Isospectral flow in Loop Algebras

Quasiperiodic flows and finite-gap solutions of the sine-Gordon equation

$$\partial^2 u \partial x^2 - \partial^2 u \partial t^2 = m^2 \sin(u), \ m \in \mathbb{R}.$$ 

have been studied by many authors and derived in a variety of ways KK, C, M, FM, Da, AA, P, Sm. Real solutions for two gaps were identified in DN and for any number of gaps in Da using Baker-Akhiai functions. In FM, AA the flow was explicitly linearized on the Jacobi variety of a hyperelliptic Riemann surface. A Liouville generating function leads to the linearization in AA, where quasiperiodic solutions of equation 1.1 are determined in terms of the integrated flows of a completely integrable finite-dimensional Hamiltonian system. However, the recovery of real solutions remains incomplete in FM and AA and the unifying rôle of the loop algebra $su(2)$ or the Adler-Kostant-Symes (AKS) theorem A, K, Sy does not apply.

In this paper we shall obtain such solutions using a general approach based upon moment maps from finite dimensional symplectic vector spaces into loop algebras as developed in AHP, AHH1, AHH2. AKS theorem and the Liouville-Arnold integration method in the context of loop algebras are central to this approach.

As shown in AHP, finite dimensional coadjoint orbits $O$ in loop algebras may be parametrized by symplectic vector spaces via such moment map embeddings. Equations of the type 1.1 are recovered as compatibility conditions for a pair of Hamiltonian Lax equations on $O$, as given by the AKS theorem. The flow on $O$ generates, as usual, a flow of line bundles defined on the underlying invariant spectral curve. Divisor coordinates defining these line bundles give a system of Darboux coordinates on $O$ according to the general scheme developed in AHH2. A Liouville generating function yields the canonical transformation to linear coordinates on the Jacobi variety of the spectral curve via the Abel map.

The new element here, relative to the generic case AHH2, is the passage to a hyperelliptic curve which is not exactly the spectral curve of the flow in $O$, but rather a quotient by an involutive automorphism extended by an additional two branch points. The quotienting is necessary because the structure of Lax pairs implies that the flow is on the twisted loop algebra $su(2)^*$, while the additional branch points are necessary to raise the genus of the curve so as to identify the Jacobi variety with the Liouville-Arnold torus. It also becomes clear in this approach that real solutions of 1.1 are equivalent to the choice of a submanifold in $O$ corresponding to a coadjoint orbit of the twisted loop algebra $su(2)^+$.

2. Darboux Coordinates for the Sine-Gordon Equation

First we apply in detail the moment map embedding method developed in AHP to the case of the loop algebra $su(2)$, in order to obtain suitable “Cartesian” coordinates on rational coadjoint orbits. Consider the vector space $\mathbb{C}^{2 \times 2}$ consisting of $2 \times 2$ complex matrices $F$. Let $F_i = (x_i, y_i), \ i = 1, 2$ be the rows of $F$. A real symplectic structure on $\mathbb{C}^{2 \times 2}$ is given by

$$\Omega = dF_2 \wedge \overline{F}_1^T + dF_1 \wedge \overline{F}_2^T.$$  \hspace{1cm} (2.1)

Now consider the submanifold

$$M = \{F \in \mathbb{C}^{2 \times 2} : F_1 \neq 0, \ F_2 \neq 0, \ F_2 \overline{F}_1^T = 0\}. $$  \hspace{1cm} (2.2)

This is not a symplectic manifold, but on $M$ we have an action of $\mathbb{C}^*$ given by

$$h(F) = (h) F_1$$

Now, considering $\mathbb{C}^{(2N) \times 2}$ as the Cartesian product of $N$ copies of $\mathbb{C}^{2 \times 2}$, with coordinates $(F_{2i-1})$

Let $su(2)$ be the loop algebra consisting of differentiable maps from a circle $K$ centered at the origin in the complex $\lambda$–plane into $su(2)$ with Lie bracket evaluated, as usual, pointwise. There is a splitting into a direct sum of subalgebras

$$su(2) = su(2)^+ \oplus su(2)^-,$$ \hspace{1cm} (2.10)
where $su(2)^+$ is the subalgebra of loops in $su(2)$ which extend holomorphically inside $K$, while $su(2)^-$ is the subalgebra of loops $X(\lambda)$ which extend holomorphically outside $K$ and normalized by the condition $X(\infty) = 0$. The Lie algebra $su(2)$ (resp., $su(2)^-$) is densely embedded into $su(2)^*$ (resp., $su(2)^{++}$) via the nondegenerate, ad-invariant bilinear form on $su(2)$ given by

$$<X,Y> := \int_K (X(\lambda)Y(\lambda)) d\lambda, \quad X,Y \in su(2). 2.11$$

Henceforth no notational distinction will be made between elements of $su(2)^*$ (resp., $su(2)^{++}$) and elements of $su(2)$ (resp., $su(2)^-$).

Fix $N$ complex numbers, $i = 1, \ldots, N$ in the interior of $K$. We define an injective moment map $\tilde{J} : M/ (C^*)^N \to \mathcal{W}$, which is locally identified with $W$, by

$$\tilde{J}(\varphi_1, \gamma_1, \ldots, \varphi_N, \gamma_N) = \lambda \sum_{i=1}^N (-)^i \gamma_i \varphi_i \bar{\gamma}_i^2$$

The fact that this is really a Poisson map with respect to the Lie–Poisson structure and that, viewed as a moment map, it generates a Hamiltonian action of the corresponding loop group $SU(2)$ on $W$, follows from the general results developed in AHP. It is also easily verified that the image of this moment map coincides with a coadjoint orbit in $su(2)^{++}$. To obtain real solutions of the sine-Gordon equation, however, we must restrict to a submanifold of $W$ whose image lies in the dual of the “twisted” subalgebra $su(2)$, consisting of fixed points of the involutive automorphism

$$\sigma(X)(\lambda) = (10)$$

In order that condition 2.13 be satisfied by the image of $\tilde{J}$, the $\gamma$’s have to come in pairs of opposite sign. There are two possibilities: either has a nonzero real part, in which case we need both $-$ and $+$, or it is purely imaginary. We may reorder these constants so that $\alpha_{i+p} = -\bar{\alpha}_i$, for the $\gamma$’s with a real part, and $\alpha_j = \sqrt{-1}$, for those purely imaginary. On $W$ we must then impose the further constraints (cf. AHP, sec. 5)

$$\varphi_{i+p} \bar{\gamma}_{i+p} = \varphi_i \bar{\gamma}_i, \quad \bar{\gamma}_{i+p}^2 = -\bar{\gamma}_i^2, \quad \varphi_{i+p}^2 = -\varphi_i^2, \quad i = 1, \ldots, p 2.14a$$

In terms of the reduced coordinates, the restriction $\tilde{J}$ of $\tilde{J}$ to $\tilde{W}$ is

$$\tilde{J} = 2\lambda (b)(\lambda) c(\lambda)$$

The flow leaves invariant the spectral curve with affine part given by

$$P(\lambda, z) = \text{det}((\lambda) - zI) = 0.2.25$$

Expanding 2.25 and using the fact that the rank of the residues of $\tilde{J}$ is equal to 1, we may write

$$P(\lambda, z) = z^2 + a(\lambda) P(\lambda) = 02.26a$$

[Remark. Equation 1.1 may equivalently be viewed as the compatibility conditions for the $x$ and $t$ flows determined by the two Hamiltonians $H_x := -H_\xi - H_\eta$, $H_t := H_\xi - H_\eta$.]

3. Quasiperiodic Solutions for the Sine-Gordon equation

The hyperelliptic spectral curve 2.25 has genus $g = 2N - 1$. However, it is invariant under the involution $(z, \lambda) \mapsto (z, -\lambda)$, and hence is a two-sheeted covering of the hyperelliptic curve $C'$, with genus $N - 1$, whose affine part given is by

$$z^2 + a(E) \tilde{P}(E) = 0, 3.1$$
where $\tilde{P}(\lambda^2) = P(\lambda)$ and $\tilde{a}(\lambda^2) = a(\lambda)$, $\lambda^2 = E$. We also define functions $\tilde{b}, \tilde{c}$ by $\tilde{b}(E) = b(\sqrt{E})$, $\tilde{c}(E) = c(\sqrt{E})$. In order to apply the Jacobi inversion method we shall need a hyperelliptic curve $\tilde{C}$ of genus $N$ with affine part given by

$$\tilde{z}^2 + E\tilde{a}(E)\tilde{P}(E) = 0.3.2$$

This is obtained from $C'$ by setting $\tilde{z} = z\lambda$, which adds branch points at $E = 0$ and $E = \infty$. Following a general method for the introduction of “spectral Darboux coordinates” developed in AHH2, we define a divisor of degree $N$ with coordinates $(E_\mu, \zeta_\mu)_{\mu=1,\ldots,N}$. The $E_\mu$ are given by the equation

$$\tilde{c}(E_\mu) - 1 = 0, 3.3$$

or, equivalently

$$\sum_{i=1}^p \left( \tilde{\gamma}_i^{22} - E + \tilde{\alpha}_i\tilde{\beta}_i^2\tilde{\alpha}_i^2 - E \right) - \sqrt{-1} \sum_{j=2p+1}^N \tilde{\gamma}_j^{22} + E - 1 = -\prod_{\mu=1}^N (E - E_\mu)\tilde{a}(E). 3.4$$

These may be viewed as complex hyperelliptic coordinates if the $(\varphi_i, \gamma_i, i = 1, \ldots, p; \gamma_j, j = 2p + 1, \ldots)$ are interpreted as Cartesian coordinates on the submanifold of $M/(\mathbb{C}^*)^N$ determined by the constrains 2.14a,b. The canonically conjugate coordinates $\zeta_\mu$ are defined by

$$\zeta_\mu = \sqrt{-\tilde{P}(E_\mu)E_\mu\tilde{a}(E_\mu)} = -2\tilde{b}(E_\mu)\sqrt{E_\mu}, 3.5$$

i.e. by the eigenvalues of the matrix $N(\lambda)\lambda^2$ at $\lambda^2 = E_\mu$. Comparing with 2.30 we see that

$$e^{\sqrt{-1}\theta} = -\prod_{\mu=1}^N (-E_\mu)^3.6$$

**Proposition 3.1** The coordinates $(E_\mu, \zeta_\mu)_{\mu=1,\ldots,N}$ form a Darboux coordinate system on the coadjoint orbit passing through $N(\lambda)$. The corresponding symplectic form is

$$\omega_N = \sum_{\mu=1}^N dE_\mu \wedge d\zeta_\mu = -d\theta. 3.8$$

**Proof** Computing the differentials of the residues of 3.4 and summing up we find, using 3.5,

$$4 \sum_{i=1}^p (\varphi_i d\tilde{\gamma}_i - \gamma_i d\tilde{\varphi}_i) + 4 \sum_{j=2p+1}^N \gamma_j d\tilde{\gamma}_j = \sum_{\mu=1}^N \zeta_\mu dE_\mu = \theta. 3.9$$

[Remark: This result could also have been obtained as in AHH2, by computing implicitly the Lie-Poisson brackets of $(E_\mu, \zeta_\mu)$ on the coadjoint orbit passing through $N(\lambda)$.] The restriction of 3.8 to the invariant level sets $P_i = c_i =$ const., $i = 0, \ldots, N - 1$ is identically zero. This defines a Lagrangian submanifold. Hence in a neighborhood of $P_i = c_i$, the one-form $\theta = dS$ may be integrated on the leaves of the Lagrangian foliation as usual to yield the Liouville generating function

$$S(P_i, E_\mu) = \sum_{\mu=1}^N \int_{E_0}^{E_\mu} \sqrt{-\tilde{P}(E)E\tilde{a}(E)dE}, 3.10$$

where $E_0 \in \tilde{C}$ is a suitably chosen base point. Derivation of $S$ with respect to the $P_i$'s gives the conjugate coordinates $Q_i$, in terms of which the flows of the Hamiltonians in the ring generated by the $P_i$'s are linear
Hamilton’s equations for $H_\xi, H_\eta$ are thus integrated to give

$$\sum_{\mu=1}^{N} \int_{E_0}^{E_\mu} E^i \sqrt{-E\tilde{a}(E)} \tilde{P}(E) dE = C_i + 2\delta_{i,0} \xi - 2\delta_{i,N-1} \eta, 3.12$$

on the Jacobi variety of $\tilde{C}$.

If the holomorphic differentials

$$\omega_i = E^i \sqrt{-E\tilde{a}(E)} \tilde{P}(E) dE, 3.13$$

were normalized, the left hand side of 3.12 would just be the Abel map. Since the $\omega_i$ form a basis of holomorphic differentials for $\tilde{C}$, choosing a basis $(a_i, b_i)_{i=1,\ldots,N}$ of $H_1(\tilde{C}, \mathbb{Z})$ such that $a_i \cdot a_i = b_i \cdot b_i = 0$, $a_i \cdot b_j = \delta_{ij}$, the matrix $M$ of integrals over the $a$-cycles of the differentials $\omega_i$,

$$M_{ij} = \int_{a_i} \omega_j, 3.14$$

is invertible. Multiplying equation 3.12 by $M^{-1}$, the flow for equation 1.1 is linearized on the Jacobi-variety of the curve $\tilde{C}$ via the Abel map

$$A(p_1 + \ldots + p_N) = U\eta + V\xi + B, 3.12'$$

where $p_\mu \in \tilde{C}$ are the points with coordinates $(\xi, E_\mu)$, and $U, V, B$ are constant vectors in $\mathbb{C}^N$ obtained by applying $M^{-1}$ to the vectors with components $-2\delta_{i,N-1}, 2\delta_{i,0}$ and $C_i$, respectively, appearing in equation 3.12. It remains to explicitly compute $u(\xi, \eta)$ as given by equation 3.7 from this linear flow. Let $\Theta$ be the theta function corresponding to the hyperelliptic curve $\tilde{C}$ and $\kappa$, the Riemann constant. From 3.7 follow in a standard way (cf. GH, Du, AHH2 (Cor. 1.7)):

**Proposition 3.2**

$$u = -2\sqrt{-1}\ln \Theta(A(0) - U\eta - V\xi - B - \kappa)\Theta(A(\infty) - U\eta - V\xi - B - \kappa) + C, 3.15$$

where $C$ is an integration constant independent of $\eta$ and $\xi$.

Proof: As usual we consider $\tilde{C}$ as a two-sheeted branched cover of $P^1(\mathbb{C})$ with projection map $\pi: \tilde{C} \to P^1(\mathbb{C})$ and branch locus $B$. Let $\{a_i, b_i\}$, $i = 1, \ldots, N$ be a basis of $H_1(\tilde{C}, \mathbb{Z})$ with common length $p_0$. Let $\pi(a_i)$ and $\pi(b_i)$ be the projections to $P^1(\mathbb{C})$. We suppose that the $a_i$’s are chosen such that the winding numbers $n(\pi(a_i), 0)$ and $n(\pi(a_i), \infty)$ are zero and such that their intersection indices are given by $a_i \cdot a_j = b_i \cdot b_j = 0$, $a_i \cdot b_j = \delta_{ij}$. Consider a polygonization $\Delta$ of $\tilde{C}$ with respect to $(a_i, b_i)$. Assume that $F(E) = \Theta(A(E) - U\eta - V\xi - B - \kappa)$ is not identically zero on $\tilde{C}$ (which holds for generically chosen $B$) so that $E_\mu \neq 0, \infty$, $\mu = 1, \ldots, N$. On $\Delta$ choose integration paths $c_0$ (resp., $c_\infty$) from one of the representations of $p_0$ on $\Delta$ (i.e., one of the vertices of $\Delta$) to $0$ (resp., $\infty$). Cut along these paths to obtain a polygonization $\Delta$. On $\Delta$ the differential

$$\varphi = \ln(-E) d\ln F(E), 3.16$$

is well defined and meromorphic. It is easily computed that

$$\sum_{\mu=1}^{N} \ln(-E_\mu) = \oint_{\partial\Delta} \varphi, 3.17$$

whereas the right hand side of 3.17 yields (see e.g. GH)

$$\oint_{\partial\Delta} \varphi = \text{constant in } \eta \text{ and } \xi$$
Conclusions:

The form of the quasiperiodic solutions 3.15 agrees with that obtained by other authors (e.g., DN) who have studied the sine-Gordon equation by a variety of methods. The new element presented here is the placing of these solutions entirely within the framework of isospectral flows in loop algebras and the AKS theorem. The reality conditions and reductions related to the invariance of the spectral curve under involutions follow naturally in this approach from the use of the twisted loop algebra $su(2)$ and the linearization of the flow via the Abel map are seen as illustrative cases of the “spectral Darboux coordinate” method developed in AHH2.

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References


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