon state, \( \Psi_1(\zeta_j) = \int \frac{d^3k}{\hbar|k|} f_j(\vec{k})\zeta_j(\vec{k}) \). We saw in the last subsection that its image on the loop space is given by \( \psi_1(\gamma) = \int f_j(\vec{k}) ds^a \) (see Eqs. (39,40)). Now, if \( \psi_1 = 0 \) for all \( \gamma \), the holonomy of \( A^a_a(\vec{k}) \) around any closed loop vanishes, whence \( A^a_a(\vec{k}) \) must be an exact 1-form. However, from (40) it is clear that \( A^a_a(\vec{k}) \) is also divergence free, whence it must vanish identically. This implies that \( f_j(\vec{k}) \) must vanish, whence, \( \Psi_1(\zeta_j) \) which we began must be zero. Thus, the restriction of the transform to a 1-photon state is indeed faithful. Let us next consider a general 2-photon state, \( \Psi_2(\zeta_j) = \int \frac{d^3k}{\hbar|k|} \int \frac{d^3k'}{\hbar|k'|} f_{jj'}(\vec{k}, \vec{k}')\zeta_j(\vec{k})\zeta_{j'}(\vec{k}') \). Its transform, \( \psi_2(\gamma) \) is given by \( \psi_2(\gamma) = \int d^3x \int d^3x' f_j d^3x d^3x' A_{aa}(\vec{x}, \vec{x}') \), where \( A_{aa}(\vec{x}, \vec{x}') \) is a “2-point field”, satisfying \( A_{aa}(\vec{x}, \vec{x}') = A_{aa}(\vec{x}', \vec{x}) \), given by the Fourier transform of \( f_{jj'}(\vec{k}, \vec{k}') \). Let us suppose that \( \psi_2(\gamma) = 0 \) for all holonomic loops \( \gamma \). Then, by choosing for

\(^4\)More precisely, the situation is the following. The form factors \( F_j(\gamma, \vec{k}) \) belong to the distributional dual of the smooth \( \zeta_j(\vec{k}) \) of compact support. The Bargmann Hilbert space is, strictly speaking, the space of weave functionals on this dual because the Gaussian measure is concentrated on distributions. The set of form factors is sparse in this dual.
\( \gamma \) loops \( \alpha, \beta \) and \( \alpha \# \beta \), one can readily show that

\[
\oint_\alpha ds^a \oint_\beta ds^{a'} \bar{A}^{(j_2)}_{aa'} = 0
\]  

(46)

for all loops \( \alpha, \beta \). Now one can repeat the argument used above for 1-photon states to conclude that \( f_{jj'}(\vec{k}, \vec{k}') \) and hence \( \Psi_2(\zeta_j) \) must vanish identically. Thus, the transform is faithful also on 2-photon states. It is clear that the argument can be extended, step by step, to \( n \)-photon states.

It therefore remains to show that a linear combination, \( \sum_n \psi_n(\gamma) \equiv T \circ \sum_n \Psi_n(\zeta_j) \), can vanish only if \( \sum_n \Psi_n(\zeta_j) \) itself vanishes. This is the second step in the argument. For simplicity, let us restrict ourselves to helicity +1 photons. The state \( \Psi(\zeta_j) \) is then of the form

\[
\Psi(\zeta_j) = \sum_{m=0}^n \int \frac{d^3\vec{k}_1}{h|\vec{k}_1|} ... \int \frac{d^3\vec{k}_m}{h|\vec{k}_m|} f_n(\vec{k}_1, ..., \vec{k}_m) \zeta_1(\vec{k}_1)...\zeta_1(\vec{k}_m)
\]  

(47)

whence its transform is of the form

\[
\psi(\gamma) = \oint_\gamma ds_1 \hat{\gamma}^{a_2}(s_1) ... \oint_\gamma ds_n \hat{\gamma}^{a_n}(s_n) \bar{A}^{(f_n)}_{a_1...a_n}
\]  

(48)

Let us suppose that \( \psi = 0 \) for all holonomic loop \( \gamma \). Choosing for \( \gamma \), loops \( \alpha, \alpha \# \alpha, \alpha \# \alpha \# \alpha, ... \), one obtains a series of constraints:

\[
f_0 + \oint_\alpha ds^a \bar{A}_a + \oint_\alpha ds^{a_1} \oint_\alpha ds^{a_2} \bar{A}_{a_1a_2} + ... + \oint_\alpha ds^{a_1} ... \oint_\alpha ds^{a_n} \bar{A}_{a_1...a_n} = 0,
\]

\[
f_0 + 2 \oint_\alpha ds^a \bar{A}_a + 2^2 \oint_\alpha ds^{a_1} \oint_\alpha ds^{a_2} \bar{A}_{a_1a_2} + ... + 2^n \oint_\alpha ds^{a_1} ... \oint_\alpha ds^{a_n} \bar{A}_{a_1...a_n} = 0,
\]

\[
f_0 + 3 \oint_\alpha ds^a \bar{A}_a + 3^2 \oint_\alpha ds^{a_1} \oint_\alpha ds^{a_2} \bar{A}_{a_1a_2} + ... + 3^n \oint_\alpha ds^{a_1} ... \oint_\alpha ds^{a_n} \bar{A}_{a_1...a_n} = 0,
\]

(49)

etc. These infinite set of conditions can be satisfied if and only if each term in the series vanishes, i.e.,

\[
\oint_\alpha ds^{a_1} ... \oint_\alpha ds^{a_n} \bar{A}_{a_1...a_n} = 0
\]  

(50)

for all loop \( \alpha \) and all \( m \in [0, n] \). Now, using the first step in the argument we conclude that \( \Psi_m(\zeta_j) \) must vanish for all \( m \), whence \( \Psi_m(\zeta_j) \) is itself zero. Thus, the transform is faithful on the dense subspace of the Bargmann...
Hilbert space spanned by all polynomials $\Psi(\zeta_j)$. This is what we set out to prove. (Since the domain space is dense, the transform can be extended to the full Bargmann Hilbert space, preserving its faithful character. However, in general the limit points are not expressible as functionals either in the Bargmann or the loop pictures.)

It is somewhat surprising that although the set of the form factors $F_j(\gamma, \vec{k})$ is sparse in the vector space of $\zeta_j(\vec{k})$, the restriction $\Psi(F_j(\gamma, \vec{k})) \equiv \psi(\gamma)$ of $\Psi(\zeta_j)$ determines $\Psi(\zeta_j)$ completely. How does this come about? Two key features of loops, connections and Bargmann representation underlie this result. First, holonomic loops serve as a “separating set” for connections: if the holonomy of the two connections around every loop is the same, the connections can differ at most by a gauge transformation. This feature is responsible for making the transform faithful on 1-photon states. The second feature—which makes the transform faithful on the full Fock space—is an integral part of the Bargmann representation: a general Bargmann state is a polynomial on the space spanned by $\zeta_j(\vec{k})$. It is because of this property of $\Psi(\zeta_j)$ that none of them is annihilated by the transform map $T$. Perhaps a rough analogy would make this point clearer. Consider the real line $R$. Although integers form a sparse set in $R$, the value of any polynomial $f(x) = \sum_{n=1}^{\infty} a_n x^n$ on the set of the integers determines $f(\gamma)$ completely: $f(m) = 0$ for all integers $m$ if and only if $a_n = 0$. The 1-photon Hilbert space, spanned by $\zeta_j(\vec{k})$ is analogous to the real line in this example, and the polynomials, to Bargmann states.

Since the map $T$ has no kernel, we can now use it to pull-back Bargmann operators to operators on the loop space. The images of the creation and annihilation operators are:

$$T \circ \hat{\zeta}_j(\vec{k}) \circ T^{-1} \psi(\gamma) = F_j(\gamma, \vec{k})\Psi(F_j(\gamma, \vec{k})) = F_j(\gamma, \vec{k})\psi(\gamma)$$  \hspace{1cm} (51)

and

$$T \circ \hat{\bar{\zeta}}_j(\vec{k}) \circ T^{-1} \psi(\gamma) = \frac{\hbar|\vec{k}|}{\delta F_j(\gamma, \vec{k})} \circ \Psi(F_j(\gamma, \vec{k})),$$

where the functional derivative is well-defined because $\Psi(F_j(\gamma, \vec{k}))$ is a polynomial in its argument. (In the next section, we will be able to rewrite this term as a “derivative” operating directly on loop functionals $\psi(\gamma)$.) It is now straightforward to re-express these operators, such as the Hamiltonian, the momentum and angular momentum operators on loop states.
There are two families of operators that are of particular interest to the loop representation. The first is the holonomy \( \hat{h}[\gamma] = \exp \oint_{\gamma} -\hat{A}_a ds^a \) of the negative frequency connection around the loop \( \gamma \). From (5), it follows that \( -\hat{A}_a(\vec{x}) \) can be expressed as a sum of annihilation operators \( \hat{\zeta}_j(\vec{k}) \):

\[
-\hat{A}_a(\vec{x}) = -\frac{1}{(2\pi)^{3/2}} \int \frac{d^3 k}{\bar{h}|\vec{k}|} e^{i \vec{k} \cdot \vec{x}} \left[ \hat{\zeta}_1(\vec{k}) m_a(\vec{k}) + \hat{\zeta}_2(\vec{k}) \bar{m}_a(\vec{k}) \right],
\]

whence \( \hat{h}[\gamma] \) can be expressed as

\[
\hat{h}[\gamma] = \exp \int \frac{d^3 k}{\bar{h}|\vec{k}|} F_j(\gamma, \vec{k}) \hat{\zeta}_j(\vec{k}).
\]

A straightforward calculation now yields the corresponding operator in the loop representation. We have:

\[
[T \circ \hat{h}[\alpha] \circ T^{-1} \psi](\beta) = \psi(\alpha \# \beta).
\]

The action is thus surprisingly simple; it involves only the operation \( \# \), which can be performed directly on the loop space, without, in the end, any reference to the transform or even the form factors of the loops. It is somewhat surprising that the operator is well defined because we have smeared the operator valued distribution \( -\hat{A}_a(\vec{x}) \) only along a 1-dimensional loop; we have effectively used a distribution –rather than a test field in \( C_0^\infty(\Sigma) \)– to smear \( -\hat{A}_a(\vec{x}) \). Indeed, had we used the real (or the positive frequency) connection in the definition of \( \hat{h}[\alpha] \), the resulting operator would not have been well-defined, it would have mapped any n-photon state to a “state with infinite norm”. This somewhat unexpected “tame” behavior of the operator in the Abelian theory may be related to the surprisingly simple renormalization properties of the Wilson-loop of the bare (non-renormalized) field operators in Yang-Mills theories discussed, e.g., in [15].

Since the holonomy operators \( \hat{h}[\alpha] \) will play an important role in the direct construction of the loop representation in the next section, let us examine their eigenvalues and eigenvectors. Since they are, in essence, exponential of annihilation operators, one would expect their eigenvectors to be coherent states. This is indeed the case. Fix a transverse connection \( C_a(\vec{x}) \) for which

\[
\int \frac{d^3 k}{\bar{h}|\vec{k}|} |C_a(\vec{k})|^2 < \infty
\]

and consider the loop state

\[
\psi_C(\gamma) = \exp \oint_{\gamma} C_a ds^a.
\]
Then we have:

\[
[T \circ \hat{h}[\alpha] \circ T^{-1}\psi_C] (\beta) = \left( \exp \oint_{\alpha} C_a ds^a \right) \psi_C(\beta).
\] (57)

Thus, \( \psi_C(\beta) \) is a simultaneous eigenstate of every holonomy operator. Therefore it is the coherent state peaked at the connection \( C_a(\vec{x}) \). Note, incidently, that the states \( |\gamma\rangle \) –the “basis states” of the loop representation– introduced at the end of Sec 3.2 are specific examples of these coherent states: They are the coherent states corresponding to (real, transverse) connections which are concentrated along the loops.

The second interesting family of operators is constructed from the positive frequency electric field \( +\hat{E}^a(\vec{x}) \). Set

\[
\hat{E}[f] \equiv \int +\hat{E}^a(\vec{x}) \bar{f}_a(\vec{x}),
\] (58)

where \( f_a \) is an equivalence class of \( C^\infty \) 1-form of compact support, modulo exact 1-forms. From (7) it follows that

\[
\hat{E}[f] = \frac{-i}{(2\pi)^{3/2}} \int \frac{d^3\vec{k}}{|\vec{k}|} \vec{f}_j(\vec{k})\hat{\zeta}_j(\vec{k}),
\] (59)

where we have set \( f_a(\vec{x}) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3\vec{k}}{|\vec{k}|} \left( f_1(\vec{k})m_{a}(\vec{k}) + f_2(\vec{k})\bar{m}_{a}(\vec{k}) + f_L(\vec{k})k_a \right) \). Using the definition (29) of the transform, it now follows that the corresponding operator in the loop representation is simply

\[
[T \circ \hat{h}[\alpha] \circ T^{-1}\psi](\gamma) = i\hbar \left( \oint_{\gamma} f_a ds^a \right) \psi(\gamma).
\] (60)

Again, the action of the operator on loop states is rather simple, one does not need form factors to express it. (That the operator is well-defined is not surprising because \( +\hat{E}^a(\vec{x}) \) has been smeared by a \( C^\infty \) test field of compact support). By inspection, the loop states

\[
\psi_{\gamma_0}(\gamma) = \begin{cases} 1 & \text{if } \gamma = \gamma_0 \\ 0 & \text{otherwise} \end{cases},
\] (61)

which are characteristic functions of a given loop \( \gamma_0 \) are simultaneous (generalized) eigenvectors of \( \hat{E}[f] \) for all \( f_a \):

\[
\hat{E}[f] \circ \psi_{\gamma_0}(\gamma) = i\hbar \left( \oint_{\gamma_0} f_a ds^a \right) \psi_{\gamma_0}(\gamma).
\] (62)
These states do not belong to the Fock space since they are not normalizable. This is not surprising because $\hat{E}[f]$ is essentially a superposition of creation operators. Note that these states, $\psi_{\gamma_0}(\gamma)$, are “complementary” to the coherent states $\psi_C(\gamma)$ —and hence, in particular, to the states $|\gamma\rangle$ introduced in section 3.2— which are eigenvectors of $\hat{h}[\alpha]$, i.e. of superpositions of annihilation operators. Indeed, while $\psi_{\gamma_0}(\gamma)$ have support on a single loop $\gamma_0$, the states $\psi_C(\gamma)$ are spread out in the loop space. Similarly, while $\psi_C$ are concentrated at a single connection $C_a$, the states $\psi_{\gamma_0}$ are completely spread out in the space of connections. This suggest that the two sets of operators, $\hat{h}[\alpha]$ and $\hat{E}[f]$, should be conjugate to one another in a suitable sense. This is indeed the case: they obey the commutation relations:

$$[\hat{h}[\alpha], \hat{E}[f]] = \left( i\hbar \oint_\gamma f_a ds^a \right) \hat{h}[\alpha].$$

(63)

The fact that the commutator is closed and has such a simple expression suggests that the algebra of these operators may serve as a useful point of departure for the construction of the loop representation. We will see in the next section that this expectation is indeed correct.

4 Loop representation

In this section we construct a loop representation of the Maxwell field directly, without any reference to the Bargmann representation of section 2 or the loop transform of section 3. It is this construction that appears to best suited for use in lattice QCD [8] and canonical quantum gravity [2]. The final picture we arrive at is the same as that obtained in section 3. Thus the representation we now construct from first principles is isomorphic with the Bargmann—and hence, also Fock—representation.

This section is also divided in to three parts. In the first we sketch the general quantization program [16, 17] that we will follow, a program that is also applicable to Yang-Mills’ and Einstein’s theories. In the second we construct the algebra of quantum operators, the loop algebra. In the third we construct a representation of this algebra on a suitably defined space of functionals of holonomic loops and and discuss some features of the resulting loop representation.
4.1 Quantization program

Consider a classical system with phase space $\Gamma$. To quantize the system we proceed in the following steps:\footnote{For a more complete discussion of the program including a treatment of constrained systems, see Refs. 15 and 16.}

1. Choose a subspace $\mathcal{S}$ of the space of complex valued functions on $\Gamma$ which is closed under the Poisson bracket operation and large enough so that any well-behaved function on $\Gamma$ can be expressed as (possibly the limit of) a sum of products of elements of $\mathcal{S}$. Elements of $\mathcal{S}$ are referred to as \textit{elementary classical variables} and are to have unambiguous quantum analogs. In the Bargmann quantization, for example, $\mathcal{S}$ is the vector space spanned by functionals $\int \frac{d^3\vec{k}}{|\vec{k}|} \hat{f}_j(\vec{k})\hat{\zeta}_j(\vec{k})$, $\int \frac{d^3\vec{k}}{|\vec{k}|} \hat{g}_j(\vec{k})\bar{\hat{\zeta}}_j(\vec{k})$, which are linear in $\hat{\zeta}_j(\vec{k})$ and $\bar{\hat{\zeta}}_j(\vec{k})$ respectively, and by constants.

2. Associate with each $f$ in $\mathcal{S}$ an \textit{elementary quantum operator} $\hat{f}$ and consider the free associative algebra generated by these abstract operators. Impose on this algebra the (generalized) canonical commutation relations:

$$[\hat{f}, \hat{g}] = i\hbar \{\hat{f}, \hat{g}\}$$

for all $f$ and $g$ in $\mathcal{S}$. In addition, if the set $\mathcal{S}$ is overcomplete, impose on the algebra also “anti-commutation relations”, namely the relations that capture the algebraic relations that exist between elements of $\mathcal{S}$. For instance if $f, g$ and $h = fg$ are all in $\mathcal{S}$, then

$$\hat{f} \cdot \hat{g} + \hat{g} \cdot \hat{f} = 2 \hat{h}.$$ 

Denote the resulting associative algebra by $\mathcal{A}$. (Note that at this stage $\mathcal{A}$ is an abstract algebra: there is, as yet, no Hilbert space for it to act upon.)

In the example of the Bargmann representation, the canonical commutation relation reduce to:

$$\left[ \int \frac{d^3\vec{k}}{|\vec{k}|} \hat{f}_j(\vec{k})\hat{\zeta}_j(\vec{k}), \int \frac{d^3\vec{k}}{|\vec{k}|} \hat{g}_j(\vec{k})\bar{\hat{\zeta}}_j(\vec{k}) \right] = \hbar \int \frac{d^3\vec{k}}{|\vec{k}|} \hat{f}_j(\vec{k})\hat{g}_j(\vec{k}).$$

There are no anti-commutation relations because the set $\mathcal{S}$ is complete but not overcomplete: there are no algebraic relations among the elements of $\mathcal{S}$. 
The loop variables we want to introduce in this section, on the other hand, are overcomplete and anti-commutation relations will be important.

3. Introduce an involution, $\ast$, on $\mathcal{A}$ by setting

\[(\hat{f})^\ast = \hat{\bar{f}}\]  

(67)

for all elementary variables $f$ (the bar denotes complex conjugation as before) and requiring that $\ast$ satisfies the defining properties of an involution: $(\hat{A} + \lambda \hat{B})^\ast = \hat{A} + \lambda \hat{B}^\ast$; $(\hat{A} \hat{B})^\ast = \hat{B}^\ast \hat{A}^\ast$ and $(\hat{A}^\ast)^\ast = \hat{A}$, for all $\hat{A}, \hat{B}$ in $\mathcal{A}$ and complex numbers $\lambda$. Denote the resulting $\ast$–algebra by $\mathcal{A}^{(\ast)}$.

4. Choose a linear representation of $\mathcal{A}$ on a complex vector space $V$. (The $\ast$–relations are ignored at this step.) In the Bargmann case, $V$ is the space of polynomials $\Psi(\zeta_j(\vec{k}))$, the operators $\int \frac{d^3k}{|k|} \hat{f}_j(\vec{k})\hat{\zeta}_j(\vec{k})$ are represented by the multiplication operators and $\int \frac{d^3k}{|k|} \hat{\zeta}_j(\vec{k})\hat{\zeta}_j(\vec{k})$ by the functional derivative along $g_j(\vec{k})$.

5. Introduce on $V$ an inner product $\langle \, , \rangle$ so that the quantum reality conditions are satisfied:

\[\langle \Psi, \hat{A}\Phi \rangle = \langle \hat{A}^\ast \Psi, \Phi \rangle\]  

(68)

for all $\Psi$ and $\Phi$ in $V$ and $\hat{A}$ in $\mathcal{A}^{(\ast)}$. Thus, it is the $\ast$–relations that are to select the inner product. In section 2 we saw that the strategy does indeed pick out an unique inner product on the Bargmann states.

The program requires two external inputs: the choice of $\mathcal{S}$ in step 1 and the choice of of the carrier space $V$ of the representation in step 4. One may make “wrong” choices and find that the program cannot be completed. However, if the choices are viable, i.e., if the program can be completed at all, the resulting inner product is unique on each irreducible sector of the representation of $\mathcal{A}$ on $V$. In the framework of this program, the textbook treatments of field theories correspond to choosing for elements of $\mathcal{A}$ the smeared-out field operators, and, for $V$, the Fock space or, alternatively, suitable functionals of fields. In the loop quantization, on the other hand, one changes this strategy.[2] Both $\mathcal{S}$ and $V$ are now constructed from holonomic loops.
4.2 A loop algebra

In this sub-section, we will carry out the first three steps in the above quantization program.

The discussion at the end of section 3 suggests that we choose as our elementary classical variables the complex-valued functionals $h[\alpha]$ and $E[f]$, labelled by holonomic loops $\alpha$ and equivalence classes of 1-form $f_a$ (where $f_a \approx f_a + \partial_a g$ for any $g$) on a spacelike 3-plane $\Sigma$, defined via

$$h[\alpha] := \exp \oint_\alpha -A_a ds^a$$

and

$$E[f] := \int_\Sigma +E^a f_a d^3 \vec{x}.$$  

Here, $-A_a$ is the negative frequency part of the real, transverse connection $A^T_a$ and $+E^a$ is the positive frequency part of the real, transverse electric field $E^a_T$. Thus, $-A_a(\vec{x}) = \frac{1}{\sqrt{2}} (A^T_a(\vec{x}) - i \Delta^{-1/2} E_T a(\vec{x}))$ and $+E^a(\vec{x}) = \frac{1}{\sqrt{2}} (E^T_a(\vec{x}) - i \Delta^{1/2} A^T a(\vec{x}))$ (Eqs. (5,7)).

The knowledge of the holonomies $h[\alpha]$ for all $\alpha$ enables one to reconstruct $-A_a(\vec{x})$ completely, while the knowledge of $E[f]$ for all $f_a$ suffices to determine $+E^a(\vec{x})$. Hence, $h[\alpha]$ and $E[f]$, provide us with an (over-) complete coordinatization of the phase space. The space $S$ of elementary classical variables required in the first step of the quantization program shall be the vector space generated by the $h[\alpha]$ and $E[f]$, subject to the obvious linear relations

$$E[f_1] + E[f_2] = E[f_1 + f_2],$$

for all $f_1$ and $f_2$. It is closed under the Poisson-bracket operation because:

$$\{h[\alpha], E[f]\} = \left( \oint_\alpha f_a ds^a \right) h[\alpha].$$

For future use, we note that the negative frequency magnetic fields $-B^a(\vec{x}) \equiv \epsilon^{abc} \partial_b -A_a(\vec{x})$ can be recovered from the holonomies $h[\alpha]$ by the obvious limiting procedure.

The next step in the quantization program is the construction of the algebra $\mathcal{A}$ of quantum operators. Let us associate with each $h[\alpha]$ in $S$ an operator $\hat{h}[\alpha]$ and with each $E[f]$ an operator $\hat{E}[f]$ and consider the associative
algebra generated by finite sums of products of these elementary quantum operators. As mentioned in section 4.1, this is an abstract algebra; there is as yet no vector space for it to act upon. On this algebra, impose the canonical commutation relations:

\[ [\hat{h}[\alpha], \hat{h}[\beta]] = 0, \quad [\hat{E}[f], \hat{E}(g)] = 0, \quad (73) \]

\[ [\hat{h}[\alpha], \hat{E}(g)] = i\hbar \left( \int_{\alpha} f_a ds^a \right) \hat{h}[\alpha]. \quad (74) \]

Further, we must incorporate in this quantum algebra the fact that \( h[\alpha] \) is over-complete, i.e. that there are algebraic relations among them: \( h[\alpha] h[\beta] = h[\alpha \# \beta] \). This is achieved by imposing on the algebra the relations

\[ \hat{h}[\alpha] \hat{h}[\beta] = \hat{h}[\alpha \# \beta] \quad (75) \]

for all holonomic loops \( \alpha \) and \( \beta \). (We do not need anti-commutators because \( \hat{h}[\alpha] \) commute.) The result is the algebra \( \mathcal{A} \) of quantum operators.\(^6\)

In analogy with the terminology used in gravity \([2, 11]\), we will call \( \mathcal{A} \) a loop algebra. Finally, let us impose the \( \ast \)-relations. Since the transverse fields \( A^T_a(\vec{x}) \) and \( E^T_a(\vec{x}) \) in the reduced phase space are all real, it follows immediately that the complex conjugate of \( +E^a(\vec{x}) \) is determined by the negative frequency magnetic field \( -B^a(\vec{x}) \) via

\[ +E^a(\vec{x}) = i \Delta^{-1/2} \text{curl} \ -B^a(\vec{x}). \quad (76) \]

Hence, in the quantum theory, the \( \ast \)-relations are given by

\[ \left( \hat{E}[f] \right)^\ast = i \hat{B}[\text{curl} \ \Delta^{-1/2} f], \quad (77) \]

where \( -\hat{B}^a(f) \) is to be obtained by taking the obvious limit of the holonomy operators \( \hat{h}[\alpha] \) and where, on the right side, we have used the unique transverse 1-form in the equivalence class to which \( f_a \) belongs. In the next section, we introduce a linear representation of \( \mathcal{A} \), find an inner product such that the \( \ast \)-relations (77) are realized by the concrete operators representing electric fields \( \hat{E}[f] \) and holonomies \( \hat{h}[\alpha] \).

\(^6\)More precisely, we consider the two sided ideals generated by equations (73,74,75) and take the quotient of the associative algebra by it.
4.3 Loop states

We can now represent the quantum algebra $\mathcal{A}$ by operators acting on suitably regular functionals $\psi(\gamma)$ on the holonomic loop space $\mathcal{HL}$. The discussion at the end of section 3.3 will motivate the various steps in the construction.

The idea is to construct the loop states simply from sums of products of $\psi(\gamma)$ of the type $\psi(\gamma) = \oint_{\gamma} f_a ds^a$. For a precise implementation of this idea, we proceed as follows. Denote by $V$ the vector space of equivalence classes of complex valued, smooth 1-forms $f_a(\vec{x})$ of compact support, where $f_a \approx f_a + \partial_a g$ for any smooth function $g$ of compact support. Denote by $V^n$ the totally symmetric, n-th rank tensor product of $V$ with itself, spanned by the fields $f_{a_1...a_n}(\vec{x}_1, ..., \vec{x}_n)$. (The symmetry property requires that $f_{a_1a_2}(\vec{x}_1, \vec{x}_2) = f_{a_2a_1}(\vec{x}_2, \vec{x}_1)$, and so on.) The carrier space $V$ underlying our representation of the loop algebra $\mathcal{A}$ is then spanned by the functionals $\psi(\gamma)$ on $\mathcal{HL}$ of the type

$$\psi(\gamma) = f_0 + \oint_{\gamma} ds^a f_a + \oint_{\gamma} ds^{a_1} \oint_{\gamma} ds^{a_2} f_{a_1a_2} + ... + \oint_{\gamma} ds^{a_1} ... \oint_{\gamma} ds^{a_n} f_{a_1...a_n}. \tag{78}$$

Here $n$ is an arbitrary (but finite) integer, $f_0$ is a complex number, $f_{a_1...a_n}$ belongs to $V^n$, and we have used the same notation as in (42) for integrals. Note that, although one may use parametrization of the loop $\gamma$ to write out the functional form of $\psi(\gamma)$ explicitly, the resulting functional is independent of the parametrization; $\psi(\gamma)$ depends only on the holonomic loop $\gamma$. Similarly, the value of $\psi(\gamma)$ remains unaltered if any of the $f_{a_1...a_n}$ is replaced by any of its gauge equivalent fields, e.g. if $f_{a_1,a_2}(\vec{x}_1, \vec{x}_2)$ is replaced by $\tilde{f}_{a_1,a_2}(\vec{x}_1, \vec{x}_2) + \tilde{f}_{a_1}(\vec{x}_1)\partial_{a_2} g(\vec{x}_2) + \tilde{f}_{a_2}(\vec{x}_2)\partial_{a_1} g(\vec{x}_1)$. Finally recall that any holonomic loop is completely determined by its form factors $F_j(\gamma, \vec{k})$, $j = 1, 2$.

We can therefore express $\psi(\gamma)$ of equation (78) as a functional of these form factors. It is simply a polynomial in $F_j(\gamma, \vec{k})$:
\[ \sum_{m=1}^{n} \int \frac{d^3 \vec{k}_1}{\hbar|k_1|} \cdots \int \frac{d^3 \vec{k}_m}{\hbar|k_m|} f_{j_1 \cdots j_m}(\vec{k}_1, \ldots, \vec{k}_m) F_{j_1}(\gamma, \vec{k}_1) \cdots F_{j_m}(\gamma, \vec{k}_m), \]  

(79)

where \( f_{j_1 \cdots j_m}(\vec{k}_1, \ldots, \vec{k}_m) \) are the Fourier coefficients of the transverse (in all indices) parts of \( f_{a_1 \cdots a_1}(\vec{x}_1, \ldots, \vec{x}_n) \). This provides an equivalent characterization of the carrier space \( V \).

Next we must represent \( \mathcal{A} \) by concrete operators on \( V \). The results of section 3 suggest the following two definitions:

\[
\left( \hat{h}[\alpha] \circ \psi \right)(\beta) := \psi(\alpha \# \beta) \quad (80)
\]

\[
\left( \hat{E}[f] \circ \psi \right)(\beta) := i\hbar \left( \oint_{\vec{k}_\beta} f_a ds^a \right) \psi(\beta) \quad (81)
\]

(for simplicity of notation we will not distinguish between elements of \( \mathcal{A} \) and operators on \( V \) representing them). It is straightforward to verify that (80) and (81) are well-defined linear mappings on \( V \) and that they provide a representation of the generalized CCR’s (74) and algebraic relations (75). Since elements of \( \mathcal{A} \) are sums of products of \( \hat{h}[\alpha] \) and \( \hat{E}[f] \) subject to these relations, we have constructed a representation of the loop algebra \( \mathcal{A} \) by linear operators acting on suitable functionals on the holonomic loop space. This completes the fourth step of the quantization program.

In the fifth and the last step we must use the \( \ast \)-relations (77) to select an inner product on \( V \); since the \( \ast \)-relations are expressed in terms of the magnetic field operator \( -\hat{B}^a(\vec{x}) \), we must first construct this operator on \( V \). Let us begin with classical holonomies. Given a point \( \vec{x} \) and a unit vector \( t^a \) at \( \vec{x} \), the negative frequency magnetic field \( -B^a(\vec{x}) \) can be recovered from holonomies via:

\[
 t^a B^a(\vec{x}) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( h[\alpha(t, \epsilon, \vec{x})] - 1 \right) \quad (82)
\]

where \( \alpha(t, \epsilon, \vec{x}) \) is a loop of area \( \epsilon \) in the 2-plane perpendicular to \( t^a \), centered in \( \vec{x} \). The action of the operator valued distribution \( -\hat{B}^a(\vec{x}) \) on \( V \) is therefore defined by

\[
\left( t^a -\hat{B}^a(\vec{x}) \right) \psi(\gamma) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[ \psi(\gamma \# \alpha(\epsilon, t, \vec{x})) - \psi(\gamma) \right]. \quad (83)
\]

(Note that \( -\hat{B}^a(\vec{x}) \) is a derivative operator on loop space; it is not a functional derivative, nor it is the area derivative loop operator [6]. Rather, it is essentially the generator of the loop group introduced by Gambini and Trias [7].)
Using this expression, it is now straightforward to compute the commutator of $\hat{B}[g] := \int \hat{B}^a(\vec{x})g_a(\vec{x})d^3\vec{x}$ and $\hat{E}[f]$:

$$[\hat{B}[g], \hat{E}[f]] = i\hbar \left( \int (\text{curl}\,g^a)f_a d^3\vec{x} \right) \mathbb{I}. \quad (84)$$

We can now determine the inner-product. Consider the state $\psi_0$ defined by $\psi_0(\gamma) = 1$ and fix normalization by setting $\langle \psi_0, \psi_0 \rangle = 1$. This state is clearly cyclic in $V$. More precisely, since

$$\hat{h}[\alpha] \circ \psi_0(\gamma) = \psi_0(\gamma) \quad (85)$$

and

$$\hat{E}[f] \circ \psi_0(\gamma) = i\hbar \left( \int_\gamma f_a ds^a \right), \quad (86)$$

it is clear that any element in $V$ can be obtained by taking linear combinations of states obtained by acting on $\psi_0(\gamma)$ repeatedly by the electric field operator $\hat{E}[f]$. Thus, the representation mapping given by (80) and (81) naturally leads us to regard $\psi_0$ as the vacuum state, $\hat{h}[\alpha]$ as the exponential of the annihilation operators and $\hat{E}[f]$ as the creation operators. The discussion of section 3 makes it clear that this interpretation is correct. However, we did not have to refer to the loop transform to arrive at this picture; the loop representation introduced here leads us directly to these conclusions.

Let us use the quantum reality conditions to determine the inner product between first excited states, $\hat{E}[f]\psi_0$ and $\hat{E}[g]\psi_0$. Using the $\ast$-relations (77) and the commutation relations (84), we have

$$\langle \hat{E}[f]\psi_0, \hat{E}[g]\psi_0 \rangle = \langle \psi_0, i\hat{B}(\text{curl}\Delta^{-1/2}\bar{f})\hat{E}(g)\psi_0 \rangle = \langle \psi_0, [i\hat{B}(\text{curl}\Delta^{-1/2}\bar{f}), \hat{E}(g)]\psi_0 \rangle = \hbar \int (\bar{f}^a \Delta^{1/2}g_a) d^3\vec{x}, \quad (87)$$

where, we have used the fact that $\hat{B}[f]$ annihilates $\psi_0$ (see Eq.(85)) and, as before, carried out the calculation in the transverse gauge. Let us define $f_a(\vec{k})$ via

$$f_a(\vec{x}) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3\vec{k}}{\hbar |\vec{k}|} f_a(\vec{k}) e^{i\vec{k} \cdot \vec{x}}. \quad (88)$$
Then, we can write the inner-products between the first excited states simply as:

\[
< \psi_f(\gamma), \psi_g(\gamma) > \equiv \oint_{\gamma} f_a ds^a, \oint_{\gamma} g_a ds^a > = \hbar \int \frac{d^3 k}{|k|} \bar{f}_a(k) g^a(k). \tag{89}
\]

It is straightforward to repeat this calculation for higher excited states. The norm of the general element \( \psi(\gamma) \) defined in (78) is given simply by

\[
||\psi(\gamma)||^2 = |f_0|^2 + \hbar \int \frac{d^3 k}{|k|} |f_a(k)|^2
\]

\[
+ \hbar^2 \int \frac{d^3 k_1}{|k_1|} \int \frac{d^3 k_2}{|k_2|} |f_{a_1 a_2}(k_1, k_2)|^2 + ...
\]

\[
+ \hbar^n \int \frac{d^3 k_1}{|k_1|} ... \int \frac{d^3 k_n}{|k_n|} |f_{a_1...a_n}(k_1, ..., k_n)|^2 \tag{90}
\]

where \( f_{a_1...a_m}(\vec{k}_1,...,\vec{k}_m) \) is the Fourier transform of the transverse part of \( f_{a_1...a_m}(\vec{x}_1,...,\vec{x}_m) \) defined via (88). Thus the quantum reality conditions do indeed suffice to determine the inner-product on \( V \). The Hilbert space of states is the Cauchy completion \( \bar{V} \) of this \( V \). It is clear from section 3.3 that \( \bar{V} \) is isomorphic with the standard Fock space of photons. Again, we were able to single out the correct scalar product without any direct reference to Poincaré group.

From the viewpoint of rigor, however, this construction is unsatisfactory in one respect: to write out the *-relations on the algebra \( \mathcal{A} \), we had to refer to the magnetic field which belongs not to the algebra \( \mathcal{A} \) but to a suitable (unspecified) completion thereof. However, this incompleteness refers to the constructive procedure used in this sub-section rather than to the arguments that establish the existence of the loop representation and its equivalence to the Bargmann (or, Fock) picture. More precisely, we can just work with the algebra \( \mathcal{A} \) introduced in section 4.2, refrain from introducing the *-relations, and simply exhibit a representation of \( \mathcal{A} \): states are given by (78), the inner-product by (90) and the generators of \( \mathcal{A} \) are represented via (80) and (81).
This gives the complete quantum description. In particular, the $*$-relations could now be deduced from the inner-product. However, now the inner product itself has to be simply postulated. In this sense, the new procedure would fail to be constructive. In the spirit of the general quantization program [16], one would like to “derive” the correct inner product by using the quantum reality conditions and to implement these conditions one must introduce on $\mathcal{A}$ a $*$-relation. Therefore, to complete the discussion presented in this section, we must either work with an appropriate completion of the algebra $\mathcal{A}$ which includes the magnetic field operator, or, alternatively, replace the algebra by another one which is closed under the $*$-relations without the need of any completion. The second of these strategies has been completed successfully. (The new algebra is based on “thickened-out loops”. It will, however, be presented elsewhere since the discussion involved is somewhat long.

To conclude this section, let us exhibit the Hamiltonian operator directly in the loop representation. The classical Hamiltonian can be expressed in terms of $+E(\vec{x})$ and its complex conjugate as:

$$H = \int_{\Sigma} d^3\vec{x} \ +E^a(\vec{x}) \ +E_a(\vec{x}).$$

(91)

Since $\hat{E}$ is a creation operator, the normal ordered quantum Hamiltonian can be written as

$$\hat{H} = \int_{\Sigma} d^3\vec{x} \ \hat{E}^a(\vec{x}) \ (\hat{E}_a(\vec{x}))^*.$$

(92)

Using the reality condition (77) we can substitute for $(\hat{E}_a)^*$ in terms of the magnetic field operator $\hat{B}_a$. The momentum space expression of the Hamiltonian operator is then given by

$$\hat{H} = \int \frac{d^3\vec{k}}{|\vec{k}|} \epsilon_{abc} \hat{E}^a(-\vec{k}) \hat{B}^b(\vec{k}) k^c.$$

(93)

To find its eigenvectors and eigenvalues, we first note the action of the magnetic field operator $\hat{B}^a(\vec{k})$ on 1-photon states, $\psi(\gamma) = F^b(\gamma, -\vec{k}')$, with momentum $-\vec{k}'$:

$$\hat{B}^a(\vec{k}) \cdot F^b(\gamma, -\vec{k}') = i\epsilon^{abc} k_c \delta^3(\vec{k}, \vec{k}')$$

(94)

which follows directly from the definition (83) of $\hat{B}^a$. Using this result in the expression of the Hamiltonian, we have:

$$\hat{H} \cdot F^b(\gamma, -\vec{k}') = \hbar |\vec{k}'| F^b(\gamma, -\vec{k}')$$

(95)
Thus, as expected, these photon states are eigenstates of the Hamiltonian with eigenvalue $\hbar |\vec{k}|$. More generally, the $n$-photon states, $\psi(\gamma) = F_{j_1}(\gamma, k_1) \ldots F_{j_n}(\gamma, k_n)$, with momenta $\vec{k}_1, \ldots, \vec{k}_n$ and polarizations $j_1 \ldots j_n$ are eigenstates of $\hat{H}$ with eigenvalue $\hbar \sum_i |k_i|$.

Finally, it is interesting to note the curious form that the number operator $\hat{N}$ takes in the loop representation. From equation (49) it is easy to see that $\hat{N}$ is given by

$$2^N \psi(\alpha) = \psi(\alpha \# \alpha).$$

(96)

5 Discussion

In this paper, we first recast the Bargmann description of free photons in terms of loops using a functional transform and then showed that the resulting loop representation can be obtained directly in the framework of a general quantization program. It is this later strategy that was first used in non-perturbative gravity [2]. The fact that one can recover the Fock description of photons working entirely with holonomic loops and 1-forms provides new support to the methods used in that program. It is also clear from the discussion of sections 3 and 4 that connections and loops provide equivalent ways of handling gauge theories. The emphasis is, however, different: while the holonomy operators are diagonal in the connection representation, it is the electric field operators that act by multiplication in the loop representation.

It is important to note that that there is not one but many loop representations. To see this, consider first non-relativistic quantum mechanics. There, one may introduce on the Hilbert space of square-integrable functions $\Psi(\lambda)$, introduce on it the operators $\hat{A}$ and $\hat{B}$ via $\hat{A} \cdot \Psi(\lambda) = \lambda \Psi(\lambda)$ and $\hat{B} \cdot \Psi(\lambda) = -i \hbar d\Psi(\lambda)/d\lambda$, and verify that they satisfy the Heisenberg commutation relations. This information does not, however, suffice to determine the physical interpretation of the states $\Psi(\lambda)$ and the operators $\hat{A}, \hat{B}$. For example, if $\lambda$ is identified with $x$, $\hat{A}$ has the interpretation of the position operator and $\hat{B}$, of the momentum operator. In this case, the states $\Psi_\alpha(\lambda) = \exp i \alpha \lambda$ are the eigenstates of the momentum operator with eigenvalue $\hbar \alpha$. If, on the other hand, $\lambda$ is identified with $p$, it is $\hat{A}$ that represents the momentum operator and $-\hat{B}$ that represents the position. The states $\Psi_\alpha(\lambda)$ are now eigenstates of the position operator with eigenvalue $-\hbar \alpha$. The situation is
similar with the loop presentation. In the Abelian case considered in this paper, one can begin with functions \( \Psi(\gamma) \) of holonomic loops \( \gamma \), define on this space two operators \( \hat{h}[\alpha] \) and \( \hat{E}[f] \) via Eqs. (80) and (81) and verify that they satisfy the (generalized) CCR. As in the case of non-relativistic quantum mechanics, by itself this procedure does not determine the representation uniquely. We have to supply, in addition, the interpretation of these operators. In this paper, we let \( \hat{h}[\alpha] \) be the holonomy operator defined by the negative frequency connection and \( \hat{E}[f] \) be the smeared out positive frequency electric fields. One could have made other choices and thus obtained other loop representations. For example, we could have let \( \hat{h}[\alpha] \) be the holonomy of the real connection and \( \hat{E}[f] \) be the smeared out real electric fields. The resulting description would then have been equivalent to that obtained by Gambini and Trias [7]. Alternatively, we could have let \( \hat{h}[\alpha] \) be the holonomy of the self dual connection and \( \hat{E}[f] \) be the anti-self dual electric field. This avenue turns out to be the most useful one in canonical gravity. (For treatment of linearized gravity –i.e. gravitons– along these lines, see [11].) The general choice is to let \( \hat{h}[\alpha] \) be the holonomy of a certain kind of connection and \( \hat{E}[f] \), the electric field canonically conjugate to that connection. The choice we made in this paper is the simplest one: with this choice, the transform of section 3 as well as the action of the operators in the resulting loop representation are well-defined without any need of regularization. The situation is more complicated in other loop representations. In particular, as we pointed out earlier, since the holonomy operator is smeared only in one dimension –along the loop– it fails to be well-defined on the Fock space if the connection is real or self dual and regularization involving “thickened-out loops” then becomes essential. In all cases, at least formally, the loop states \( \psi(\gamma) \) can be thought of as the matrix elements \( <0| \hat{h}[\gamma]|\Psi> \) of the holonomy operators between the vacuum state and a given state \( \Psi \). It is only in the representation analysed in this paper that \( \hat{h}[\gamma] \) are the exponentials of annihilation operators and the interpretation is precise. Furthermore, now \( \psi(\gamma) \) can, in addition, be interpreted in terms of a coherent state basis.

An interesting feature of any loop representation is the way in which it deals with gauge invariance. In section 4.2, we defined \( h[\alpha] \) and \( E[f] \) in terms of the transverse parts of the positive and negative frequency fields (Eqs. (69) and (70). Note, however, that we could have dropped the restriction to the transverse part without changing the final result. For, the holonomy of a connection depends only on its transverse part and, by its very definition
(81), the quantum operator $\hat{E}[f]$ also depends only on the transverse part of $f$. In the non-abelian case, the details of the situation are more complicated. However, it is again true that gauge invariance is automatically implemented in the loop representation and does not have to be imposed as a restriction on quantum states. Thus, the loop representation captures the physical degrees of freedom of a gauge theory in a natural way. It is this simplicity that provided a primary motivation for the use of loop space methods in gauge theories by Madelstam, Migdal, Polyakov, Wilson and others [6] over the past three decades.

In a sense, however, the tradition of using loops as basic objects goes back substantially further—in fact, all the way to Faraday! For, gauge theories can be said to have originated in Maxwell’s work which formalized Faraday’s intuitive picture of electromagnetism as a theory of “lines of force” trapped in space. In absence of sources, each line of force is a closed loop. It turns out, quite remarkably, that this picture of a classical field has direct analogs in the loop formulation of the quantum theory.

To see this, recall first that there exist two quantum states that can be naturally associated with any given loop $\gamma$. First is the (generalized) coherent state, $C_{F(\gamma)}(\zeta_j) \equiv \exp \int \frac{d^3k}{(2\pi\hbar)^3} \bar{F}_j(\gamma, \vec{k}) \zeta_j(\vec{k})$. We saw at the end of section 3.2 that the expression $\Psi(\gamma)$ of a state in the loop representation can be interpreted as the scalar product between the state $\Psi$ and $C_{F(\gamma)}(\zeta_j)$. Now, this coherent state is peaked precisely at the classical field configuration in which the (real, transverse) connection is concentrated along the loop $\gamma$.

Thus, the number $\Psi(\gamma_o)$ represents the probability amplitude of seeing a loop-like classical excitation of the connection along $\gamma_o$ when the quantum system is in the state $\Psi$. The second state associated with a closed loop $\gamma_o$ is $\Psi_{\gamma_o}(\gamma)$, the characteristic function of the loop $\gamma_o$, that we introduced in section 3.3. This is the eigenstate of the electric field operators (appropriate to the specific loop representation under consideration) $\hat{E}(f)$ with eigenvalues $i\hbar \int_\gamma f_\alpha dS^\alpha$. These states represent the simplest excitations of the electric field. Why are the excitations associated with closed loops $\gamma_o$ rather than points? In the quantum theory of scalar fields, for example, the simplest excitations are localized at points. Why is the same not true of the electric field? The reason is that the electric field is constrained to satisfy the Gauss law and divergence-free vector fields cannot be supported on points. The simplest geometrical structure on which they can have support are precisely
closed loops! Thus, Faraday’s vision of the electromagnetic field in terms of lines of force is realized in two different ways in the in the loop description of quantum electro-magnetism.

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