A topological quantum field theory of non-abelian differential forms is investigated from the point of view of its possible applications to description of polynomial invariants of higher-dimensional two-component links. A path-integral representation of the partition function of the theory, which is a highly on-shell reducible system, is obtained in the framework of the antibracket-antifield formalism of Batalin and Vilkovisky. The quasi-monodromy matrix, giving rise to corresponding skein relations, is formally derived in a manifestly covariant non-perturbative manner.
Introduction

Metric-independent gauge theories with a non-trivial classical action, so-called topological field theories of the Schwarz type (see Ref. 1, for a comprehensive review of all topological theories), play an important role in “physics” approach to polynomial invariants of knots and links. The first paper\(^2\) establishing this direction of research deals with an application of Chern-Simons theory to the Jones (or more generally to the Homfly) polynomial. The second important example of a theory of this type is provided by gauge theory of (non-abelian) differential forms, so-called BF-theory.\(^3\)–\(^8\) Although the pure BF-theory is now a well-established system, it seems that the problem of physical observables and/or corresponding possible topological invariants has not yet been fully satisfactory solved in its context. For example, in Refs. 4 and 5 an additional restricting flatness condition is imposed on the curvature, whereas Ref. 6 exclusively deals with abelian observables, yielding gaussian linking numbers. Ref. 7 only sketches an idea, and a particular four-dimensional version of Ref. 8 does not explicitly relate the observables introduced to possible topological invariants. In this work we formally derive, in the framework of BF-theory, skein relations for some higher-dimensional two-component links.

Roughly speaking, there are two kinds of difficulties related to BF-theory in an arbitrary dimension \(d\). The first rather conceptual difficulty concerns the problem of “physical observables” measuring linking phenomena in higher dimensions. In the standard (three-dimensional) Chern-Simons case, one traditionally introduces the Wilson loops, but obviously this method does not work in higher dimensions. To encode topologically interesting information in the framework of higher-dimensional theory one should introduce topological “matter” multiplets living on the submanifolds corresponding to the higher-dimensional knots/links under consideration. Since the purpose of our paper is to derive skein relations for a two-component link (the link consisting of the two
components, $K$ and $C$, of dimension $d - 2$ and 1, respectively) we will supplement the standard BF-action with two sets of topological matter multiplets. The second more technical difficulty concerns the problem of covariant quantization. BF-theory, as a highly on-shell reducible system, requires a treatment in the framework of the antibracket-antifield formalism of Batalin and Vilkovisky. Solutions of this problem, at least in the case of the pure BF-theory, are presented in Ref. 11 (see also Ref. 12).

In the first “classical” part of the work, we will define a total classical action of the full theory, i.e. BF-system plus the topological matter part, derive classical equations of motion and find classical symmetries of the action. The second “quantum” part is devoted to the BRST quantization of the system in the framework of the formalism of Batalin and Vilkovisky. We will present an explicit and elegant form of the solution of the master equation, as well as a compact form of the BRST $s$ operator, playing here an auxiliary role. A covariant path-integral representation of the partition function appears as a straightforward consequence of the formalism used. In the third “topological” part, appealing to reader’s imagination and using the Stokes theorem, we derive (in a non-perturbative way) the quasi-monodromy matrix giving rise to skein relations corresponding to an arbitrary pair of irreducible representations of an arbitrary compact semisimple Lie group $G$. Appendix contains some useful formulas valid also in a more general case.

1. Classical action

In $d$ dimensions ($d \geq 2$), the classical action of gauge theory of non-abelian differential forms (BF-theory) is defined as

$$ S_{BF}^{cl} = \frac{1}{\lambda} \int_S \text{Tr}(BF), \quad (1.1) $$

where the coupling constant $\lambda$ can assume an arbitrary non-zero complex value, $S$ is a $d$-dimensional sphere (formally, one could also try to consider a more general manifold),
$B$ is a Lie-algebra valued $(d-2)$-form, $B = T^a B^a$, $F$ is the curvature two-form, and the normalization of the generators $T^a$ of the compact semisimple Lie group $G$ is $\text{Tr}(T^a T^b) = \frac{1}{2} \delta^{ab}$. For differential forms all products are exterior ones. It is interesting to note that contrary to Chern-Simons theory the coupling constant $\lambda$ is not constrained to integer values. Accordingly, the corresponding parameter in skein relations is not constrained to integer values either.

The classical action of matter part of gauge theory of non-abelian differential forms consists of the two pieces $(d \geq 3)^7$:

(1)
\[
S_{\Omega}^{\text{cl}} = \frac{1}{2} \int_{\mathcal{K}} \left( \bar{\Theta} d_A \Omega + d_A \bar{\Omega} \Theta + \bar{\Theta} B \Theta \right),
\]
where $\Theta$ and $\bar{\Theta}$ are zero-forms, $\Omega$ and $\bar{\Omega}$ are $(d-3)$-forms (all the forms are in an irreducible representation $R_1(G)$ with the generators $t^a_1$), $d_A$ is the exterior covariant derivative, $d_A \Omega \equiv d\Omega + A\Omega$, $d_A \bar{\Omega} \equiv d\bar{\Omega} - A^T \bar{\Omega}$, $A \equiv t^a_1 A^a$, and $\mathcal{K}$ is a $(d-2)$-dimensional closed submanifold imbedded in $S$, a $(d-2)$-knot (the first component of the link $\mathcal{L}$);

(2)
\[
S_\eta^{\text{cl}} = \frac{1}{2} \int_{\mathcal{C}} \bar{\eta} d_A \eta,
\]
where $\eta$ and $\bar{\eta}$ are zero-forms in an irreducible representation $R_2(G)$, and $\mathcal{C}$ is a one-dimensional loop imbedded in $S$, a standard knot (the second component of the link $\mathcal{L}$).

Then the classical action of the whole theory is given as the sum
\[
S^{\text{cl}} = S_{BF}^{\text{cl}} + S_{\Omega}^{\text{cl}} + S_\eta^{\text{cl}}.
\]
Hence the corresponding classical equations of motion are of the form
\[
d_A B + \lambda (\bar{\Theta} t_1 \Theta - \bar{\Theta} t_1 \Omega) \delta(\mathcal{K}) - \lambda \bar{\eta} t_2 \eta \delta(\mathcal{C}) = 0,
\]
\[
F + \lambda \bar{\Theta} t_1 \Theta \delta(\mathcal{K}) = 0,
\]
\[
d_A \Omega + B \Theta = 0, \quad d_A \bar{\Omega} + \bar{\Theta} B = 0,
\]
where $\delta(K)$ and $\delta(C)$ is a Dirac-delta two-form and a $(d-1)$-form, respectively (see Appendix (1)). A subset of all solutions of the classical equations of motion, important for further symmetry analysis, is given by

$$d_A B = F = d_A \Omega = d_A \bar{\Omega} = \Theta = \bar{\Theta} = \eta = \bar{\eta} = 0. \quad (1.5)$$

In general case, i. e. $d \geq 4$, the action (1.3) enjoys four kinds of local gauge symmetries:

(1) Ordinary gauge symmetry ($d \geq 2$)

$$\delta_1 A = -\frac{1}{2\lambda}d_A \sigma_1 \equiv -\frac{1}{2\lambda}(d\sigma_1 + [A, \sigma_1]),$$

$$\delta_1 B = \frac{1}{2\lambda}[\sigma_1, B],$$

$$\delta_1 \Omega = \frac{1}{2\lambda}\sigma_1 \Omega, \quad \delta_1 \bar{\Omega} = -\frac{1}{2\lambda}\bar{\Omega}\sigma_1,$$

$$\delta_1 \Theta = \frac{1}{2\lambda}\sigma_1 \Theta, \quad \delta_1 \bar{\Theta} = -\frac{1}{2\lambda}\bar{\Theta}\sigma_1,$$

$$\delta_1 \eta = \frac{1}{2\lambda}\sigma_1 \eta, \quad \delta_1 \bar{\eta} = -\frac{1}{2\lambda}\bar{\eta}\sigma_1, \quad (1.6)$$

where $\sigma_1$ is a zero-form in $R_{Adj}(G)$.

(2) B-symmetry ($B$ stands for $B$-field or Bianchi)

$$\delta_2 A = \delta_2 \Theta = \delta_2 \bar{\Theta} = \delta_2 \eta = \delta_2 \bar{\eta} = 0, \quad (1.7a)$$

$$\delta_2 B = \frac{1}{2\lambda}d_A \sigma_2, \quad (1.7b)$$

$$\delta_2 \Omega = -\frac{1}{2\lambda}\sigma_2 \Theta, \quad \delta_2 \bar{\Omega} = -\frac{1}{2\lambda}\bar{\Theta}\sigma_2, \quad (1.7c)$$
where $\sigma_2$ is a $(d - 3)$-form in $R_{\text{Adj}}(G)$. This symmetry emerges for $d \geq 3$, and is reducible for $d \geq 4$. The reducibility follows from the fact that we can perform an additional transformation

$$\delta_2' \sigma_2 = \frac{1}{2\lambda} d_A \sigma_2', \quad (1.8)$$

which is an on-shell symmetry transformation of (1.7b). Namely

$$\delta_2' \delta_2 B = \frac{1}{4\lambda^2} d_A^2 \sigma_2' = \frac{1}{4\lambda^2} [F, \sigma_2'] = 0, \quad (1.9)$$

where the solution (1.5) of the equations of motion has been used. For $d \geq 5$, we can repeat this procedure performing further transformations

$$\delta_2'' \sigma_2' = \frac{1}{2\lambda} d_A \sigma_2'', \quad (1.10)$$

which is an on-shell symmetry transformation of (1.8), and so on. We conclude, that according to the Batalin-Vilkovisky quantization scheme, we deal with $(d - 3)$-stage on-shell reducible gauge symmetry.

(3) “Matter” gauge symmetry of $\Omega$

$$\delta_3 A = \delta_3 \bar{\Omega} = \delta_3 \bar{\Theta} = \delta_3 \Theta = \delta_3 \eta = \delta_3 \bar{\eta} = 0, \quad (1.11a)$$

$$\delta_3 B = \frac{1}{2} \lambda \bar{\Theta} t_1 \sigma_3 \delta(K), \quad (1.11b)$$

$$\delta_3 \Omega = -\frac{1}{2} d_A \sigma_3, \quad (1.11c)$$

where $\sigma_3$ is a $(d - 4)$-form in $R_1(G)$. This symmetry emerges for $d \geq 4$, and it appears to be reducible for $d \geq 5$. The obvious on-shell symmetry transformation of (1.11c) is

$$\delta_3' \sigma_3 = -\frac{1}{2} d_A \sigma_3'. \quad (1.12)$$

Accordingly, the symmetry is $(d - 4)$-stage on-shell reducible.

(4) “Matter” gauge symmetry of $\bar{\Omega}$

$$\delta_4 A = \delta_4 \bar{\Omega} = \delta_4 \bar{\Theta} = \delta_4 \Theta = \delta_4 \eta = \delta_4 \bar{\eta} = 0,$$
\[ \delta_4 B = \frac{1}{2} \lambda \delta(\mathcal{K}) \tilde{\sigma}_4 t \Theta, \]
\[ \delta_4 \tilde{\Omega} = -\frac{1}{2} d_A \tilde{\sigma}_4, \]

where \( \tilde{\sigma}_4 \) is a \((d-4)\)-form in \( R_1(G) \). This symmetry is, in general, also \((d-4)\)-stage on-shell reducible.

We have chosen a bit strange normalization of our gauge transformations because we want to achieve exact agreement with the non-degeneracy condition (A2.3). One should also note that the \( B \)-symmetry would be broken out if \( \partial \mathcal{K} \neq \emptyset \), and the “matter” symmetries would be broken out by the last term in (1.2a) if the submanifold \( \mathcal{K} \) had self-intersections. But since we would like to interpret \( \mathcal{K} \) as a knot we should assume that \( \mathcal{K} \) is closed, and that it has no self-intersections.

2. Quantum action

Covariant quantizing on-shell reducible gauge systems can be approached by means of the Batalin-Vilkovisky antifield-antibracket procedure.\(^{10}\) The final result of such a procedure is a covariant path-integral representation of the partition function \( Z \). The problem is essentially solved if one succeeds in finding a proper non-degenerate solution \( S \) (an extended classical action, or a classical part of the quantum action \( W \)) of the master equation

\[ (S, S) = 0, \]  

where the antibracket is defined in Appendix (2).

Instead of trying to solve the master equation perturbatively we simply postulate that the solution is of the form analogous to the form of the classical action \( S^{\text{cl}} \) (strictly speaking, a minimal part of \( S \)), and then it is given by

\[ S = S_{BF} + S_{\Omega} + S_{\eta}, \]  

\[ 2.1 \]
with
\[
S_{BF} = \frac{1}{\lambda} \int_S \text{Tr}(bf),
\]  
(2.3a)

\[
S_{\Omega} = \frac{1}{2} \int_K (\bar{\partial}d_\alpha \omega + d_\alpha \bar{\omega} \theta + \bar{\partial}b\theta) = \frac{1}{2} \int_K \left( \bar{\partial}d_A \omega + d_A \bar{\omega} \theta + \bar{\partial}B \Theta \right),
\]  
(2.3b)

where
\[
d_\alpha \omega \equiv d\omega + a\omega,
\]
\[
d_\alpha \bar{\omega} \equiv d\bar{\omega} - a^T \bar{\omega},
\]
\[
f \equiv d^2_a \equiv da + a^2,
\]  
(2.4)

and
\[
S_\eta = \frac{1}{2} \int_C \bar{\eta}d_\gamma \eta = \frac{1}{2} \int_C \bar{\eta}d_A \eta = S_{\eta}^{cl}.
\]  
(2.3c)

The fields entering (2.3) are five-graded (four-graded with respect to the four ghost numbers corresponding to the four gauge symmetries) non-homogeneous forms in respective representations of \(G\). They constitute a minimal sector of the theory, and they assume the following explicit form:

\[
a \equiv \sum_{g=0}^{1} A^g_{1-g} 0 0 0 + \sum_{g=0}^{d-2} B^g_{g+2} -1 -g 0 0 ,
\]

\[
b \equiv \sum_{g=0}^{1} A^g_{d-1+g} 0 0 0 - \sum_{g=0}^{d-2} B^g_{d-2-g} 0 0 0 ,
\]

\[
\omega \equiv \sum_{g=0}^{d-3} \Omega^g_{d-3-g}, \quad \bar{\omega} \equiv \sum_{g=0}^{d-3} \bar{\Omega}^g_{d-3-g},
\]

\[
\theta \equiv \Theta + \sum_{g=1}^{d-2} \Omega^g_{1} 0 0 0 -g , \quad \bar{\theta} \equiv \bar{\Theta} + \sum_{g=1}^{d-2} \bar{\Omega}^g_{1} 0 0 0 -g ,
\]  
(2.5)

where the following identifications have been assumed for the classical fields \(\phi^{cl} = \{A, B, \Omega, \bar{\Omega}, \Theta, \bar{\Theta}\}\),

\[
A \equiv A^0_{1} 0 0 0 , \quad B \equiv B^0_{d-2} 0 0 0 ,
\]

\[
\Omega \equiv \Omega^0_{d-3} 0 0 0 , \quad \bar{\Omega} \equiv \bar{\Omega}^0_{d-3} 0 0 0 ,
\]

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\[ \Theta \equiv \Theta^0_0^0_0^0, \quad \bar{\Theta} \equiv \bar{\Theta}^0_0^0_0^0, \quad (2.6) \]

and for the minimal sector \( \phi^{\text{min}} \)

\[ A^{\text{min}} = \{ A^1_1^0_0_0_0, A^1_0_0_0_0 \}, \]
\[ B^{\text{min}} = \{ B^0_0_0_0_0, B^0_1_0_0_0, \ldots, B^0_d^{-2}_0_0_0 \}, \]
\[ \Omega^{\text{min}} = \{ \Omega^0_0_0_0_0, \Omega^0_0_0_1_0, \ldots, \Omega^0_0_0_0_{d-3} 0 \}, \]
\[ \bar{\Omega}^{\text{min}} = \{ \bar{\Omega}^0_0_0_0_0, \bar{\Omega}^0_0_0_0_1, \ldots, \bar{\Omega}^0_0_0_0_{d-3} 0 \}. \quad (2.7) \]

The total degrees “Deg” (A3.4) of our forms are

\[ \text{Deg}_a = \deg A = 1, \]
\[ \text{Deg}_b = \deg B = d - 2, \]
\[ \text{Deg}_\omega = \text{Deg} \bar{\omega} = \deg \Omega = \deg \bar{\Omega} = d - 3, \]
\[ \text{Deg}_\theta = \text{Deg} \bar{\theta} = \deg \Theta = \deg \bar{\Theta} = 0. \quad (2.8) \]

It is implicitly assumed that only the integrands with all ghost numbers equal to zero, and with form degrees equal to the dimensions of respective manifolds survive in (2.3).

Now, we can introduce a BRST operator \( s \). To preserve self-consistency, the action of \( s \) on the fields should be defined according to (A3.2). In a compact notation,

\[ sa = \frac{1}{2\lambda} f + \frac{1}{2} \bar{\theta} t_1 \theta \delta(K), \]
\[ sb = \frac{1}{2\lambda} d_a b + \frac{1}{2} (\bar{\omega} t_1 \theta - \bar{\theta} t_1 \omega) \delta(K) - \frac{1}{2} \bar{\eta} t_2 \eta \delta(C), \]
\[ s\omega = \frac{1}{2} (d_a \omega + b\theta), \]
\[ s\bar{\omega} = \frac{1}{2} (d_a \bar{\omega} + \bar{\theta} b), \]
\[ s\theta = (-)^{d+1} \frac{1}{2} d_a \theta, \]
\[ s\bar{\theta} = (-)^{d+1} \frac{1}{2} d_a \bar{\theta}. \]
\[ s\bar{\eta} = s\bar{\eta} = 0, \quad (2.9) \]

where

\[ s\Theta = s\bar{\Theta} = 0. \quad (2.9a) \]

Performing a very straightforward calculation we can easily check that (see (A3.1))

\[ (S, S) \equiv sS = 0, \quad (2.10) \]

provided the obvious additional topological condition \( \mathcal{K} \cap \mathcal{C} = \emptyset \) is satisfied. To perform the calculation one should make use of the Stokes theorem, the generalized Bianchi identity,

\[ d_a f = 0, \quad (2.11) \]

as well as the formulas (A3.5). It appears that our BRST operator \( s \) is automatically nilpotent, \( s^2 = 0 \) (see (A3.3)). Also, it can be easily checked that \( S \) possesses the correct classical limit in the sense of Batalin and Vilkovisky

\[ S(\phi, \phi^*)|_{\phi^*=0} = S^{\text{cl}}(\phi^{\text{cl}}), \quad (2.12) \]

where the collective symbol \( \phi^* \) denotes all antifields, and that it satisfies the condition of non-degeneracy (A2.3)

\[ \frac{\delta_l \delta_r S}{\delta A^*_d \delta A_0} \bigg|_{\phi^*=0} = -\frac{1}{2\lambda} \ast d_A \delta, \]

\[ \frac{\delta_l \delta_r S}{\delta B^*_1 \delta B_{d-2}} \bigg|_{\phi^*=0} = \frac{1}{2\lambda} \ast d_A \delta, \quad 1 \leq g \leq d - 2, \]

\[ \frac{\delta_l \delta_r S}{\delta \Omega^*_g \delta \Omega_{d-3}} \bigg|_{\phi^*=0} = -\frac{1}{2} \ast d_A \delta, \quad 1 \leq g \leq d - 3, \]

\[ \frac{\delta_l \delta_r S}{\delta \bar{\Omega}^*_g \delta \bar{\Omega}_{d-3}} \bigg|_{\phi^*=0} = -\frac{1}{2} \ast d_A \delta, \quad 1 \leq g \leq d - 3, \quad (2.13) \]

where the first two covariant derivatives act in the adjoint representation \( R_{\text{Adj}}(G) \), the second two ones in \( R_1(G) \), and \( \delta \) denotes the ordinary Dirac-delta.
The extended classical action should be supplemented with the auxiliary term $S_{\text{aux}}$. The form of $S_{\text{aux}}$ is universally given for an arbitrary theory in Ref. 10, and in our case

$$S_{\text{aux}}(\phi^*_\text{aux}, \Pi^\phi) = \int_S \text{Tr}(A_{\text{aux}}^* \Pi^A + \sum_i B_{\text{aux}}^* i \Pi^B_i) + \frac{1}{2} \int_K \sum_j (\Omega_{\text{aux}}^* j \Pi^\Omega_j + \bar{\Omega}_{\text{aux}}^* j \Pi^\bar{\Omega}_j),$$

(2.14)

where $\Pi^\phi = \{\Pi^A, \Pi^B_i, \Pi^\Omega_j, \Pi^\bar{\Omega}_j\}$ consists of a multiplet of Lagrange multipliers (Nakanishi-Lautrup-Stueckelberg fields), and $\phi^*_\text{aux} = \{A_{\text{aux}}^*, B_{\text{aux}}^* i, \Omega_{\text{aux}}^* j, \bar{\Omega}_{\text{aux}}^* j\}$ denotes a multiplet of antifields in the auxiliary sector. Thus all the fields appearing in the full action $S$, minimal and auxiliary, constitute the four triangles of ghosts corresponding to the four gauge symmetries. The explicit form of the form degrees and of the ghost numbers uniquely follows from the Batalin-Vilkovisky prescription and the duality condition. From (A3.2), we directly obtain

$$s\phi^*_I = \frac{1}{2} \Pi^\phi_I,$$

$$s\phi^*_\text{aux} = s\Pi = 0,$$

(2.9b)

where $I$ denotes a respective index.

One should stress that our extended classical action satisfies not only the classical master equation (2.1) but also the quantum one

$$\frac{1}{2} (S, S) = i \Delta_{\text{BV}} S \equiv i \frac{\delta_r \delta_l S}{\delta \phi^I \delta \phi^*_I} = 0.$$  

(2.15)

To check it we can observe that the RHS of (2.15) vanishes identically by virtue of antisymmetry of the structure constants. Strictly speaking, the RHS of (2.15) is not well-defined mathematically, but it already vanishes for a regularized version. Hence the full quantum action $W = S + S_{\text{aux}}$.

As a final step of the quantization procedure we should define the gauge fermion $\Psi(\Phi)$ that satisfies some conditions of non-degeneracy. In turn, the gauge fermion defines the antifields,

$$\phi^*_I = * \frac{\delta_r \Psi}{\delta \phi^I}.$$  

(2.16)
Since we are not going to perform any perturbative calculations we need not to fix a concrete form of $\Psi$. In fact, our further analysis is independent of a particular form of $\Psi$, and exclusively rests only on its existence.

Thus the partition function of our theory can be written in the following covariant path-integral representation

$$Z = \int D\phi D\Pi \exp(iW)|_\Sigma,$$

where $\phi = \{\phi^{\text{min}}, \phi^{\text{aux}}\}$, and the symbol $\Sigma$ indicates that we should eliminate antifields using (2.16).

As the gauge fixing procedure (the gauge fermion $\Psi$) unavoidably introduces the metric tensor, the question arises as to the metric independence of $Z$. Since the metric tensor enters $Z$ only through $\Psi$ it follows from the theorem of Batalin and Vilkovisky (on gauge-independence) that $Z$ should also be metric independent.

3. Invariant polynomials

In this section, we will approach the problem of invariant polynomials for higher-dimensional links consisting of the two components, $K$ and $C$, of the dimension $d - 2$ and $1$, respectively. From topological point of view, our approach is purely formal, and presumably it concerns only some subclass of genuinely higher-dimensional invariants of links, where possibly some matrices corresponding to the simplex equation rather than to the Yang-Baxter one would appear. However, our approach is fully motivated, as we consider non-trivial physical observables, which describe some linking phenomena in $d$ dimensions in a non-trivial way.

The prototype of our two-component link is the pair consisting of the closed manifold $K$ and the loop $C$, which enter our theory through the topological matter action $S_\Omega + S_\eta$. As we would like to derive the topological invariants in the form of invariant polynomials we should derive corresponding skein relations. To this end, we have to
compare some number of copies \( L_k \) \((k = 1, 2, \ldots, N)\) of the link \( L \), entering our skein relation, appropriately differing inside \( d \)-dimensional balls \( B \) \((N \geq 3, \text{ and } N \text{ depends on the pair of irreducible representations, } R_1(G) \text{ and } R_2(G)\). An explanation of the word “appropriately” implicitly follows from our further construction. To calculate the contributions to the functional integral coming from \( B \)'s that are different for the different copies \( L_k \) we will use the Stokes theorem. First of all, we postulate that each \( B \) contains two connected (but obviously disjoint) pieces of the corresponding copy of \( L \), say \( K' \) and \( C' \). For simplicity, suppose that only the pieces \( C' \) are different, whereas \( K' \) are identical for all \( L_k \). In order to facilitate comparison of the different situations one should assume some standard position of \( C' \) in each \( B \). It is a two-dimensional surface \( D \) inside each \( B \) swept out by \( C' \) that differs \( C \) for different copies of \( L \). Analytically, the difference is expressed by the following integral

\[
\Delta S = \frac{1}{2} \int_{C' = \partial D} \bar{\eta} d_A \eta = \frac{1}{2} \int_D (d_A \bar{\eta} d_A \eta + \bar{\eta} F \eta), \tag{3.1}
\]

where the Stokes theorem has been used. Now we suppose that the intersection of \( D \) and \( K' \) is exactly in \( k - 1 \) points for each \( L_k \). As the theory is topological (metric-independent) we can deform the links freely without any influence on the path integral, provided we avoid intersections, which constitute a kind of singularities. The contributions coming from the intersections can be easily calculated, and they are the only analytical trace of our construction. Since the dimension of \( K' \) is \( d - 2 \), and the dimension of \( D \) is 2, the dimension of \( K' \cap D \) is, in general position, 0. In fact, we can assume

\[
K' \cap D = \begin{cases} \emptyset, & \text{for } k = 1, \\ \bigcup_{l=1}^{k-1} P_l, & \text{for } 2 \leq k \leq N, \end{cases} \tag{3.2}
\]

where \( P_l \) are intersection points. Now, we can perform the following substitution in (3.1)

\[
F^a \longrightarrow -2i \lambda \star \frac{\delta}{\delta B^a}, \tag{3.3}
\]

provided the order of terms in (2.17) is such that the functional derivative (3.3) can act on \( S_{BF} \) yielding the curvature \( F \). We can observe that (3.3) is a translation operator
in a function space. Functionally integrating by parts with respect to $B$ we obtain, as a result of a translation in the last term of $S_\Omega$, the quasi-monodromy “operator”

$$M = \exp \left[ \frac{\lambda}{2i}(\bar{\theta}t_1^a \theta)(\bar{\eta}t_2^a \eta)(x_i) \right], \quad (3.4)$$

for each $\mathcal{P}_l$ with coordinates $x_l$ (summation with respect to $a$).

Since there are other terms entering the path integral that could possibly give a contribution to (3.4), one should explain why this is not the case. In particular, only the “potential” term in (3.1) is expected to give a contribution to the path integral (exactly in the form (3.4)). To understand this fact we can think of a lattice formulation of our theory, where $D$ is a plaquette. As usual, $F$ should live on plaquettes, whereas $A$ should live on bonds. Since $K$ pierces the plaquette $D$ rather than a bond there should be no contribution from the “kinetic term”, which just resides on the bond. One can also observe that the field $B$ enters as well the gauge fermion $\Psi$, and therefore $\Psi$ could be affected by (3.3), but due to the theorem of Batalin and Vilkovisky on the $\Psi$-independence of $Z$ this change of $\Psi$ is inessential. Summing up, the whole contribution coming from the intersection(s) is contained in the monodromy operator (3.4).

Now, it is necessary to calculate matrix elements of $M$. To this end, we should be provided with appropriate scalar products. Obviously, these scalar products should be already contained in the theory rather than given from outside. Tracing the standard method of the derivation of path-integral representation of a partition function from the operator formulation we can easily decipher the form of the scalar products from the form of the “kinetic” terms. Namely, there are the following three (normalized) non-trivial scalar products for the matter fields:

$$\langle f \cdot g \rangle_{\bar{\eta} \eta} \equiv \frac{1}{2\pi i} \int f g e^{i\bar{\eta} \eta} d\bar{\eta}d\eta,$$  \hspace{1cm} (3.5a)

$$\langle f \cdot g \rangle_{\bar{\theta} \omega} \equiv \frac{1}{2\pi i} \int f g e^{i\bar{\theta} \omega} d\bar{\theta}d\omega,$$  \hspace{1cm} (3.5b)

$$\langle f \cdot g \rangle_{\bar{\omega} \theta} \equiv \frac{1}{2\pi i} \int f g e^{i\bar{\omega} \theta} d\bar{\omega}d\theta.$$  \hspace{1cm} (3.5c)
The matrix elements of $M$ are given by the following fourfold scalar product

$$M = (\bar{\eta} \bar{\omega} \cdot M \cdot \omega \eta) = \exp \left( \frac{\lambda}{2i} t_1^a \otimes t_2^b \right).$$  \hspace{1cm} (3.5)$$

Interestingly, the algebraic form of $M$ is identical to the standard, three-dimensional one. Since our approach applies to three dimensions as well, we have obtained, as a by-product of our analysis, the most general (where the parameter $\lambda$ is unconstrained) form of the quasi-monodromy matrix.

Having a particular compact semi-simple Lie group $G$ and a pair of irreducible representations $R_1(G)$ and $R_2(G)$ we can automatically yield a corresponding skein relation, following the recipe given in. For example, for the fundamental representations of $\text{SU}(N)$ we obtain the Homfly polynomial, which specializes to the Jones polynomial after putting $N = 2$. For $\text{SO}(N)$ we obtain the Dubrovnik-Kauffman polynomial, whereas non-fundamental representations provide us, as a rule, with the Akutsu-Wadati polynomials.

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**Appendix**

(1) A Dirac-delta $n$-form $\delta(\mathcal{N})$, $0 < n < d$, where $\mathcal{N}$ is a $(d-n)$-dimensional submanifold of a $d$-dimensional manifold $\mathcal{M}$, is defined through the relation

$$\int_{\mathcal{M}} (...) \delta(\mathcal{N}) \equiv \int_{\mathcal{N}}(...),$$  \hspace{1cm} (A1.1)

where “…” denote some $(d-n)$-form, i. e. $\delta(\mathcal{N})$ constraints integration on $\mathcal{M}$ to the submanifold $\mathcal{N}$. If we parametrize $\mathcal{N}$ with \{x^{n+1}, x^{n+2}, \ldots, x^d\} in a locally cartesian
coordinate system \( \{x^1, x^2, \ldots, x^n, x^{n+1}, \ldots, x^d \} \), \( \delta(N) \) will assume the following simple explicit local form

\[
\delta(N) = \delta(x^1) \delta(x^2) \ldots \delta(x^n) \epsilon_{12\ldots n} dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n. \tag{A1.2}
\]

From (A1.2) it directly follows that

\[
\delta^2(N) = 0. \tag{A1.3}
\]

The square of the Dirac-delta \( n \)-form \( \delta(N) \) is zero in a meaningful way because it vanishes already in a regularized version due to antisymmetry of differential forms. Presumably, a mathematically more rigorous description in terms of de Rham’s currents would also be possible.\(^{14}\)

(2) In the framework of the antibracket-antifield formalism of Batalin and Vilkovisky, the antibracket of arbitrary two functions \( X \) and \( Y \) on the extended phase space of variables \( \{\phi^I, \phi^*_I\} \) is defined according to Ref. 10 as

\[
(X, Y) \equiv \frac{\delta_r X}{\delta \phi^I} \frac{\delta_l Y}{\delta \phi^*_I} - \frac{\delta_r X}{\delta \phi^*_I} \frac{\delta_l Y}{\delta \phi^I}, \tag{A2.1}
\]

where, in the condensed notation, \( I = (i, x) \) denotes discrete as well as continuous indices, and \( r \) (\( l \)) means right (left) derivative. We assume that form degrees of antifields are determined by duality. In such a convention, the explicit form of (A2.1) is given by

\[
(X, Y) \equiv \int \left( \frac{\delta_r X}{\delta \phi^I(z)} * \frac{\delta_l Y}{\delta \phi^*_I(z)} - \frac{\delta_r X}{\delta \phi^*_I(z)} * \frac{\delta_l Y}{\delta \phi^I(z)} \right) dz, \tag{A2.2}
\]

where the dualizing density star operator “\(*\)” does not contain the metric tensor, and it takes into account the total degree rather than the form one. One should notice that the formula (A2.2) possesses a good geometrical meaning because the whole integrand is a \( d \)-form.
The condition of non-degeneracy of a solution of the master equation assumes, in our convention, the form
\[
\delta_l\delta_r S|_{\phi^*=0} = *Z(x, y), \tag{A2.3}
\]
where \(Z(x, y)\) is an integral kernel of gauge transformation, and the functional differentiation is performed with respect to an appropriate pair of (anti)ghosts.\(^{10}\)

(3) It is not absolutely necessary, but technically very convenient and in accordance with tradition, to introduce a BRST operator \(s\). It follows from the definition of \(s\),

\[
(X, S) \equiv sX, \tag{A3.1}
\]
where \(S\) satisfies the master equation, that in order to preserve the self-consistency, we should put

\[
s\phi^I = *\frac{\delta_l S}{\delta \phi^I} \equiv *\delta S, \quad s\phi^*_I = -*\frac{\delta_l S}{\delta \phi^I} \equiv -*\delta S. \tag{A3.2}
\]

Our BRST operator \(s\) is nilpotent automatically. This fact is a consequence of the Jacobi identity

\[
s^2 X \equiv ((X, S), S) \equiv \frac{1}{2}(-)^{\text{Deg}X}((S, S), X) \equiv \frac{1}{2}(-)^{\text{Deg}X}(sS, X) = 0, \tag{A3.3}
\]
with the total degree “Deg” defined as

\[
\text{Deg}X \equiv \text{deg} X + \text{gh} X, \tag{A3.4}
\]
where “deg” is the ordinary form degree, and “gh” is the sum of all four ghost numbers.

There are also some other useful for our further discussion identities, e. g.

\[
\{s, d\} \equiv 0,
\]

\[
s(XY) \equiv sXY + (-)^{\text{Deg}X}XsY,
\]

\[
XY \equiv (-)^{\text{Deg}X\text{Deg}Y}XY. \tag{A3.5}
\]
References


