United Nations Educational Scientific and Cultural Organization
and
International Atomic Energy Agency
THE ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

**THE SOLITON SOLUTION OF BBGKY'S CHAIN
OF QUANTUM KINETIC EQUATIONS FOR SYSTEM OF
PARTICLES, INTERACTING BY DELTA POTENTIAL**

M. Yu. Rasulova¹

*The Institute of Nuclear Physics Uzbekistan Academy of Sciences,
Ulughbek, Tashkent 702132, Uzbekistan²*

and

The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy.

Abstract

The BBGKY's chain of quantum kinetic equations that describes the system of Boltzmann particles interacting by delta potential is solved with the help of nonlinear Schrödinger's equations. The solution of the chain is defined in terms of soliton.

MIRAMARE – TRIESTE

August 2001

¹E-mail: rasulova@suninp.tashkent.su

²Permanent address.

*This paper is dedicated to
Prof. Joel Lebowitz's 70th birthday*

From the second quarter of the 20th century, when the BBGKY's chain of kinetic equations [N.N.Bogoliubov, 1970]-[J.Yvon, 1935] had been formulated the problem of solution of chain kinetic equations is still a challenging problem. A big step in the solution of all chain was done in the seventies by Prof.D.Y.Petrina and his colleagues [D.Ya. Petrina,1972]-[D.Ya. Petrina, 1979]. They proved the existence and uniqueness of solution of BBGKY's chain for classic and quantum cases. But the problem of obtaining BBGKY's chain explicit solution for physical purposes remained.

Brilliant results of solution of nonlinear equations, in particular nonlinear Schrödinger equation [C.S.Gardner, J.M.Green, M.D.Kruskal, R.M.Miura, 1967]-[B.A.Dubrovin, C.P.Novikov,1974], and results of investigation of systems interacting by potential in the form of delta function [E.H.Lieb, W.Liniger, 1963]-[C.N.Yang, C.P.Yang,1969] opened prospects for obtaining explicit solution of BBGKY's chain. Based on some physical assumption on potential (weak potential) and the density (high density) the heuristic soliton solution of Vlasov's quantum equation and BBGKY's chain of quantum kinetic equations [D.Ya.Petrina, V.Z.Enolskiy,1976] have been obtained.

In the beginning of the nineties for the first time by operator method, later on basis of relation following from self-consistent condition by M.Yu.Rasulova [M.Yu.Rasulova,1991] it was proved that BBGKY's chain, describing the evolution of Bose [M.Yu.Rasulova,1997]-[M.Yu.Rasulova, B.I.Sattarov,1991] and Fermi [Rasulova M.Yu., Muradov T.T., 1998] particles interacting by delta potential ($\kappa > 0$) has solution in terms of Bethe ansatz.

This paper is dedicated to soliton solution of BBGKY's chain of quantum kinetic equations for particles, interacting by delta potential in the domain $x_i \neq x_j, i, j = 1, 2, \dots, N, \dots$. Here x_i, x_j are the coordinates and N is the quantity of particles.

With this purpose in view we consider the infinite quantum system of identical particles with mass $m = \frac{1}{2}$, interacting by pair potential $\Phi(x_i - x_j)$ and distributed over one dimensional space with density $\frac{1}{v}$. The evolution of this system is described by the sequences of density matrices

$$F(t) = \{F_1(t, x; x'), \dots, F_s(t, x_1, \dots, x_s; x'_1, \dots, x'_s), \dots\}$$

which satisfy the BBGKY's quantum kinetic equations [N.N.Bogoliubov, 1970]:

$$\begin{aligned}
i \frac{\partial}{\partial t} F_s(t, x_1, \dots, x_s; x'_1, \dots, x'_s) &= (H(x_1, \dots, x_s) F_s(t, x_1, \dots, x_s; x'_1, \dots, x'_s) - \\
&- F_s(t, x_1, \dots, x_s; x'_1, \dots, x'_s) H(x'_1, \dots, x'_s)) + \frac{1}{v} \int \sum_{1 \leq i \leq s} (\Phi(x_i - x_{s+1}) - \Phi(x'_i - x_{s+1})) \times \\
&\times F_{s+1}(t, x_1, \dots, x_s, x_{s+1}; x'_1, \dots, x'_s, x_{s+1}) dx_{s+1},
\end{aligned} \tag{1}$$

with initial condition:

$$F_s(t, x_1, \dots, x_s; x'_1, \dots, x'_s) /_{t=0} = F_0(x_1, \dots, x_s; x'_1, \dots, x'_s), \tag{2}$$

with the normalization condition:

$$\frac{1}{V^s} \int_V \int_V \int_V F_s(t, x_1, \dots, x_s; x_1, \dots, x_s) dx_1 \dots dx_s = \frac{1}{v^s}. \tag{3}$$

In (1) $F_s(t, x_1, \dots, x_s; x'_1, \dots, x'_s)$ is the density matrix, x is the particle coordinate, $[,]$ is the Poisson bracket, $2m=1$, $\hbar = 1$, $t \geq 0$ is the time, v -density of particles, the Hamiltonian $H_s(x_1, \dots, x_s)$ is given by

$$H_s(x_1, \dots, x_s) = - \sum_{i=1}^s \frac{\partial^2}{\partial x_i^2} + \sum_{i < j} \Phi(x_i - x_j).$$

It should be noted, that density matrix $(F_s(t, x_1, \dots, x_s; x'_1, \dots, x'_s))_{s=1}^\infty$ must satisfy the condition of self-consistency:

$$F_s(t, x_1, \dots, x_s; x'_1, \dots, x'_s) = \int F_{s+1}(t, x_1, \dots, x_s, x_{s+1}; x'_1, \dots, x'_s, x_{s+1}) dx_{s+1};$$

$$F_s(t, x_1, \overset{x_i}{\underset{\vee}{\dots}}, x_{s+1}; x'_1, \overset{x'_i}{\underset{\vee}{\dots}}, x'_{s+1}) = \int F_{s+1}(t, x_1, \dots, x_s, x_{s+1}; x'_1, \dots, x'_s, x'_{s+1})_{x_i=x'_i} dx_i \tag{4}$$

for any $i = 1, 2, \dots, s+1$; $s = 1, 2, \dots$, where $(x_1, \overset{x_i}{\underset{\vee}{\dots}}, x_{s+1}) = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{s+1})$.

We consider the chain of BBGKY's quantum kinetic equations with interaction potential of the delta function type:

$$\begin{aligned}
i \frac{\partial}{\partial t} F_s(t, x_1, \dots, x_s; x'_1, \dots, x'_s) &= (H(x_1, \dots, x_s) \times \\
&\times F_s(t, x_1, \dots, x_s; x'_1, \dots, x'_s) - F_s(t, x_1, \dots, x_s; x'_1, \dots, x'_s) H(x'_1, \dots, x'_s)) + 2\kappa \int \sum_{1 \leq i \leq s} (\delta(x_i - x_{s+1}) - \\
&- \delta(x'_i - x_{s+1})) F_{s+1}(t, x_1, \dots, x_s, x_{s+1}; x'_1, \dots, x'_s, x_{s+1}) dx_{s+1},
\end{aligned}$$

where $\kappa \geq 0$, the Hamiltonian $H_s(x_1, \dots, x_s)$ is given by

$$H_s(x_1, \dots, x_s) = - \sum_{i=1}^s \frac{\partial^2}{\partial x_i^2} + 2\kappa \sum_{i < j} \delta(x_i - x_j).$$

Let us consider the domain of phase space, where $x_k \neq x_l$ and $x'_k \neq x'_l$ for any $k \neq l$. Then, the last equation is transformed into:

$$i \frac{\partial}{\partial t} F_s(t, x_1, \dots, x_s; x'_1, \dots, x'_s) = \int \sum_{1 \leq i \leq s} \left(-\frac{\partial^2}{\partial x_i^2} - \frac{\partial^2}{\partial x_i'^2} \right) F_s(t, x_1, \dots, x_s; x'_1, \dots, x'_s) +$$

$$+ 2\kappa \sum_{1 \leq i \leq s} (F_{s+1}(t, x_1, \dots, x_s, x_i; x'_1, \dots, x'_s, x_i)) - (F_{s+1}(t, x_1, \dots, x_s, x'_i; x'_1, \dots, x'_s, x'_i)). \quad (5)$$

The problem is to find the solution $(F_s(t, x_1, \dots, x_s; x'_1, \dots, x'_s))_{s=1}^{\infty}$ of an infinite chain of equations (2) in the domain, where $x_k \neq x_l$ and $x'_k \neq x'_l$ for any $k \neq l$, $k, l = 1, 2, \dots$, in addition $F_s(t, x_1, \dots, x_s; x'_1, \dots, x'_s)$ will satisfy the initial conditions (2), the normalization condition (3) and the condition of self-consistency (4). It is known, that matrix $F_s(t, x_1, \dots, x_s; x'_1, \dots, x'_s)$ is related to density matrix $\rho(t, x_1, \dots, x_N; x'_1, \dots, x'_s, x_{s+1}, \dots, x_N)$ of a system of N particles by [N.N.Bogoliubov, 1970]:

$$F_s(t, x_1, \dots, x_s; x'_1, \dots, x'_s) = S p_{s+1, \dots, N}(\rho(t, x_1, \dots, x_N; x'_1, \dots, x'_N),$$

$$\rho(t, x_1, \dots, x_N; x'_1, \dots, x'_N) = \sum_{k_1, \dots, k_N; k'_1, \dots, k'_N} C(k_1, \dots, k_N) \Psi_{k_1, \dots, k_N}(t, x_1, \dots, x_N) \times$$

$$\times \Psi_{k'_1, \dots, k'_N}^*(t, x'_1, \dots, x'_N) C^*(k'_1, \dots, k'_N). \quad (6)$$

Here we take $\Psi_{k_1, \dots, k_N}(x_1, \dots, x_N)$ in the form

$$\Psi_{k_1, \dots, k_N}(x_1, \dots, x_N) = \sqrt{\frac{\prod_{(f)} (k_f!)}{N!}} \varphi_{f_1}(x_1) \cdot \varphi_{f_N}(x_N),$$

where k_i is the f_i s number, $k_i = 0, 1, 2, \dots$, $\sum_f k_f = N$ f-type of states and $\varphi_{f_i}(x_i)$ -the function with the following properties:

$$\int \varphi_{f'}(x) \varphi_{f''}(x) dx = \delta(f' - f''),$$

$$\delta(f' - f'') = \begin{cases} 1, & f' = f'', \\ 0, & f' \neq f''. \end{cases}$$

In the domain, where $x_k \neq x_l$ and $x'_k \neq x'_l$ for any $k \neq l$, $k, l = 1, 2, \dots$ (6) has the form

$$F_s(t, x_1, \dots, x_s; x'_1, \dots, x'_s) = \sum_{k_1, \dots, k_s; k'_1, \dots, k'_s} C(k_1, \dots, k_s) \times$$

$$\times \sqrt{\frac{\prod_{(f)} (k_f!)}{s!}} \exp(it f_1^2) \varphi_{f_1}(x_1) \cdots \exp(it f_s^2) \varphi_{f_s}(x_s) \times$$

$$\times \sqrt{\frac{\prod_{(f')} (k_{f'}')!}{s!}} \varphi_{f'_1}^*(x'_1) \exp(-it f_1'^2) \cdots \times$$

$$\times \varphi_{f'_s}^*(x'_s) \exp(-it f_s'^2) C^*(k'_1, \dots, k'_s). \quad (7)$$

For the wave function of s-particles $\Psi_{k_1, \dots, k_s}(t, x_1, \dots, x_s)$, the condition

$$\int \Psi^*(t, x_1, \dots, x_s) \Psi(t, x_1, \dots, x_s) dx_1 \cdots dx_s = 1.$$

takes place. Whence it follows:

$$\begin{aligned} & \sum_{k_1, \dots, k_s} |C(k_1, \dots, k_s)|^2 = 1, \\ & \sum_{k_1, \dots, k_s, k_{s+1}; k'_1, \dots, k'_s} C(k_1, \dots, k_s, k_{s+1}) C^*(k'_1, \dots, k'_s, k_{s+1}) \\ & = \sum_{k_1, \dots, k_s; k'_1, \dots, k'_s} C(k_1, \dots, k_s) C^*(k'_1, \dots, k'_s). \end{aligned} \quad (8)$$

The self-consistency condition is not difficult to check with (8). Let's introduce the notations $\varphi_f(t, x) = \exp(itf^2)\varphi_f(x)$, $\varphi_{f'}^*(t, x') = \varphi_{f'}^*(x')\exp(-itf'^2)$.

Theorem 1. To satisfy equation (5) with the condition (2) it is necessary and sufficient that the following equations take place:

$$i \frac{\partial}{\partial t} \varphi_{f_i}(t, x_i) = - \frac{\partial^2}{\partial x_i^2} \varphi_{f_i}(t, x_i) + 2\kappa \varphi_{f_i}(t, x_i) |\varphi_{f_i}(t, x_i)|^2, \quad (9)$$

$$\varphi_{f_i}(t, x_i) |_{t=0} = \varphi_{f_i}(x_i). \quad (10)$$

$$i \frac{\partial}{\partial t} \varphi_{f'_i}^*(t, x'_i) = - \frac{\partial^2}{\partial x_i^2} \varphi_{f'_i}^*(t, x'_i) + 2\kappa \varphi_{f'_i}^*(t, x'_i) |\varphi_{f'_i}^*(t, x'_i)|^2, \quad (11)$$

$$\varphi_{f'_i}^*(t, x_i) |_{t=0} = \varphi_{f'_i}^*(x_i). \quad (12)$$

Proof Let (9) and (11) take place. Take $F_s(t, x_1, \dots, x_s; x'_1, \dots, x'_s)$ in form (7), and make the differentiation on t and multiply by i . In place of $i \frac{\partial}{\partial t} \varphi_{f_i}(t, x_i)$, $i \frac{\partial}{\partial t} \varphi_{f'_i}^*(t, x'_i)$ we substitute right parts of the (9) and (11). Further, we obtain the equation (5) using the relation (8).

One can also show the opposite. We can state that the solution of the equation (5) is reduced to the solution of non-linear equations for $\varphi_{f_i}(t, x_i)$, $\varphi_{f'_i}^*(t, x'_i)$. We take the functions $F_s(t, x_1, \dots, x_s; x'_1, \dots, x'_s)$, $F_{s+1}(t, x_1, \dots, x_s, x_{s+1}; x'_1, \dots, x'_s, x'_{s+1})$ in form (7). Having differentiated $F_s(t, x_1, \dots, x_s; x'_1, \dots, x'_s)$ on t, and using (8) we obtain (10).

Similarly, in order to obtain the boundary conditions for functions (10), (12) we substitute in (2) $F_s(t, x_1, \dots, x_s; x'_1, \dots, x'_s)$ by (7).

Therefore, we can conclude: if the density matrix (7) is a solution of the equation (5) with the boundary condition (2), $(\varphi_{f_i}(t, x_i))_{i=1}^\infty$, $(\varphi_{f'_i}^*(t, x'_i))_{i=1}^\infty$ are solutions of corresponding equations (9), (11) satisfying to conditions (10) and (12).

Theorem 2. The solution of the chain of quantum kinetic BBGKY's equations with delta

potential in a domain of phase space where $x_k \neq x_l, x'_k \neq x'_l$ for any $k \neq l$ one can find by soliton solution of nonlinear Schrödinger equation:

$$\varphi(t, x) = \sqrt{\frac{2}{\kappa}} \frac{(\lambda + i\nu)^2 + \exp[2\nu(x - x_0 - 2\lambda t)]}{1 + \exp[2\nu(x - x_0 - 2\lambda t)]}, \quad (13)$$

at $\kappa > 0$ or

$$\varphi(t, x) = \sqrt{\frac{2}{|\kappa|}} \frac{\exp[-4i(\xi^2 - \eta^2)t - 2i\xi x + i\Omega]}{ch[2\eta(x - x_0) + 8\eta\xi t]}, \quad (14)$$

at $\kappa < 0$ [B.E.Zaharov,1971],[B.E.Zaharov, A.B.Shabat,1971],[D.Ya.Petrina, V.Z.Enolskiy,1976].

Proof: As is known from [B.E.Zaharov,1971], [B.E.Zaharov, A.B.Shabat,1971], [D.Ya.Petrina, V.Z.Enolskiy,1976], functions (13),(14) are soliton solutions of the nonlinear Schrödinger equation with initial conditions (10), (12) at $\kappa > 0$ and $\kappa < 0$, respectively. Here the case $\kappa < 0$ is not considered for systems of Bose particles [E.H.Lieb, W.Liniger, 1963]. In (13) parameter λ characterises the amplitude and velocity of soliton , ν is expresses via parametr λ as:

$$\nu = \sqrt{1 - \lambda^2}.$$

Moreover, this relation is valid:

$$\frac{\kappa}{2} |\varphi(t, x)|^2 = 1 - \frac{\nu^2}{ch^2\nu(x - x_0 - 2\lambda t)}.$$

If $\kappa < 0$, than $N < const$ and $\varphi(t, x) \rightarrow 0$ at $x \rightarrow \pm\infty$. In formula (14) ξ is the amplitude, η is the velocity, x_0 is the center of position and Ω is the phase of the soliton. The relation between κ and η in this case is defined from norm condition in the form:

$$\int |\varphi(t, x)|^2 dx = N,$$

where N is the number of particles.

Therefore the density matrix has the following form:

$$\begin{aligned} F(t, x_1, \dots, x_s; x'_1, \dots, x'_s) &= \sum_{k_1, \dots, k_s; k'_1, \dots, k'_s} C(k_1, \dots, k_s) \times \\ &\times \sqrt{\frac{\Pi_{(f)}(k_f!)}{s!}} \varphi_{f_1}(t, x_1) \cdots \varphi_{f_s}(t, x_s) \times \\ &\times \sqrt{\frac{\Pi_{(f')}(k'_{f'}!)}{s!}} \varphi_{f'_1}^*(t, x'_1) \cdots \varphi_{f'_s}^*(t, x'_s) C^*(k'_1, \dots, k'_s), \end{aligned}$$

where φ is defined by (13) or (14). This result also may be easily obtained using relation of Vlasov equation and BBGKY's chain defined by Markowich and others

[P.A.Markowich, C.A.Ringhofer, C.Schmeiser,1990].

Acknowledgments

I would like to acknowledge the INTAS program, the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy and Head of Condensed Matter Section Prof. Yu Lu for grant and hospitality.

References

- [N.N.Bogoliubov, 1970] Problems of Dynamical Theory in Statistical Physics. In: Selected papers **2**, Kiev, Naukova Dumka, 1970.
- [M.Born, H.S.Green, 1949] A General Kinetic Theory of Fluids. Cambridge University Press, Cambridge 1949.
- [J.G.Kirkwood, 1946] The statistical mechanical theory of transport processes, J.Chem.Phys. **14**, 180 1946, **15**, 72 1947.
- [J.Yvon, 1935] La theorie statistique des fluides of l'equation d'etat, Actualites Scientifiques et Industrielles, No.203, Hermann et Cie, Paris, 1935.
- [D.Ya. Petrina, 1972] On solution of the kinetic Bogoliubov equations: quantum statistics, *Teor.Mat.Fiz.*, **13**, 391-405, 1972.
- [D.Ya.Petrina, A.K.Vidybida, 1975] Cauchy problem for the Bogoliubov chain of equations, *Trudy MIAN SSSR*, **136**, 370-378, 1975.
- [M.Yu.Rasulova, 1976] The Cauchy problem for the kinetic Bogoliubov equations. Quantum case, *Docl.Acad.Nauk Uz.SSR*, **2**, 6-9, 1976.
- [D.Ya. Petrina, 1979] Mathematical description of the evolution of infinite systems of classical statistical physics: locally perturbed one-dimensional systems, *Teor.Math. Fiz.*, **38**, 230-25, 1979.
- [C.S.Gardner, J.M.Green, M.D.Kruskal, R.M.Miura, 1967] *Phys.Rev.Lett.*, V.**19**, P.1095, 1967.
- [F.Cologero, 1978] (editor) Nonlinear evolution equations solvable by the spectral transform (Collection of papers), London: Pitman, 1978.
- [P.D.Lax, 1968] *Commun.Pure Appl.Math.*, **21**, P.467, 1968.
- [B.E.Zaharov, 1971] *JTEP*, **60**, (3), P.993, 1971.
- [B.E.Zaharov, A.B.Shabat, 1971] *JTEP*, **61**, P.118, 1971.
- [B.A.Dubrovin, C.P.Novikov, 1974] *JTEP*, **67**, Vip.12, 2131, 1974.

- [E.H.Lieb, W.Liniger, 1963] *Phys.Rev.***130**, 1605, 1963.
- [E.K.Sklianin, L.D.Faddeev, 1979] *Dokl.Acad. Nauk SSSR* **243**, 1430, 1979.
- [E.K.Sklianin, 1979] *Dokl.Acad.Nauk SSSR* **244**, 1337, 1979,**16**, 2515, 1977.
- [E.K.Sklianin, 1980] *Zapiski LOMI*, **95**, 1980.
- [H.B.Thacker, 1975] *Phys.Rev.***D11**,838, 1975; **14**, 3508, 1976; **16**, 2515, 1977.
- [H.B.Thacker, D.W.Wilkinson, 1979] *Phys.Rev.* **D19**, 3660, 1979.
- [P.B.Creamer, et al.,1979] Fermilab-Pub-79/75-THY, Sept. 1979.
- [C.N.Yang, C.P.Yang,1969] *Journ.of Math.Phys.* **10**, 1115, 1969.
- [D.Ya.Petrina, V.Z.Enolskiy,1976] On Oscillations of One-Dimensional Systems, *Preprint ITP-76-17R*, Kiev 1976.
- [M.Yu.Rasulova,1991] *Preprint INP Acad.of Scien. R.Uzb.***P-9-540**, 1991; *Jour. Nonl. Math. Phys.*, **8**,Suppl.,2001.
- [M.Yu.Rasulova, B.I.Sattarov,1991] *Preprint INP Acad. of Scien.R.Uzb.* **P-11-549**, 1991.
- [M.Yu.Rasulova,1997] *Reports on Mathem.Physics*, **40**,No3,P.551, 1997.
- [Rasulova M.Yu., Muradov T.T., 1998] *Uzbekskiy fizicheskiy Journal*, **2**, 1998.
- [P.A.Markowich, C.A.Ringhofer, C,Schmeiser,1990] *Semiconductor Equations*, Springer-Verlag Wien New York,1990,p.19.