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where \( \Delta(x) \) is the Regge trajectory function.

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as a power series in a strong coupling constant $\overline{\alpha}_s = \alpha_s N_c / \pi$

\[
A(s, t) \sim -i \sum_{N=2}^{\infty} (is\overline{\alpha}_s)^N s^{1+\frac{N\epsilon_N}{2N_c}} \beta_A^{(N)}(t) \beta_B^{(N)}(t).
\]  

(1)

Above, $N$ denotes the number of reggeized gluons, called Reggeons, propagating in the $t$-channel and the residue factors, $\beta_A^{(N)}$, measure the overlap of the wave function of the compound state of $N$ reggeized gluons with the wave functions of two scattered particles. The parameter $\epsilon_N$ is defined as the maximal energy for the Schrödinger (BKP) equation

\[
\mathcal{H}_N \Psi (\{ \vec{z}_k \}) = \epsilon_N \Psi (\{ \vec{z}_k \})
\]

(2)

where $\Psi (\{ \vec{z}_k \})$ is the Reggeon wave function and $\vec{z}_k$ denotes two-dimensional transverse coordinates of $k^{th}$ reggeized gluon. $\mathcal{H}_N$ is the effective QCD Hamiltonian describing pair-wise interaction between $N$ reggeons.

In the multi-colour limit, this Hamiltonian simplifies significantly [10, 11] leading to

\[
\mathcal{H}_N = \sum_{k=0}^{N-1} H(\vec{z}_k, \vec{z}_{k+1}) \quad \text{where} \quad \vec{z}_0 \equiv \vec{z}_N.
\]

(3)

It describes [8] the nearest neighbour interaction of the Reggeons and has a hidden cyclic and mirror permutational symmetry. Moreover, it possesses the set of the $(N-1)$ integrals of motion, which are the eigenvalues of conformal charges, $\hat{q}_n$ and $\hat{\bar{q}}_n$ [7],

\[
[\mathcal{H}_N, \hat{q}_n] = [\hat{q}_n, \hat{\bar{q}}_n] = [\mathcal{H}_N, \hat{\bar{q}}_n] = [\hat{\bar{q}}_n, \hat{q}_m] = 0, \quad n, m = 2, \ldots, N.
\]

(4)

Thus, this system is completely integrable. The lowest integral may be expressed in terms of the conformal $SL(2)$ weight of the state $h$ as $\hat{q}_2 = -h(h-1)$. The Hamiltonian (3) is equivalent to the XXX Heisenberg spin magnet [7].

Our solution of the Schrödinger equation (2) is based on the method of the Baxter Q-operator [7]. It relies on the existence of the operator $Q(u, \bar{u})$ depending on the pair of complex spectral parameters $u$ and $\bar{u}$ and satisfying the following relations. It commutes with itself for different values of the spectral parameters and with the integrals of motion

\[
[Q(u, \bar{\nu}), Q(v, \bar{\nu})] = [\mathcal{I}_N(u, \{ \bar{\nu}_n \}), Q(v, \bar{\nu})] = [\mathcal{I}_N(\bar{\nu}, \{ \bar{\nu}_n \}), Q(v, \bar{\nu})] = 0,
\]

(5)

\footnote{bar does not denote complex conjugation for which we use an asterisk.}
where
\[ \hat{t}_N(u, \{\bar{q}_n\}) = 2u^N + \bar{q}_2 u^{N-1} + \ldots + \bar{q}_N, \]

and \(u, v\) are two complex spectral parameters. It also has to satisfy the Baxter equations

\[
\begin{align*}
\hat{\ell}_N(u, \{\bar{q}_n\})Q(u, \bar{\mu}) &= u^N Q(u + i, \bar{\mu}) + u^N Q(u - i, \bar{\mu}) \\
\hat{\ell}_N(\bar{u}, \{\bar{q}_n\})Q(u, \bar{\mu}) &= (\bar{\mu} + i)^N Q(u, \bar{\mu} + i) + (\bar{\mu} - i)^N Q(u, \bar{\mu} - i).
\end{align*}
\]

Furthermore, the \(Q\)-operator has prescribed analytical properties, i.e. known pole structure, and asymptotic behavior at infinity. The above conditions fix the \(Q\)-operator uniquely and allow us to quantize the integrals \(q_k\) [7, 13, 14, 15]. It turns out that it is possible to express the Hamiltonian (3) in terms of the Baxter \(Q\)-operator [7]. Combining together the solutions of the Baxter equations and the quantum conditions for \(q_k\) with the Schrödinger equation (2) we can calculate the energy spectrum.

For \(N = 3\) there exist two integrals of motion, \(q_2\) and \(q_3\). The quantized values of \(q_3\) exhibit some structure that can be seen on Figure 1 (left panel) [9]. The circled crosses denote the ground states. They have the highest energy for fixed \(N\). The spectrum of quantized \(q_3\) at \(N = 3\) may be approximated by the following formula [12]

\[
[q_3^{\text{quant}}(\ell_1, \ell_2)]^{1/3} = \frac{\Gamma^3(2/3)}{2\pi} \left( \frac{1}{2} \ell_1 + i \frac{\sqrt{3}}{2} \ell_2 \right)
\]

**Fig. 1.** Quantized values of the integrals of motion at \(\hbar = 1/2\) for different number of reggeons \(N = 3\) (left) and \(N = 4\) (right).
where \( \ell_1, \ell_2 \in \mathbb{Z} \) and \( \ell_1 + \ell_2 \) is even. Similar structure is observed for \( N = 4, h = 1/2 \) and \( q_3 = 0 \) (plot 1 on the right)

\[
[q^{\text{upper}}_{\ell_1, \ell_2}]^{1/4} = \frac{\Gamma^2(3/4)}{2\sqrt{\pi}} \left( \frac{1}{\sqrt{2} \ell_1} + i \frac{1}{\sqrt{2} \ell_2} \right).
\]  

(9)

All these states have continuation in \( \nu_h = \Im(h) \), so, the eigenvalues of \( \{q_k\} \) form linear trajectories in the full \( q_k \) space [16, 17]. The spectrum of \( q_N \) has similar structure for higher \( N \).

<table>
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<th>( q_4 )</th>
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<th>( q_6 )</th>
<th>( q_7 )</th>
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Table 1. Quantum numbers \( q_k \) and energy, \( \epsilon_N(\{q_k\}) \), of the \( N \)-Reggeons states in the multi-colour QCD for \( h = 1/2 \).

For the ground states (tab. 1), the integrals of motion are either purely real or purely imaginary. Moreover, the odd integrals are equal to zero for even \( N \)'s. The numbers at \( N = 2 \) and \( N = 3 \) agree with the previously published ones [2, 5, 8, 15, 17], while the results for higher \( N \) are new. The energy is positive for even \( N \)'s and negative for odd \( N \)'s and it is shown in a plot 2. The contribution of these states to the scattering amplitude increases with \( s \) for the positive energy \( \epsilon_N \) and decreases for the negative energy \( \epsilon_N \). The exact values of the energy are denoted by crosses on the left panel of fig. 2. The upper and the lower curves stand for the functions \( 1.8402/(N - 1.3143) \) and \( -2.0594/(N - 1.0877) \), respectively.

The right panel in the figure 2 shows the dependence of the energy \( \epsilon(\nu_h) \) along the ground state trajectory for different number of particles \( 2 \leq N \leq 8 \). These functions are symmetrical in \( \nu_h \). The maximum energy is in \( \nu_h = 0 \), at large \( \nu_h, \epsilon_8 > \cdots > \epsilon_3 > \epsilon_2 \).

Summarizing we found the spectrum of the multi-Reggeon compound states in QCD. In the Pomeron sector (even \( N \)) the intercept of states is bigger than 1 but smaller than the intercept of the BFKL Pomeron: \( \alpha_2 > \alpha_4 > \cdots > 1 \). In the odderon sector (odd \( N \)) the intercept of the states is smaller than 1 but it increases with \( N \): \( \alpha_3 < \alpha_5 < \cdots < 1 \).

Recently Lipatov and de Vega [18] found another set of the solutions for \( N = 3, 4 \) which differ from our expressions. This discrepancy requires further studies.
Fig. 2. The dependence of the ground state energy, $\epsilon_N$, on the number of particles $N$ and on the $\nu_h$.

REFERENCES


