TWO DIMENSIONAL BARYONS IN
THE LARGE N LIMIT

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Abstract

We propose a bilocal field theory for mesons in two dimensions obtained as a kind of non local bosonization of two dimensional QCD. Its semi-classical expansion is equivalent to the $1/N_c$ expansion of QCD. Using an ansatz we reduce the classical equation of motion of this theory in the baryon number one sector to a relativistic Hartree equation and solve it numerically. This (non topological) soliton is identified with the baryon.
Its widely believed that quantum chromodynamics (QCD) is the theory of strong interactions. However the degrees of freedom of its original lagrangian (quarks and gluons) do not correspond to the physical particles (hadrons). Besides, many properties of the physical particles have an intrinsically non-perturbative character, making difficult to make predictions starting from first principles. One possible strategy is to construct effective theories valid in the low energy domain. In four dimensions the spontaneous breakdown of chiral symmetry leads to the non-linear sigma model that describes the lightest mesons (pions), that appear in this context as Goldstone bosons. An interesting fact about this model is the presence of soliton solutions (the so called Skyrme term [1] has to be included to stabilize these solutions). After the inclusion of the Wess-Zumino term [2], this soliton is a fermion, has baryon number one, and consequently, can be identified with the nucleon. In this way, although nucleons are not explicit in the lagrangian, their properties can be studied, with reasonable agreement with experiment [3]. Arguments using the $1/N_c$ expansion [4]also suggest that nucleons are solitons [5]. In some aspects two dimensional QCD resembles the four dimensional case (for example, it is confining) but it is much simpler, providing a place to test ideas too difficult to apply directly in four dimensions, and it has been numerically solved using a discrete light cone quantization [6]. One can take an attitude similar to the four dimensional case and consider low energy effective theories [7], even though chiral symmetry is not spontaneously broken. But in two dimensions one can do better: to write an equivalent lagrangian involving only meson fields without relying on low energy approximations. The price to pay for this is that the mesons fields are not local. To see this let us start with the QCD action with one flavor

$$S = \int d^2 x \sqrt{-g} \left[ -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu \ a} + \bar{q}(i\partial_{\mu} - gA_{\mu})\gamma_{\mu}q \right].$$

Let us write

$$q = \begin{pmatrix} \eta \\ \chi \end{pmatrix},$$

and choose the gauge $A_- = 0$. Only $\chi$ is a true degree of freedom. In fact, we can use $\eta$ and $A_+$ equations of motion

$$i\partial_- \eta = \frac{m}{2}\chi, \quad \partial_-^2 A_+^a = -\frac{g}{2} :\chi T^a \chi :,$$

$$2$$
to write $\eta$ and $A_+^a$ in terms of $\chi$

\[ \eta(x) = -i \frac{m}{2} \int dy h(x - y) \chi(y) \]
\[ A_+^a(x) = \frac{g}{2} \int dy G(x - y) : \chi^+ T^a \chi(y) : , \tag{4} \]

where

\[ h(x) = \frac{1}{2} \text{sgn}(x), \]
\[ G(x) = -\frac{1}{2} |x|. \tag{5} \]

The hamiltonian (generator of $x^+$ translations) we obtain is

\[ H = -i \frac{m^2}{4} \int dx dy h(x - y) \chi^+_m(x) \chi_m(y) \]
\[ + \frac{g^2}{4} \int dx dy G(x - y) : \chi^+_m(x) T^a_{mn} \chi_n(x) : : \chi^+_p(y) T^a_{pq} \chi_q(y) :, \tag{6} \]

where $m, n \ldots$ are color indices. The normal order is in relation to the vacuum defined as

\[ \chi(p)|0> = 0 \]
\[ \chi^+(-p)|0> = 0, \tag{7} \]

with

\[ \chi(x) = \int \frac{dp}{2\pi} e^{-ipx} \chi(p). \tag{8} \]

Defining

\[ M(x^+, x^-, y^-) = \frac{1}{N_c} : \chi^+_m(x^+, x^-) \chi_m(x^+, y^-) :, \tag{9} \]

we can write the hamiltonian\,(6), after some rearrangement of the interaction term as

\[ H = -i \frac{\mu^2 N_c}{4} \int dx dy h(x - y) M(x,y) \]
\[ - \frac{g^2 N_c^2}{4} \int dx dy G(x - y) M(x,y) M(y,x), \tag{10} \]

with

\[ \mu^2 = m^2 - \frac{g^2 N_c}{\pi}. \tag{11} \]
The bilocal field \( M(x, y) \) creates a quark-antiquark pair (a meson) at two causally connected points so a theory in terms of these objects do not violate causality. The commutation relation for the fields \( M(x, y) \) are those of the infinite dimensional general linear group with a central extension

\[
[M(x, y), M(z, u)] = \frac{1}{N_c} [\delta(y - z)(M(x, u) - \epsilon(x - u)) - \delta(x - u)(M(z, y) - \epsilon(z - y))],
\]

where

\[
\epsilon(x) = \frac{i}{2\pi} \mathcal{P}(\frac{1}{x}).
\]

One particular representation of (12) corresponds to two dimensional QCD. In order to have a theory with no negative norm states and positive energy this representation should be chosen unitary and of highest weight. We can choose the representation this way only when \( N_c \) is an integer [8] (from the point of view of meson theory that is the only reason this should be so). The hamiltonian (10) and the commutation relations (12) define a quantum theory of mesons equivalent to QCD, but written entirely in terms of color singlets. It is also Lorentz invariant with the transformation rules

\[
x^+ \rightarrow \lambda x^+ \\
x^- \rightarrow \lambda^{-1} x^- \\
M(x, y) \rightarrow \lambda M(x, y),
\]

with \( \lambda \) related to the boost velocity \( v \) by \( \lambda = (1 - v)(1 - v^2)^{-1/2} \). Notice that the factor \( 1/N_c \) plays the role of \( \hbar \). Consequently the classical version of the theory defined above corresponds to QCD at \( N_c \rightarrow \infty \). The classical equation of motion for \( M(x, y) \) is

\[
\partial_+ M(x, y) = \frac{\mu^2}{4} \int dz M(x, z) h(z - y) - h(x - z) M(z, y) \\
- \frac{ig^2 N_c}{2} \int dz G(z - x) M(x, z)(M(z, y) - \epsilon(z - y)) \\
- G(z - y) M(z, y)(M(x, z) - \epsilon(x - z)).
\]

The vacuum expectation value of \( M(x, y) \) in QCD is zero. If we expand (15) around \( M(x, y) = 0 \) and keep up to linear terms we have, in momentum space

\[
\partial_+ M(p, q) = \frac{i\mu^2}{4} \left( \frac{1}{q} - \frac{1}{p} \right) M(p, q) \\
- \frac{ig^2 N_c}{4} \int dk G(k)(\text{sgn}(p) M(p - k, q - k) - \text{sgn}(q) M(p + k, q + k)).
\]
Defining
\[ P_- \equiv p - q \]
\[ \xi \equiv \frac{p}{P_-} \]
\[ \eta(p/P_-) \equiv M(p, q) \] (17)
and remembering that the Fourier transform of \( G(x) \) is a distribution defined by
\[ \int dp\, G(p) f(p) = \int dp \, \frac{1}{p^2} (f(p) - f(0)). \] (18)
we have
\[ m_{\text{meson}}^2 \eta(\xi) = -(m^2 - \frac{N_c g^2}{\pi})(\frac{1}{\xi} + \frac{1}{1 - \xi})\eta(\xi) \]
\[ - i g^2 N_c \int d\xi' (\text{sgn}(\xi) - \text{sgn}(\xi - 1)) \frac{1}{(\xi - \xi')^2} (\eta(\xi') - \eta(\xi)). \] (19)
This is just the equation derived by \( \text{t'Hooft} \) [9] summing the ”rainbow” diagrams whose solution gives the meson spectrum (up to order of \( 1/N_c \)). As shown by \( \text{t'Hooft} \) there are no scattering states, implying confinement.

The baryon number can be written as
\[ B = \int dx M(x, x). \] (20)
Thus the nucleon should be identified with the lowest energy static solution of (15) such that \( B \) defined by (20) is equal to 1. Due to the complexity of (15) we will use numerical analysis and the ansatz
\[ M(x, y) = \frac{1}{N_c} \Psi^*(x)\Psi(y) + \epsilon(x - y), \] (21)
with
\[ \int dx |\Psi(x)|^2 = 1, \] (22)
where \( \Psi \) is a classical bosonic variable. With this ansatz the equation of motion (15) reduces to an equation for \( \Psi \)
\[ \partial_+ \Psi(x) = \frac{m^2}{4} \int dz h(z - x) \Psi(z) + i g^2 \int dz G(z - x)|\Psi(z)|^2 \Psi(x) \] (23)
Note we have now $m$ instead of the previous renormalized mass $\mu$. A static $M(x, y)$ does not imply a static $\Psi(x)$: $\Psi(x)$ can change in time by a phase

$$\partial_0 M(x, y) = 0 \rightarrow \Psi(x) = \psi(x^+ - x^-)e^{-if(x^+ + x^-)}.$$  

This phase can be chosen arbitrarily, without change of physics. We will choose a phase growing linearly with time $f(x^+ + x^-) = \mathcal{E}(x^+ + x^-)/2$. Combining with (23) we arrive at

$$\partial_- \psi(x) + \frac{m^2}{4} \int dz h(z - x) \psi(z) + i \frac{g^2}{2} \int dz G(z - x)|\psi(z)|^2 \psi(x) = -i \mathcal{E} \psi(x). \quad (25)$$

This is the Hartree equation for relativistic particles interacting through a Coulomb potential in the light cone variables. This can be seen more easily if we write (25) in momentum space. It will be more useful though to obtain (25) from a variational principle first. In fact (25) is equivalent to minimizing

$$E = \int dx i \psi^*(x) \partial_+ \psi(x) - i \frac{m^2}{4} \int dx dy \ h(x - y) \psi^*(x) \psi(y)$$

$$- \frac{g^2}{4} \int dx dy G(x - y) |\psi(x)|^2 |\psi(y)|^2$$

under the constraint

$$\int dx |\psi(x)|^2 = 1. \quad (27)$$

Notice that the functional $E$ is just the energy $P_+ + P_-$, $P_+$ being the hamiltonian $H$ written in terms of $\psi$. In momentum space $E$ is given by

$$E = \int \frac{dp}{2\pi} (p + \frac{m^2}{4p}) |\psi(p)|^2$$

$$- \frac{g^2}{4} \int \frac{dp dq dk}{8\pi^3} G(p) \psi^*(k) \psi^*(q) \psi(k + p) \psi(q - p). \quad (28)$$

The symmetry $\psi(x) \rightarrow \psi^*(-x)$ of (25) implies that $\psi(p)$ is real for the ground state. Also, in order to have a positive definite energy $\psi(p)$ should vanish for negative $p$. This way we have

$$E = \int \frac{dp}{2\pi} (p + \frac{m^2}{4p}) |\psi(p)|^2$$

$$- \frac{g^2}{4} \int \frac{dp dq dk}{8\pi^3} \frac{1}{p^2} [\psi(k) \psi(q) \psi(k + p) \psi(q - p) - \psi(k)^2 \psi(q)^2]. \quad (29)$$
A discrete version of the energy above was minimized numerically using the method of the gradient [10], this means, starting from some arbitrary initial configuration the code changes it following the negative of the gradient of the discrete energy. The constraint (27) was imposed in two steps: 1) projecting out the component of the gradient normal to the constraint and 2) normalizing the state after each ”time” step. Some examples of wave functions for diverse values of the parameter \(m/g\) are shown in figure 1. They are concentrated around a mean value of \(p_-\) and are smooth, which implies that in position space \(\psi\) vanishes at infinity faster than any polynomial. Notice that the two terms in \(E\) have competing effects. The kinetic term is minimized with a \(\delta\) function around \(p = m/2\). The potential energy though favors a more spread out wave function. In fact we can see that the larger the value \(g\) for fixed \(m\), the broader the function is. In the weak coupling, non relativistic limit \(m/g \rightarrow \infty\), the peak of the wave function is roughly around \(p = m/2\). For the strong coupling region though, the peak is at larger values of \(p\), reaching a value \(\sim g/4\) at \(m = 0\). At this point the mass of the baryon is different from zero, signaling that the chiral limit does not correspond to \(m = 0\) (remember that \(m\) is the bare mass of the quark). This is disturbing and is probably related to the fact that in (21) the limit \(\Psi(x) = 0\) gives \(M(x, y) \neq 0\). The ground state of the baryon number one sector is not only \(N_c\) quarks on top of the vacum, but contains distortions even far away from the baryon. Notice it is the \(\epsilon(x–y)\) term in (21) that produces the mass renormalization back from \(\mu\) to \(m\). For negative values of \(m^2\) the energy is still positive as long as \(m^2\) is larger than some critical value \(\sim -2g^2/3\). Beyond this point \(E\) is not bounded from below. This behavior of the baryon mass as function of \(m/g\) is shown in figure 2. For large values of \(m/g\) the mass of the baryon (divided by \(N_c\), the number of quarks in the baryon) approaches the quark mass, as it should be for a non relativistic system. It is a little larger because in two dimensions the binding energy is positive. We also solved (23) linearised around the wave function \(\psi_0\) of the (ground state) baryon. Now \(\psi(p)\) does not need to be real (because it describes excited states) and we have one equation for the real part, another for the imaginary part. So, if

\[
\psi(p) = \psi_0(p) + \sigma_R(p) + i\sigma_I(p),
\]
then we have

\[
(p + \frac{m^2}{4p})\sigma_R(p) - \frac{g^2}{2} \int \frac{dq \, dk}{(2\pi)^2} G(k - p)(2\psi_0(k + q - p)\psi_0(q)\sigma_R(k) \\
+ \sigma_R(k + q - p)\psi_0(q)\psi_0(k)) \\
= \mathcal{E}\sigma_R(p)
\]  

and

\[
(p + \frac{m^2}{4p})\sigma_I(p) - \frac{g^2}{2} \int \frac{dq \, dk}{(2\pi)^2} G(k - p)(2\psi_0(k + q - p)\psi_0(q)\sigma_I(k) \\
- \sigma_I(k + q - p)\psi_0(q)\psi_0(k)) \\
= \mathcal{E}\sigma_I(p).
\]

The energies $\mathcal{E}$ correspond to the excited states of the baryon. Typical examples of the wave function of the low lying states are shown in figure 3 and 4. Notice that there are two states nearly degenerate with the ground state. The source of this degeneracy is the presence of two symmetries in (25). The first is phase invariance

\[
\psi(x) \to e^{i\alpha}\psi(x)
\]

that implies that $i\psi_0(p)$ will be a solution with the same ground state energy. The second is translation invariance

\[
\psi(x) \to \psi(x - x_0)
\]

that implies that $ip\psi(p)$ is also degenerate with the ground state. Translation symmetry is broken by the lattice so this degeneracy is slightly lifted in our numerical results, as can be seen in figure 3. The numerical procedure to find the n-th excited state was to minimise the energy in the subspace orthogonal to all states with lower energy, so instead of $ip\psi_0(p)$ we have a linear combination of $ip\psi_0(p)$ and $i\psi_0(p)$ orthogonal to $\psi_0(p)$.

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REFERENCES

Figure Captions

Figure 1: Examples of wave functions of the ground state. $P$ is measured in arbitrary units and $g = 400$.

Figure 2: Energy as a function of $m/g$ for positive and negative values of $m^2$ (at the left of 0). Energy is measured in arbitrary units and $g = 30$. Negatives values energy are meaningless because the process of minimisation never reached an end.

Figure 3: Ground state and two other states degenerate with it.

Figure 4: Three examples of excited states.