Exactly Solvable Potentials and Quantum Algebras

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Abstract

Self-similar potentials and corresponding symmetry algebras are briefly discussed.

The following scheme describes applications of quantum-algebraic structures within the context of ordinary quantum mechanical spectral problems. Let $T$ be an operator of the affine transformation on line

$$Tf(x) = \sqrt{q}f(qx + a), \quad x \in \mathbb{R},$$

where $0 < q \leq 1$ and $a \in \mathbb{R}$ are scaling and translation parameters respectively. $T$ is unitary in $L^2$-space, $T^\dagger = T^{-1}$. We introduce two factorization operators

$$A^+ = (p + iW(x))T, \quad A^- = T^{-1}(p - iW(x)) = (A^+)\dagger,$$

where $W(x)$ is an arbitrary real function (superpotential). They define two Hamiltonians

$$H_- = A^+A^- = p^2 + W^2(x) - W'(x) = p^2 + U_-(x),$$

$$H_+ = \frac{1}{q^2}A^-A^+ = p^2 + \frac{1}{q^2}W^2\left(\frac{x-a}{q}\right) + \frac{1}{q}W'\left(\frac{x-a}{q}\right) = p^2 + U_+(x).$$

These may be unified into one $2 \times 2$ Hamiltonian

$$\mathcal{H} = \begin{pmatrix} H_- & 0 \\ 0 & H_+ \end{pmatrix} = p^2 + V(x) + B(x)\sigma_3, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where $V(x)$ and $B(x)$ are linear combinations of $U_\pm(x)$. Introducing supercharges

$$\mathcal{Q}^- = \begin{pmatrix} 0 & 0 \\ A^- & 0 \end{pmatrix}, \quad \mathcal{Q}^+ = \begin{pmatrix} 0 & A^+ \\ 0 & 0 \end{pmatrix},$$

we obtain a deformation of the $sl(1|1)$ superalgebra

$$\mathcal{Q}^+\mathcal{Q}^- + q^{-2}\mathcal{Q}^-\mathcal{Q}^+ = \mathcal{H}, \quad (\mathcal{Q}^\pm)^2 = 0, \quad \mathcal{H}\mathcal{Q}^\pm = q^{\pm 2}\mathcal{Q}^\pm\mathcal{H}. \quad (5)$$

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1 Talk presented at the XIXth International Colloquium on Group Theoretical Methods in Physics, Salamanca (Spain), June 29 - July 4, 1992

2 On leave of absence from the Institute for Nuclear Research, Moscow, Russia
The limit $q \to 1$ restores conventional relations. However, even in this case we have a
generalization of the standard supersymmetric models because the shift parameter $a$
enters the potential $U_+(x)$, i.e. still $A^\pm$ are infinite order differential operators.

Comparing (5) with the undeformed algebra one can see that double degeneracy is lifted,
the energy split is proportional to $1 - q^2$. Subhamiltonians $H_\pm$ are not isospectral but their
eigenvalues are related via scaling. Vacuum energy of $\mathcal{H}$ is formally semipositive, $E_0 \geq 0$,
the equality is reached when $A^-$ (or $A^+$) has a normalizable zero mode.

Let us take “magnetic field” $B(x)$ in (4) to be homogeneous, $U_-(x) - U_+(x) = 2B = \text{const}.$
This leads to the mixed finite-difference-differential equation determining superpotential,

$$W''(x) + qW'(qx + a) + W^2(x) - q^2W^2(qx + a) + 2Bq^2 = 0. \quad (6)$$

For convenience we rewrite (6) in the form

$$f'(x) + qf'(qx) + f^2(x) - q^2f^2(qx) = k, \quad (7)$$

where $f(x) = W(x + a/(1 - q))$ and $k = -2Bq^2$. Parameter $k$ may be removed from (7)
by rescaling but we keep it as a unique (positive) dimensional parameter. Note also that if
$x$ and $q$ are complex variables then it is natural to consider only $|q| \leq 1$ region due to the
symmetry $f(x, q^{-1}) = iqf(qx/i, q)$.

For this special case, one can write instead of (3)

$$A^+A^- = H + k/(1 - q^2), \quad A^-A^+ = q^2H + k/(1 - q^2), \quad (8)$$

where

$$H = p^2 + f^2(x) - f'(x) - k/(1 - q^2). \quad (9)$$

Now it is easy to see that $A^\pm$ and $k$ form a $q$-deformed Heisenberg-Weyl algebra [1],

$$A^-A^+ - q^2A^+A^- = k, \quad [A^\pm, k] = 0. \quad (10)$$

The Hamiltonian $H$ $q$-commutes with $A^\pm$

$$HA^\pm = q^{\mp 2}A^\pm H. \quad (11)$$

As a result, from the normalized vacuum state $|0\rangle$, $A^-|0\rangle = 0$, a series of energy eigenstates
is generated

$$|n\rangle = \frac{(A^+)^n}{\sqrt{k^n[n]!}}|0\rangle, \quad \langle n|m\rangle = \delta_{nm}, \quad [n]! = \prod_{i=1}^{n} \frac{1 - q^{2i}}{1 - q^2},$$

$$A^+|n\rangle = k^{1/2}\sqrt{\frac{1 - q^{2(n+1)}}{1 - q^2}} |n + 1\rangle, \quad A^-|n\rangle = k^{1/2}\sqrt{\frac{1 - q^{2n}}{1 - q^2}} |n - 1\rangle,$$

$$H|n\rangle = E_n|n\rangle, \quad E_n = -\frac{k}{1 - q^2}q^{2n}. \quad (12)$$

This gives the whole discrete spectrum provided $A^\pm$ respect boundary conditions of the
problem.
It is convenient to consider antisymmetric superpotential, \( f(-x) = -f(x) \), which is given by the power series

\[
f(x) = \sum_{j=1}^{\infty} c_j x^{2j-1}, \quad c_j = \frac{q^{2j} - 1}{q^{2j} + 1} \frac{1}{2j - 1} \sum_{l=1}^{j-1} c_{j-l} c_l, \quad c_1 = \frac{k}{1 + q^2}.
\]  

(13)

Odd and even \( c_j \) have different signs and the series converges for arbitrarily large \( x \) with \( f(\infty) > 0 \). After analytical continuation of (13) to the imaginary axis (or, for \( q > 1 \)) all expansion coefficients are positive and a pole singularity develops at some point \( x_0 \). In both cases the state \( |0\rangle \propto \exp(-f^2 f(y)dy) \) is normalizable by topological arguments (for the singular superpotential the space region should be restricted to \([-x_0,x_0]\)). Only first solution corresponds to (12). In the second case operator \( T \) moves positions of poles and zeros so that \( A^+\)-action brings singularities into wave functions. Therefore (12) is not valid for \( q > 1 \).

Various limits of \( q \) recover well-known exactly solvable potentials. At \( q = 0 \) a subcase of the Rosen-Morse problem arises, \( f(x) \propto \tanh x \). Analytical continuation (or, the limit \( q \to \infty \) at fixed \( c_1 \)) gives Pöschl-Teller problem, \( f(x) \propto \tan x \). The limit \( q \to 1 \) restores customary harmonic oscillator potential (note that for \( W(x) \) this is a non-trivial limit). If \( q \) is complex, then hermiticity properties of \( A^\pm \) are broken. Nevertheless, there are interesting physically relevant real potentials at \( q^n = 1 \). The \( q = -1 \) case is equivalent to \( q = 1 \). At \( q^3 = 1 \) the Schrödinger equation coincides with the simplest Lamé equation for equianharmonic Weierstrass function. Corresponding spectral problem, defined by the requirement for wave functions to vanish in singular points, is known to be solvable. This example provides ordinary differential calculus realization of cyclic representations of the \( q \)-oscillator algebra (10), the associated \( q \)-special functions being the ordinary Lamé functions.

Equation (7) was found by A.Shabat [2] after substitution of the self-similarity ansatz \( f_j(x) = q^j f(q^j x) \), \( k_j = q^{2j} k \), into the chain of coupled Ricatti equations

\[
f_j'(x) + f_{j+1}'(x) + f_j^2(x) - f_{j+1}^2(x) = k_j, \quad j \in \mathbb{Z},
\]  

(14)

which arises in factorization method. He also described general structure of this system for real \( x \) and \( 0 \leq q < 1 \) through the \( n \)-soliton, \( n \to \infty \), approximations. Connection with the \( q \)-oscillator algebra was established in Ref.[3]. Associated deformation of supersymmetric quantum mechanics was discussed in Ref.[4]. Appearance of the equianharmonic Weierstrass function at \( q^3 = 1 \) was described in Ref.[5].

In the rest part of this paper we outline a generalization of the Shabat’s potential. First we note that periodic conditions

\[
f_{j+N}(x) = f_j(x), \quad k_{j+N} = k_j,
\]  

(15)

provide finite-dimensional truncation of the chain (14). Odd \( N \) cases, endowed by the restriction \( \sigma \equiv k_1 + k_2 + \ldots + k_N = 0 \), are known to be integrable in terms of the hyperelliptic functions defining finite-gap potentials [6]. If \( \sigma \neq 0 \), then one has essentially more complicated situations. At \( N = 1 \) this gives harmonic oscillator problem. The \( N = 2 \) system coincides with the conformal quantum mechanical model,

\[
f_{1,2}(x) = \frac{1}{2}(\pm \frac{k_1 - k_2}{k_1 + k_2} 1 + \frac{k_1 + k_2}{2} x).
\]  

(16)
Already $N = 3$ case leads to transcendental potentials, namely, $f_j$ start to depend on the solutions of Painlevé-IV equation [7].

Applying the ideas of deformed supersymmetry to the whole chain (14) the author have found the following $q$-periodic closure

$$f_{j+N}(x) = qf_j(qx), \quad k_{j+N} = q^2k_j.$$  \hspace{1cm} (17)

These conditions describe $q$-deformation of the finite-gap and related potentials discussed in Refs.[6,7]. The Shabat’s system is generated at $N = 1$. Because of the highly transcendental character of self-similarity and relation with the Painlevé equations, all potentials associated to (17) may be called as $q$-transcendental ones.

Algebraically, $q$-transcendental potentials are characterized by the ladder relations

$$HA^+ - q^2A^+H = \sigma A^+, \quad A^-H - q^2HA^- = \sigma A^-,$$  \hspace{1cm} (18)

$$A^+ = (p + if_1)(p + if_2)\ldots(p + if_N)T, \quad A^- = (A^+)\dagger,$$

$$H = (p + if_1(x))(p - if_1(x)),$$

and the identities

$$A^+A^- = \prod_{i=1}^{N}(H - \lambda_i), \quad A^-A^+ = \prod_{i=1}^{N}(q^2H + \sigma - \lambda_i),$$  \hspace{1cm} (19)

where parameters $\lambda_i$ are defined by the equalities $k_i = \lambda_{i+1} - \lambda_i, \lambda_1 = 0, \lambda_{1+N} = \sigma$. From (19) a particular $q$-commutator of $A^+$ and $A^-$ may be easily fixed.

Let $A^\pm$ be well defined operators. If $\sigma > 0$ then $A^-$ is the lowering operator for the discrete energy states. The number of normalizable independent solutions of the equation $A^-|E\rangle = 0$ determines the number ($\leq N$) of geometric series composing the spectrum of $H$. In the $q \to 0$ limit only few of the levels (solitons) remain in the spectrum. If in the $q \to 1$ limit the potentials remain to be smooth, then spectral series become equidistant. A more detailed consideration of the $q$-transcendental potentials will be given elsewhere [8]. Here we just mention that at $N = 2$ one has $A^-A^+ - q^4A^+A^- \propto H$, which together with (18) describes a particular “quantization” of the conformal algebra $su(1, 1)$ (see, e.g., Ref.[9]). By the same reason this self-similar system may be called as $q$-deformed conformal quantum mechanics.

The author is indebted to A.Shabat for valuable discussions and helpful comments. This research was supported by the NSERC of Canada.

References


