The mechanism of false vacuum decay is presently well understood[1]. Quantum-mechanical tunnelling from a false vacuum to a true one proceeds via formation of a true vacuum bubble inside the false vacuum background. Energy gained by forming the true vacuum bubble less the wall energy is transferred to the wall kinetic energy, driving the wall expand asymptotically to the speed of light. The effects of gravity on the false vacuum decay we studied by Coleman and DeLuccia[2]. In particular, they found that a Minkowski false vacuum cannot decay into an anti-de Sitter (AdS) true vacuum unless the matter vacuum energy difference $\Delta V$ is sufficiently large. The result can be generalized to vacuum decay between two AdS vacua with corresponding matter vacuum energies $V_{\text{true}} < V_{\text{false}} < 0$. 
must be satisfied in order for a true vacuum bubble to form. Here, \( \kappa \equiv 8\pi G_N \) and \( \sigma \) denotes the bubble wall energy per unit area.

The study of false vacuum decay in \( N = 1 \) supergravity theories is interesting on its own. However, when the matter fields are associated with the low energy scale, the splitting of the non-degenerate supersymmetric vacua is generically small; i.e., \( \sim \kappa \). The effective Lagrangian of superstring vacua is described by an \( N = 1 \) supergravity as well. However, in superstring theories, other fields, e.g., dilaton and moduli, and gravity are on an equal footing, so the effects of gravity can yield distinctly new features. In particular, when the vacuum expectation values (VEV’s) of the moduli fields are of the order of \( M_{pl} \), the non-perturbatively induced potential of moduli fields is significantly modified by gravity. In this case \( \Delta V = \mathcal{O}(V) \) and prior to this study[3] not much could be said about the stability of such superstring vacua.

We point out that quantum tunnellings between any supersymmetric vacua in \( N = 1 \) supergravity are absolutely impossible[3] by establishing a Bogomol’nyi bound for the bubble wall energy density. In particular, vacuum decay from a supersymmetric Minkowski vacuum to an AdS supersymmetric vacuum is not possible at all. This in turn applies to the case of tunnelling in the moduli sector of string theory when the non-perturbative moduli potential is turned on.

We consider four-dimensional \( N = 1 \) local supersymmetry with one chiral superfield \( T \). The bosonic sector of the four-dimensional \( N = 1 \) supergravity Lagrangian reads

\[
e^{-1}L = -\frac{1}{2\kappa}R + K_{TT}g^{\mu\nu}\nabla_\mu T \nabla_\nu \bar{T} - V(T, \bar{T})
\]

in which the supergravity scalar potential \( V(T, \bar{T}) \) is defined as

\[
V \equiv e^{\kappa K}|K^{TT}|D_TW|^2 - 3\kappa|W|^2
\]

here \( e = |\det g_{\mu\nu}|^{\frac{1}{2}}, K(T, \bar{T}) \) is Kähler potential and \( D_TW \equiv e^{-\kappa K}(\partial_T e^{\kappa K}W) \). Newton’s constant appears consistently in the combination \( \kappa = 8\pi G_N \). Supersymmetry preserving minima of the scalar potential (3) satisfy \( D_TW = 0 \). This in turn implies (see eq. (3)) that the supersymmetry preserving vacua have either zero vacuum energy (Minkowski space-time) when \( W = 0 \), or constant negative vacuum energy \( -3\kappa e^{\kappa K}|W|^2 \) (AdS space-time) when \( W \neq 0 \). Thus, the tunnelling process between supersymmetric vacua corresponds to tunnelling either between Minkowski and AdS space-times or between two AdS space-times of different cosmological constants.

In superstring theories, the scalar field \( T \) e.g., corresponds to a modulus field arising from compactification, and its non-perturbatively induced superpotential \( W \) is assumed to reflect the underlying target space modular invariance[5, 6] under the \( PSL(2, \mathbb{Z}) \) duality transformations:

\[
T \rightarrow \frac{aT - ib}{icT + d} , \ ad - bc = 1 , \ {a, b, c, d} \in \mathbb{Z}.
\]

In this case[6], \( K = -3\kappa \ln[(T + \bar{T})] \). The superpotential \( W \) is a modular function of weight \(-3\) under \( PSL(2, \mathbb{Z}) \) defined in the fundamental domain \( \mathcal{D} \) of the \( T \)-field. One of the simplest choices for a modular invariant superpotential is \( W(T) = j(T) \eta^{-6}(T) \) where \( \eta(T) \) is the Dedekind function: a modular function of weight \( 1/2 \) and \( j(T) \) is a modular-invariant function.[7] In this case[6] the scalar potential for the \( T \) field has two supersymmetric minima at \( T = 1 \ (V < 0) \) and \( T = \rho \equiv e^{i\pi/6} \).
The bubble formation is studied by using an $O(4)$ symmetric Ansatz\cite{1} for a bounce solution interpolating between the true(AdS) and false(Minkowski or AdS) vacuum. The metric for this Ansatz in Euclidean space is\cite{2}

$$ds^2 = d\xi^2 + R(\xi)d\Omega_3^2 = B(\xi')(d\tau^2 + dr^2 + r^2d\Omega_2^2)$$ (5)

where $\xi$ is the Euclidean radial distance from an arbitrary origin and $\xi^2 = \tau^2 + r^2$. The second line of (5) follows after a redefinition of the radial coordinate $\xi$ into $\xi'$: $d\xi'/\xi' = d\xi/\sqrt{R(\xi)}$. Note that only with coordinate $\xi'$ we can clearly attach the meaning to $r = \sqrt{x^2 + y^2 + z^2}$ as the radius of the two sphere. The classical evolution of the materialized bubble is described by the Wick rotation back to Minkowski space-time, i.e., by changing the Euclidean time $\tau$ back to Minkowski time $t$.

It is most convenient to study the energy density of the bubble wall at the moment of its actual formation, i.e. at Euclidean time $\tau = 0$. At this moment the bubble is instantaneously at rest\cite{1,2}; the time derivative of the matter field $\partial_T T \equiv (\tau/\xi')\partial_\xi T$ and the metric coefficient $\partial_\tau B \equiv (\tau/\xi')\partial_\xi B$ both vanish at $\tau = t = 0$. As it turns out\cite{3} for the minimal energy configuration of the bubble, the metric coefficient $B$ and the matter field $T$ satisfy first order differential equations. This in turn justifies the choice that at $\tau = t = 0$ the matter field $T$ and the metric coefficient $B$ of the minimal energy configuration are only functions of $r = \sqrt{x^2 + y^2 + z^2}$, i.e., the radius of the two-sphere. At the moment of the actual bubble formation one is thus working with a specific spherically symmetric metric Ansatz:\footnote{We calculate explicitly in the Lorentzian instead of Euclidean signature. The conclusion about the positive minimal energy stored in the bubble wall or the minimal action theorem for the bounce solution are equivalent since time independence is assumed throughout.}

$$ds^2 = B(r)(dt^2 - dr^2 - r^2d\Omega_2^2)$$ (6)

For the purpose of studying the minimal energy configuration of the bubble wall energy density we introduce supersymmetry charge density: \footnote{Our conventions are the following: $\gamma^\mu = \epsilon^a_\mu \gamma^a$ where $\gamma^a$ are the usual Dirac matrices satisfying $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$; $\epsilon^a_\mu \epsilon^b_\nu = \delta^a_b$; $a = 0,..,3$; $\mu = t, x, y, z$; $\gamma^1 = \gamma^t$; $(+, -, - , -)$ space-time signature.}

$$Q[\epsilon'] = 2\int_{\partial\Sigma} d\Sigma_{\mu\nu}(\epsilon'\gamma^{\mu\nu}\psi_\lambda)$$ (7)

where $\Sigma$ is a space-like hypersurface enclosing the bubble wall. Here, $\epsilon'$ is a commuting Majorana spinor and $\psi_\rho$ is the spin 3/2 gravitino field. Taking a supersymmetry variation of $Q[\epsilon']$ with respect to another commuting Majorana spinor $\epsilon''$ yields

$$\delta_\epsilon Q[\epsilon'] = \{Q[\epsilon'], Q[\epsilon]\} = \int_{\partial\Sigma} N^{\mu\nu} d\Sigma_{\mu\nu}$$

$$= 2\int_{\Sigma} \nabla_\nu N^{\mu\nu} d\Sigma_\mu$$ (8)

where we introduced the generalized Nester's form\cite{4}

$$N^{\mu\nu} = \tilde{\epsilon}'\gamma^{\mu\nu}\nabla_\rho\epsilon$$ . (9)

The supercovariant derivative appearing in Nester's form is

$$\nabla_\rho\epsilon \equiv \delta_\epsilon\psi_\rho = [2\nabla_\rho + ie^{aK/2}(WP_R + \bar{W}P_L)\gamma_\rho \equiv \text{Im}(K_T\partial_\rho T)\gamma_\rho] \epsilon$$ (10)
follows from Stoke’s law.

We can describe the energy stored in the bubble wall (or equivalently the minimal action stored in the wall of the bounce solution) using a thin wall approximations. Such an approximation is valid in the case when the radius $R$ of the bubble is much larger than its thickness $2\Delta R$, and becomes exact when $R \to \infty$. The boundary condition on the metric coefficient is $B(r = R) = 1$, which serves as a suitable choice for normalizing the metric. This in turn defines the surface of the large radius bubble to be $4\pi R^2$. In the thin wall approximation, in the region with $r \sim R$, the metric coefficients do not change appreciably over the range of the domain wall. The boundary $\partial \Sigma$ are two boundaries of two-sphere, one at $R - \Delta R$ and the other at $R + \Delta R$, with the constraint that $R \gg 2\Delta R$. In this limit, the spherical domain wall approaches the planar domain wall.

Analysis of the surface integral in (8) yields two terms: (1) The ADM mass of the configuration, denoted $4\pi R^2 \cdot \sigma$ and (2) The topological charge, denoted $4\pi R^2 \cdot C$. Here, $\sigma$ and $C$ denote the ADM mass density and the topological charge density of the bubble wall. The minimum topological charge density, which corresponds to a supersymmetric configuration, is given by

$$|C| = 2 \frac{1}{\sqrt{3}} \left| (\zeta |W e^{\frac{\kappa K}{2}}|)_{r = R + \Delta R} - (\zeta |W e^{\frac{\kappa K}{2}}|)_{r = R - \Delta R} \right|$$

where $\zeta = \pm 1$. $\zeta_{R - \Delta R} = -\zeta_{R + \Delta R}$ if $W$ goes through 0 somewhere as $r$ traverses an interval $(R - \Delta R, R + \Delta R)$ and $\zeta_{R - \Delta R} = \zeta_{R + \Delta R}$ otherwise (see discussion of the minimal energy solution in the following.) The second line in eq. (11) follows from the properties of the supersymmetric vacua, namely, as we have shown earlier for the supersymmetric minimum, $V = -3\kappa |W|^2 e^{\kappa K}$.

The volume integral in eq. (8) can be shown to be positive definite. This in turn implies

$$\sigma \geq |C|$$

which is saturated if and only if $\delta Q[e] = 0$. It can be shown that saturation of this bound implies that the bosonic background is supersymmetric and $\delta \psi_\mu = 0$ and $\delta \chi = 0$. For the tunnelling from Minkowski ($V_{false} = 0$) or AdS ($V_{false} \neq 0$) to AdS ($V_{true} \neq 0$) inequality (12) yields:

$$\frac{3}{4} \kappa \sigma^2 \geq 3 \kappa (|W e^{\kappa K}|_{R + \Delta R} - |W e^{\kappa K}|_{R - \Delta R})$$

$$= (\sqrt{-V_{true}} - \sqrt{-V_{false}})^2.$$  

Inequality, (13), is the central result. Notice that the positive energy bound (13) for the minimal energy density of the thin domain wall has precisely the opposite inequality sign as eq. (1) of the Coleman-DeLuccia bound for the existence of a bubble instanton. In other words, vacuum tunnelling is not allowed in $N = 1$ supergravity since the available vacuum energy difference is not sufficient to materialize the tunnelling bubble. For the marginal case when the bound (13) is saturated, such a domain wall configuration is supersymmetric. However, even in this case, since the matter

† In the case that the bubble forming between the two AdS space-times has $W$ going through 0 somewhere in between, the supersymmetric configuration satisfies $3/4 \kappa \sigma^2 = 3 \kappa (|W e^{\kappa K}|_{R + \Delta R} + |W e^{\kappa K}|_{R - \Delta R}) = (\sqrt{-V_{true}} + \sqrt{-V_{false}})^2$ and so the Coleman-DeLuccia bound is never saturated; i.e. tunnelling is super-suppressed. Therefore, we do not discuss this case further.
wall energy plus the gravitational energy of the AdS space time inside the bubble wall, the result is an infinitely large radius of the critical tunnelling bubble as was shown explicitly in [3].

The solution in the limit $R \to \infty$ can be found by examining the equations of motion, i.e., the first order Bogomol’nyi equations $\delta \chi = 0$ and $\delta \psi_{\mu} = 0$ which are necessary conditions for a supersymmetric bosonic configuration. Technical details for the derivation of these equations in the spherical frame are presented in Ref. [3]. The upshot of the analysis is that the constraints on the spinor arising from the Bogomol’nyi equations are incompatible unless the wall is located strictly at $R \to \infty$. Indeed, this corresponds to a planar static supersymmetric domain wall [8, 9],† which can be identified with the vacuum bubble in the limit as the bubble radius of curvature $R \to \infty$.

In this limit the equations of motion for the matter field $T(r)$ and the metric coefficient $B(r)$ can be written as:

$$\partial_r T(r) = \zeta \sqrt{B} |W| e^{\kappa K/2} K^{TT} \frac{D_T W}{W},$$
$$\partial_r \left( \frac{1}{\sqrt{B}} \right) = \kappa \zeta |W| e^{\kappa K/2}. \quad (14)$$

The constraint on the phase of $W$ along with the first of (14) imposes the geodesic equation [8]:

$$\text{Im} \left( \partial_r T \frac{D_T W}{W} \right) = 0. \quad (15)$$

This result implies that in the limit $\kappa \to 0$, $W(r)$ lies in the $W$ plane on a straight line that extends through the origin. Explicit solutions in $R \to \infty$ limit are identical to those of the planar domain wall.

In the case of a planar domain wall the wall can be put in the $(x, y)$ plane. $T(z)$ and $B(z)$ satisfy the same eqs. (14), with $r$ being replaced by $z$. Explicit solutions for $T(z)$, $B(z)$ have been obtained in specific cases in Ref. [9].

The wall interpolating between a Minkowski vacuum ($W_{z=+\infty} = 0$) and an AdS vacuum ($W_{z=-\infty} \neq 0$) is most interesting. These vacua have a distinct space-time topology; Minkowski topology is $\mathbb{R}^4$ and AdS topology $S^1(time) \times \mathbb{R}^3(space)$. The metric coefficient is then of the form:

$$B(z) \to 1, \quad z \to \infty;$$
$$B(z) \to \frac{3}{\kappa |V_{AdS}| z^2}, \quad z \to -\infty, \quad (16)$$

where $V_{AdS} = -3\kappa |W|^2 e^{\kappa K} |z|^{-\infty}$.

In such a domain wall background the geodesic motion of massive particles in the $z$ direction† satisfies the world-line equation [9]:

$$\left( \frac{dz}{dt} \right)^2 + \frac{A}{\epsilon^2} = 1. \quad (17)$$

with $\epsilon$ being energy per mass.

A convenient way to understand massive particle motion is to consider a particle with a given initial coordinate velocity $v_0$ at some coordinate $z_0$; from (17) $\epsilon$ for such a particle is

† Examples of global supersymmetric domain walls are given in Refs. [10, 11].

‡ The metric is invariant under $x, y$ boosts and thus without loss we can move to an inertial frame in which there is no motion in these directions.
potential \( V(z) \equiv (1 - v^2) - A(z_0)(1 - v_0^2) \). Again, points where \( V(z) = 0 \) are turning points.

For particles incident upon the wall from the Minkowski side, passage through to the AdS side is always allowed. However, the reverse motion requires the initial velocity to satisfy \( v^2 > 1 - A(z_o) \); otherwise there is a turning point and the particle returns to the AdS side.

One can understand the repulsive nature of these space-times on the AdS side by calculating the force on a test particle which has a fixed position \( z \) (fiducial observer). This force can be obtained through the geodesic equation and yields the magnitude of the acceleration \( |\alpha| \equiv |\int f_\alpha f^\alpha|/m^2 = (1/2 A(z_0)^2) = (\kappa|\mathcal{W}|e^{2K})^2 \). Away from the wall, the proper acceleration has the constant magnitude.

In this region, integration of (17) yields the hyperbolic world line for freely falling test particles \( z^2 - t^2 = a^{-1}e^{-2} \), i.e. they are Rindler particles. On the other hand on the Minkowski side of the walls, free test particles experience no gravitational force even though there is an infinite object nearby.

One can understand the no-force result for the particles living on the Minkowski side of the walls through the formalism of singular hypersurfaces.[12] A straightforward calculation† yields a negative effective gravitational mass/area due to the wall whereas AdS has exactly the opposite positive gravitational mass. Thus the observer on the Minkowski side of the wall does not feel any gravitational force.

In conclusion we have studied the issue of false vacuum decay in supergravity theory. We have found that the supersymmetric vacua are stable against false vacuum decay into other supersymmetric vacua nonperturbatively, i.e., to all orders in \( \kappa = 8\pi G_N \) expansion. For example, the AdS supersymmetric vacuum is not connected via quantum tunnelling to a supersymmetric Minkowski vacuum. The results are exact in \( G_N \) and complete perturbative analysis (in the leading order of \( G_N \)) by Weinberg.[14]

The technique to obtain the minimal surface energy of the vacuum bubble and the consequent absence of tunnelling between the supersymmetric vacua is complementary to the positive energy theorems for supersymmetric AdS vacua.[15] Namely, the minimal total energy of supersymmetric vacua is absolutely zero.[15] Such vacua are therefore degenerate and consequently there can be no tunnelling. This conclusion complements and conforms to the Bogomol’nyi bound of eq. (13) for the minimal energy density of the bubble wall. in this sense the problem of vacuum degeneracy in a locally

As a consequence, the degeneracy of supergravity vacua yields static domain walls.[8, 9] For example, there is a domain wall configuration interpolating between a supersymmetric Minkowski vacuum, whose topology is \( \mathbb{R}^4 \), and a supersymmetric AdS vacuum, whose topology is \( S^1(\text{time}) \times \mathbb{R}^3(\text{space}) \), thus yielding a meaning to vacuum degeneracy for such topologically distinct vacua as supersymmetric Minkowski and AdS vacua. Study of interesting space-time effects, e.g. description of geodesically complete space-times and existence of Cauchy horizons, in such domain wall backgrounds is underway.[16]

The work presented here has been done in collaboration with S. Griffies and S.-J. Rey. I would like to thank them as well as R. Davis and H. Soleng for discussions. The research was supported in part by the U.S. DOE Grant DE-AC02-76-ERO-3071, and by a junior faculty SSC fellowship.


† See [13] for an example of this formalism applied to a planar geometry.