Abstract

We investigate the problem of gauge invariance of the effective potential in Chern–Simons systems. Working at the one-loop level, we show explicitly that the picture the subject of gauge invariance has already constructed in scalar electrodynamics gets unchanged in the Chern–Simons territory.

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In this paper we return\textsuperscript{1–4} to the problem of gauge invariance of the effective potential. Here the novelty appears in three-dimensional spacetime and relies on the possibility of introducing the Chern–Simons term. Evidently, we do not expect the Chern–Simons term to change drastically the standard picture the subject of gauge invariance has already constructed. Apart from its intrinsic interest, however, a specific investigation seems particularly desirable in view of extending former results to this territory. To this end here we report mainly on introducing simple and explicit arguments, which reflect the general way gauge invariance plays it role when the standard gauge dynamics is changed to include the Chern–Simons term.

Recent and very interesting progress on self-dual Chern–Simons\textsuperscript{5–8} and Maxwell–Chern–Simons\textsuperscript{9,10} systems have been gotten. Unfortunately, however, almost nothing has been done concerning gauge invariance when the investigation goes beyond the classical level.

To investigate gauge invariance of the effective potential, we then concentrate on the Chern–Simons systems defined by

\begin{equation}
L_{CS} = \frac{1}{4} \kappa \varepsilon_{\mu\nu\lambda} A^\mu F^{\nu\lambda} + (\partial_\mu + ieA_\mu)\bar{\phi}(\partial^\mu - ieA^\mu)\phi - V(\phi),
\end{equation}

and

\begin{equation}
L_{MCS} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + L_{CS}.
\end{equation}

Here \(V(\phi)\) is the potential for the scalar fields, which can be of up to sixth order in \(\phi\) and is supposed to present spontaneous symmetry breaking. We notice that the above systems differ from the ones considered in Refs. [8,10]. Then we remark that neither self-duality nor
the presence of fermions plays any specific role on the gauge invariance issues we shall be concerned in the following.

To prepare the above systems to the calculation of the effective potential, we set $\varphi = (\phi_1 + i\phi_2)/\sqrt{2}$ and use $\phi^2 = \phi_1^2 + \phi_2^2 = \phi_a\phi_a$ to write in Euclidean spacetime

$$L^E_{CS} = -\frac{i}{4} \kappa \varepsilon_{\mu \nu \lambda} A_\mu F_{\nu \lambda} + \frac{1}{2} \partial_\mu \phi_a \partial_\mu \phi_a + \frac{1}{2} e^2 \phi^2 A_\mu A_\mu - e A_\mu \varepsilon_{ab} \phi_a \partial_\mu \phi_b + V(\phi^2), \quad (3)$$

and

$$L^E_{MCS} = \frac{1}{4} F_{\mu \nu} F_{\mu \nu} + L^E_{CS}. \quad (4)$$

Now, under an infinitesimal gauge transformation the fields change as $\delta A_\mu = -\partial_\mu \omega$ and $\delta \phi_a = -e \omega \varepsilon_{ab} \phi_b$. Then instead of working with (3) and (4) we have to deal with

$$L^{eff}_{CS} = L^E_{CS} + L_g \quad (5)$$

and

$$L^{eff}_{MCS} = \frac{1}{4} F_{\mu \nu} F_{\mu \nu} + L^{eff}_{CS}. \quad (6)$$

Here $L_g = L_f + L_c$, and the gauge-fixing ($L_f$) and gauge-compensating ($L_c$) contributions are generically given by

$$L_f = \frac{1}{2} f^2(A, \phi), \quad L_c = \bar{c} \frac{\delta f}{\delta \omega} c. \quad (7, 8)$$

To better explore gauge invariance we choose to work with general R gauges. In this case the gauge-fixing function is

$$f(A, \phi) = \xi^{-\frac{1}{2}} (\partial_\mu A_\mu + e \varepsilon_{ab} \nu_a \phi_b), \quad (9)$$
where $\xi$ and $\mathbf{v} = (v_1, v_2)$ are the gauge parameters. Then we get the gauge-fixing and gauge-compensating contributions to $L_g$ as

$$L_g = \frac{1}{2} \xi^{-1} \left( \partial_\mu A_\mu + e \varepsilon_{ab} v_a \phi_b \right)^2 + \xi^{-\frac{1}{2}} \left[ \partial_\mu \bar{c} \partial_\mu c + e^2 v_a \phi_a \bar{c} c \right].$$

(10)

To calculate the effective potential one usually shifts the scalar fields. Without loosing generality we choose to shift $\phi_1 \rightarrow \bar{\phi} + \phi_1$. In this case we set $v_1 = v$ and $v_2 = 0$, which immediately satisfies the (good gauge) condition first introduced by Fukuda and Kugo.\textsuperscript{11} The classical or zero-loop potential is then given by $V(\bar{\phi})$. To obtain the one-loop contributions we have to collect the quadratic terms in $L^{\text{eff}}$. Here we have, after leaving out the bar over the classical field,

$$V_{CS}^{(1)}(\phi) = V_H^{(1)}(\phi) + V_c^{(1)}(\phi) + V_{gG}^{(1)}(\phi),$$

(11)

and

$$V_{MGCS}^{(1)}(\phi) = V_H^{(1)}(\phi) + V_c^{(1)}(\phi) + V_{MgG}^{(1)}(\phi),$$

(12)

where $V_H^{(1)}(\phi)$ and $V_c^{(1)}(\phi)$ are the Higgs and ghost fields contributions, respectively. They are given by

$$V_H^{(1)}(\phi) = \frac{1}{2} \int \frac{d^3 k}{(2\pi)^3} \ln \left( k^2 + \frac{d^2 V}{d\phi^2} \right),$$

(13)

and

$$V_c^{(1)}(\phi) = -\int \frac{d^3 k}{(2\pi)^3} \left[ \ln (k^2 + e^2 v \phi) - \frac{1}{2} \ln \xi \right].$$

(14)

The contributions $V_{gG}^{(1)}(\phi)$ and $V_{MgG}^{(1)}(\phi)$ come from the gauge and Goldstone fields, which are coupled. To get them explicitly we write the corresponding quadratic contributions to each Lagrangian in the general form

$$\frac{1}{2} \Phi_i^i M_{ij} \Phi_j,$$

(15)
where $\Phi^t = (A_1 A_2 A_3 \phi_2)$ is the transpose of the column vector $\Phi$ and $M$ is a 4 by 4 matrix, which can be written as, after going to momentum space,

$$
M = \begin{pmatrix}
\alpha \delta_{\mu \nu} + \beta k_\mu k_\nu + \kappa \varepsilon_{\mu \nu \lambda} k_\lambda & i e \xi^{-1} (\xi \phi - v) k_\mu \\
-i e \xi^{-1} (\xi \phi - v) k_\nu & k^2 + (1/\phi)(dV/d\phi)
\end{pmatrix}.
$$

(16)

Here we recall that in the 't Hooft/R$\xi$ gauge $v = \xi \phi$. In this case the gauge-Goldstone coupling vanishes, as we can immediately see from (16). In the general case we then recognize that the Goldstone term, $k^2 + (1/\phi)(dV/d\phi)$, does not depend on the particular Chern–Simons or Maxwell–Chern–Simons system one is considering. This is also true for $i e \xi^{-1} (\xi \phi - v) k_\mu$, the coupling between the gauge and Goldstone fields. The gauge term has the general structure $\alpha \delta_{\mu \nu} + \beta k_\mu k_\nu + \kappa \varepsilon_{\mu \nu \lambda} k_\lambda$. However, $\alpha$ and $\beta$ depend on the particular model one is working with: for the Chern–Simons system we have

$$
\alpha_{CS} = e^2 \phi^2, \quad \beta_{CS} = \xi^{-1};
$$

(17a, b)

for the Maxwell–Chern–Simons system they are given by

$$
\alpha_{MCS} = k^2 + e^2 \phi^2, \quad \beta_{MCS} = \xi^{-1} (1 - \xi).
$$

(18a, b)

Now, after calculating the determinants we get

$$
V_{gG}^{(1)} = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \left\{ \ln(\kappa^2 k^2 + e^4 \phi^4) + \ln[(k^2 + e^2 v \phi)^2 + \frac{1}{\phi} \frac{dV}{d\phi}(k^2 + \xi e^2 \phi^2)] - \ln \xi \right\},
$$

(19)

and

$$
V_{MgG}^{(1)} = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \left\{ \ln(\kappa^2 k^2 + (k^2 + e^2 \phi^2)^2) + \ln[(k^2 + e^2 v \phi)^2 + \frac{1}{\phi} \frac{dV}{d\phi}(k^2 + \xi e^2 \phi^2)] - \ln \xi \right\}.
$$

(20)
To write the effective potentials to the Chern–Simons and Maxwell–Chern–Simons systems we collect the results already obtained to get, up to the one-loop order

\[
V_{CS}^1(\phi) = V(\phi) + \frac{1}{2} \int \frac{d^3 k}{(2\pi)^3} \{ \ln \left( k^2 + \frac{d^2 V}{d\phi^2} \right) + \ln \left( k^2 + e^4 \phi^4 / \kappa^2 \right) + \\
\ln[(k^2 + e^2 v \phi)^2 + \frac{1}{2} \frac{dV}{d\phi}(k^2 + \xi e^2 \phi^2)] - \ln(k^2 + e^2 v \phi)^2 \},
\]  
(21)

and

\[
V_{MCS}^1(\phi) = V(\phi) + \frac{1}{2} \int \frac{d^3 k}{(2\pi)^3} \{ \ln \left( k^2 + \frac{d^2 V}{d\phi^2} \right) + \ln(k^2 + m^2) + \ln(k^2 + m^2) + \\
\ln[(k^2 + e^2 v \phi)^2 + \frac{1}{2} \frac{dV}{d\phi}(k^2 + \xi e^2 \phi^2)] - \ln(k^2 + e^2 v \phi)^2 \}.
\]  
(22)

In the above result we have set \( \kappa^2 k^2 + (k^2 + e^2 \phi^2)^2 = (k^2 + m^2_+)(k^2 + m^2_-) \); then

\[
m^2_\pm = e^2 \phi^2 + \frac{1}{2} \kappa^2 \pm \frac{1}{2} \kappa^2 \sqrt{1 + 4e^2 \phi^2 / \kappa^2}.
\]  
(23)

To investigate gauge invariance we now use (21) and (22). Here we immediately see that both results present the same gauge-dependent contribution; explicitly

\[
V^{(1)}(\phi; \xi, v) = \frac{1}{2} \int \frac{d^3 k}{(2\pi)^3} \{ \ln[(k^2 + e^2 v \phi)^2 + \frac{1}{2} \frac{dV}{d\phi}(k^2 + \xi e^2 \phi^2)] - \ln(k^2 + e^2 v \phi)^2 \}.  
\]  
(24)

More importantly, this result is exactly what we have already found in standard scalar electrodynamics. Then the proof of gauge invariance of the effective potential is already given in Refs [3, 4]. Here we recall that in the Nielsen way to check gauge invariance the quantities

\[
C_\xi(\phi; \xi, v) = \xi \frac{\partial \phi}{\partial \xi}, \quad C_v(\phi; \xi, v) = v \frac{\partial \phi}{\partial v},
\]  
(25a, b)

play a basic role in constructing the identities

\[
\xi \frac{dV}{d\xi} = \xi \frac{\partial V}{\partial \xi} + C_\xi \frac{\partial V}{\partial \phi} = 0,
\]  
(26a)
and

\[ v \frac{dV}{dv} = v \frac{\partial V}{\partial v} + C_v \frac{\partial V}{\partial \phi} = 0, \quad (26b) \]

which ensure gauge invariance of the effective potential. This subject has been extensively discussed in the past. Then we omit details to write, up to the one-loop order,

\[ C^{(1)}_\xi(\phi; \xi, v) = -\frac{1}{2} \xi e^2 \phi \int \frac{d^3 k}{(2\pi)^3} \left\{ \frac{1}{(k^2 + e^2 v \phi)^2} + \frac{1}{\phi} \frac{dV}{d\phi} (k^2 + \xi e^2 \phi^2) \right\} \quad (27a) \]

and

\[ C^{(1)}_v(\phi; \xi, v) = e^2 v \int \frac{d^3 k}{(2\pi)^3} \left\{ \frac{(k^2 + \xi e^2 \phi^2)}{(k^2 + e^2 v \phi)} [(k^2 + e^2 v \phi)^2 + \frac{1}{\phi} \frac{dV}{d\phi} (k^2 + \xi e^2 \phi^2)] \right\} \quad (27b) \]

Note that both \( C^{(1)}_\xi(\phi; \xi, v) \) and \( C^{(1)}_v(\phi; \xi, v) \) are finite in three dimensions, although renormalization does not change our conclusions. To understanding why gauge invariance goes the above way, we recall that the BRST symmetry the effective Lagrangian (6) engenders remains unchanged when one discards the Maxwell or the Chern–Simons term. However, in order not to change the way we are doing this investigation, we simply note that \( C^{(1)}_\xi(\phi; \xi, v) \) and \( C^{(1)}_v(\phi; \xi, v) \) only depend on the ghost, Goldstone and gauge-Goldstone propagators. And these propagators do not change when one goes from standard scalar electrodynamics to Chern–Simons scalar electrodynamics. To make this point clear, let us now write down the propagators. Here after some algebraic manipulations we get, for the Higgs field

\[ \Delta_H(k) = \frac{1}{k^2 + d^2 V/d\phi^2}, \quad (28) \]
for the ghost field

$$\Delta_g(k) = \xi^{-\frac{1}{2}} D_v^{-1}(k),$$  \hspace{1cm} (29)

for the Goldstone field

$$\Delta_G(k) = D_v^{-1}(k) D_\xi(k),$$  \hspace{1cm} (30)

for the gauge-Goldstone field

$$\Delta_{G\mu}(k) = i e (\xi \phi - v) D_v^{-1}(k) k_\mu,$$  \hspace{1cm} (31)

for the gauge field

$$\Delta_{CS}^{\mu\nu}(k) = \frac{e^2 \phi^2}{\kappa^2} D_\kappa^{-1}(k) \{ \delta_{\mu\nu} + (\xi - 1) D_\xi^{-1}(k) k_\mu k_\nu - \frac{\kappa}{e^2 \phi^2} \varepsilon_{\mu\nu\lambda\kappa} k_\lambda +$$

$$\xi \left( \frac{\kappa^2}{e^2 \phi^2} - 1 \right) D_v^{-1}(k) D_\xi(k) k_\mu k_\nu + e^2 (\xi \phi - v)^2 D_v^{-1}(k) D_\xi^{-1}(k) D(k) k_\mu k_\nu \},$$  \hspace{1cm} (32)

and

$$\Delta_{MCS}^{\mu\nu}(k) = D_+^{-1}(k) D_-^{-1}(k) D(k) \{ \delta_{\mu\nu} + (\xi - 1) D_\xi^{-1}(k) k_\mu k_\nu - \kappa D^{-1}(k) \varepsilon_{\mu\nu\lambda\kappa} k_\lambda +$$

$$\xi \kappa^2 D_v^{-1}(k) D_\xi(k) k_\mu k_\nu + e^2 (\xi \phi - v)^2 D_v^{-1}(k) D_\xi^{-1}(k) D(k) k_\mu k_\nu \}. \hspace{1cm} (33)$$

For simplicity, in the above results we have set

$$D(k) = k^2 + e^2 \phi^2, \hspace{1cm} D_\kappa(k) = k^2 + e^4 \phi^4 / \kappa^2, \hspace{1cm} (34a, b)$$

$$D_\xi(k) = k^2 + \xi e^2 \phi^2, \hspace{1cm} D_v(k) = k^2 + e^2 v \phi, \hspace{1cm} (34c, d)$$

$$D_+(k) = k^2 + m_+^2, \hspace{1cm} D_-(k) = k^2 + m_-^2, \hspace{1cm} (34e, f)$$

$$D_v(k) = (k^2 + e^2 v \phi)^2 + \frac{1}{\phi} \frac{dV}{d\phi} (k^2 + \xi e^2 \phi^2), \hspace{1cm} (34g)$$
and

\[ D_{\xi v}(k) = k^2 + \xi^{-1} e^2 v^2 + \frac{1}{\phi} \frac{dV}{d\phi}. \]  \hspace{1cm} (34h)

Then we note that the propagator for the gauge field is the only to depend on the particular system we are working with; the others do not change anything and, more importantly, they are equal to the corresponding propagators one gets in standard scalar electrodynamics\(^3\).

Some remarks are now in order: First, in \( \Delta_{\mu\nu}^{MCS} \) put \( \kappa = 0 \) to reproduce the result one has in standard scalar electrodynamics.\(^3\) Second, yet in \( \Delta_{\mu\nu}^{MCS} \) use \( v = \xi \phi \) to obtain the result in the 't Hooft/R\(^{\xi}\) gauge.\(^12\). Third, set \( v = 0 \) and \( \xi = 0 \) in \( \Delta_{\mu\nu}^{CS} \) to get the result in Landau gauge; here, compare with Ref. \([8]\), apart from the difference in the spacetime metric and in the way the scalar field is there shifted.

To conclude, we have explicitly shown that the gauge-dependent contribution to the effective potential at the one-loop level remains unchanged when one goes from standard scalar electrodynamics to Chern–Simons scalar electrodynamics, despite the presence of the Maxwell term in the last case. Then the proof of gauge invariance of the effective potential follows in the same way it does when the gauge field presents standard dynamics.

As a final comment, we recall that the Chern–Simons term gets a surface contribution after a gauge change. Then we have to care about the boundary conditions at the border of the underlying manifold one is working with. Such a problem appears, for instance, when thermal effects are introduced, but this was already considered in Ref. \([13]\).
REFERENCES


