ON THE SPECTRUM, NO GHOST THEOREM AND MODULAR INVARIANCE
OF $W_3$ STRINGS,

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Abstract

A spectrum generating algebra is constructed and used to find all the physical states of the $W_3$ string with standard ghost number. These states are shown to have positive norm and their partition function is found to involve the Ising model characters corresponding to the weights 0 and 1/16. The theory is found to be modular invariant if, in addition, one includes states that correspond to the Ising character of weight 1/2. It is shown that these additional states are indeed contained in the cohomology of $Q$. 
1 Introduction

The Zamolodchikov $W_3$ algebra can be used to construct a $W_3$ [1] string theory in much the same way that the Virasoro algebra can be used to construct a bosonic string theory. Although there are, with hindsight, a number of ways of constructing the bosonic string, the path most suited to our present purposes is as follows: starting from the Virasoro algebra we construct the BRST charge and demand its square vanish. This condition implies that $c = 26$ and the intercept is one [2]. We then find a realization of the Virasoro algebra with $c = 26$, for example 26 free scalars $x^\mu,\mu = 0, 1, \ldots, 25$. The physical states are the nontrivial cohomology classes of $Q$ which are also subject to a ghost constraint [3]. Given the physical states and the conformal operators, one can then construct the scattering amplitudes.

The construction of a $W_3$ string along these lines was first suggested in reference [4]. For the $W_3$ algebra the BRST charge had previously been constructed [5] and found to square to zero if $c = 100$ and the intercept is 4. There exist [6] realizations of $W_3$ which contain an arbitrary number of scalar fields, which have $c = 100$, and can be used as the basis for a $W_3$ string[7][8]. An important general feature of these realizations is that they require a background charge. The physical states should be the non-trivial cohomology classes of $Q$, subject also to an analogue of the ghost condition. Unfortunately, the cohomology of $Q$ is unknown; however, it has been tacitly assumed that, like the bosonic string, the physical states are essentially contained in a state of a given ghost number and even of a given ghost type. We will refer to states of this particular form as those of standard ghost type. Some clarification of the form of the physical states of standard ghost type was given in references [7] [8]. In reference [9], however, a systematic analysis of the physical states of this type was undertaken at low levels by solving the physical state conditions explicitly and then finding which of these states were null. We now summarise these results. Given a $W_3$ string made from $D + 1$ free bosonic fields $\varphi$, $x^\mu, \mu = 0, 1, \ldots, D - 1$, the energy-momentum tensor $T(z)$ and $W_3$ current $W(z)$ are of the form

\begin{align*}
T &= -\frac{1}{2} (\partial \varphi)^2 - Q_B \partial^2 \varphi + \hat{T} \\
W &= \frac{1}{3} (\partial \varphi)^3 + Q_B \partial \varphi \partial^2 \varphi + \frac{1}{3} Q_B^2 \partial^3 \varphi + 2 \partial \varphi \hat{T} + Q_B \partial \hat{T}
\end{align*}  

(1.1)  

(1.2)
where
\[ T = -\frac{1}{2} \partial x^\mu \partial x^\nu \eta_{\mu\nu} - \alpha_\mu \partial^2 x^\mu \]

and the background charges are given by
\[ 12Q_B^2 = 74 - \frac{1}{2} , \quad 12\alpha \cdot \alpha = 26 - \frac{1}{2} - D \quad (1.3) \]

Demanding that \( Q \) vanish on the states of standard ghost type implies that the part of the state which is generated by the action of the bosonic oscillators alone must satisfy
\[ (L_0 - 4) |\psi\rangle = 0 , \quad (W_0) |\psi\rangle = 0 , \quad L_n |\psi\rangle = 0 , \quad W_n |\psi\rangle = 0 , \quad n \geq 1 \quad (1.4) \]

where in terms of the currents \( L_n \) and \( W_n \) are given by \( T(z) = \sum_n L_n z^{n-2} \) and \( W(z) = \sum_n W_n z^{n-3} \)

Although it was found that there existed physical states involving all types of oscillators, those physical states that involved the oscillators \( \alpha_n \) contained in \( \varphi \) were null. This was only established at low levels, but we will assume, in this paper that it is true at all levels. The non-null states were therefore contained in the subspace \( \tilde{H} \) of the complete Fock space \( H \) generated by the oscillators \( \alpha_n^\mu \) contained in \( x^\mu \) alone. Further, as a consequence of the \( W_0 \) constraint the physical states contained in \( H \) had their momentum in the \( \varphi \) direction frozen to fixed values. As a result, such physical states \( |\tilde{\psi}\rangle \in \tilde{H} \) obeyed the constraints
\[ \tilde{L}_n |\tilde{\psi}\rangle = 0 , \quad n \geq 1 , \quad (\tilde{L}_0 - a^i) |\tilde{\psi}\rangle = 0 \quad (1.5) \]

where the intercept \( a^i \) can only take the two values 1 and \( \frac{15}{16} \) and \( \tilde{T} = \sum_n \tilde{L}_n z^{n-2} \).

The effective central charge for the states in \( \tilde{H} \) is that for the Virasoro operators \( \tilde{L}_n \), namely \( 26 - \frac{1}{2} \). The physical states that satisfy equation (1.4) also include some null states. For example, in the intercept 1 and \( \frac{15}{16} \) sectors the lowest level null states are of the form \( \tilde{L}_{-1} |\Omega_1\rangle \) and \( (\tilde{L}_{-2} + \frac{4}{7} \tilde{L}_{-1}^2) |\Omega_2\rangle \) respectively. The count of physical degrees of freedom, that is physical states minus null states, was found up to and including level 2. We refer the reader to reference [9] for further details of the above. It was also noticed as a phenomenological observation, that the central charges and intercepts arising in \( W_N \) strings were related to the central
charges and weights that occur in the minimal models [7][8] and that at low levels some of the physical states had positive norm [8].

The purpose of the present paper is to find all the physical states of the $W_3$ string contained, at any level, in the subspace $\tilde{H}$. We will show that all these states have positive norm, so establishing a no ghost theorem in this sector. We also divide the physical states into those of positive definite norm and those that are null. It is clear from the distribution of null states, described above at the lowest level, that the count of states is not that which would follow from a straightforward light-cone analysis. One might, naively, think that the background charge used in the above $W_3$ string construction is responsible for this failure; however, in section 2 we examine, as a toy model, a bosonic string with any number of scalars and a background charge. We find that there is no obstacle to constructing a light-cone formulation provided, as usual, that $c = 26$ and the intercept is 1. These conditions do not hold for the above physical states in $\tilde{H}$. Consequently, to find the spectrum we must use an alternative construction.

In section 3 we give the spectrum generating algebra of the $W_3$ string in the subspace $\tilde{H}$ and use it to derive the physical states. This calculation is carried out firstly for the $W_3$ string that is constructed from $\varphi$, as usual, and from 25 free scalars $x^\mu$ and one real fermion $\psi$. This string does not require a background charge, except in the $\varphi$ direction, since the fields $x^\mu$ and $\psi$ possess a central charge $c = 26 - \frac{1}{2}$ without it. As a result the spectrum of this string is free from the interpretational difficulties associated with the masses in the presence of the background charge. The calculation of the spectrum is also performed for the $D+1$ scalar $W_3$ string described above. One of the interesting features of the spectrum of these strings is that their partition functions involve the Ising model characters corresponding to the conformal weights 0 and $\frac{1}{16}$. They arise from the structure of the null states, which are associated with a Virasoro-like oscillator $C_n$. For the former string, these characters are not to be confused with that arising from the real fermion.

An important consistency condition of the usual bosonic and superstrings, in addition to the absence of a conformal anomaly, is the requirement of modular invariance. Having derived the spectrum of the $W_3$ strings we can, in section 4, examine whether their cosmological constant are modular invariant. We find
that they are not modular invariant if we take into account the physical states of standard ghost type, but that were the spectrum to possess additional states associated with an Ising character for weight $\frac{1}{2}$, then they would be modular invariant.

As we have explained the physical states we found in $\tilde{H}$ may not be the full set of states of the $W_3$ string for two reasons: there may be non-trivial cohomology classes of $Q$ which are not of standard ghost type and even amongst those of standard ghost type there may exist, above level 2, non-null physical states not contained in $\tilde{H}$, that is with $\varphi$ oscillators present. The structure of the required states shows that the former possibility must occur if modular invariance is to hold. In section 5 it is shown that these required additional states do indeed occur in the cohomology of $Q$.

2. A Toy Model

One of the more unusual features of $W_3$ string theories, as formulated so far, is that they possess a background charge. The usual bosonic string can also be constructed for any number of scalar fields provided one introduces a background charge which is tuned to give $c = 26$, i.e. $D + 12\alpha^2 = 26$. The energy momentum for such an open string is

$$ T = -\frac{1}{2} \partial x \cdot \partial x - \alpha \cdot \partial^2 x $$

where we have rotated to Euclidean world sheet coordinates. The number of states at any given level is most easily found in the usual 26 dimension string by using the light-cone formalism [10].

We now examine the extent to which we can apply the light-cone method to the string with a background charge. We will not assume, at least initially that it is critical, i.e. $c$ may not necessarily be 26. We can, as usual, impose the choice $\partial x^+ = \bar{\partial} x^+ = c/2$ where $c$ is a constant which we take to be $2\alpha' p^+$. We can then solve $T = 0$ for $\partial x^-$ namely:

$$ \partial x^- = -\frac{1}{c} \left[ \sum_{i=1}^{D-3} \partial x^i \partial x^i + \partial y \partial y + 2\alpha \partial^2 y \right] $$

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where $x^\mu = (y, x^\mu, \mu = 0, 1, \ldots, D - 2)$ and we have taken the background charge to be in the $y$ direction. In terms of oscillator the above equation becomes

$$\alpha^-_n = -\frac{1}{\sqrt{2\alpha p^+}}(l_n - e\delta_{n,0}) \quad (2.3)$$

where $e$ is a constant.

$$l_n =: \frac{1}{2} \sum_{m}^{D-3} \sum_{i=1}^{D-3} \alpha_{n-m}^i \alpha_m^i + \alpha_{n-m} \alpha_m - i(n+1)\alpha \alpha_n \quad (2.4)$$

and

$$i \partial y = \sqrt{\frac{\alpha'}{2}} \sum_{n} \alpha^\mu z^{-n} \quad (2.5)$$

and similarly for $x^\mu$. The Hamiltonian of the system is $H = -\sqrt{2\alpha p^+} \alpha^-$ and the $J^\pm$ generators are of the form

$$J^\pm =: (\frac{1}{\sqrt{2\alpha'}}(x^- \alpha^-_0 - x^-_0 \alpha^-_0) + i \sum_{n=1}^{\infty} \frac{1}{n} (\alpha^-_n \alpha^-_n - \alpha^-_n \alpha^-_n)) \quad (2.6)$$

The background charge restricts the string to have only $SO(D - 2, 1)$ Lorentz invariance; however, we should demand that this symmetry is preserved under quantization. The problematic commutation relation is $[J^-, J^-]$ which can be evaluated using the relations

$$[l_n, l_m] = (n - m)l_{n+m} + \frac{(D - 2 + 12\alpha^2)}{12} n(n^2 - 1)\delta_{n+m,0}$$

$$[x^i_0, l_n] = \sqrt{2\alpha'}i\alpha^i_n, \quad [\alpha^i_n, \alpha^j_m] = -m\alpha^j_{n+m}$$

$$[x^-_0, p^+] = i \quad (2.7)$$

One finds that these relations depend on the presence of the background charge only through the central charge of $l_n$ and so the commutator $[J^-, J^-]$ will vanish if the intercept, $e = 1$ and $c = D + 12\alpha^2 = 26$. Assuming these conditions are met, then quantization is straightforward and we can immediately state that the number of states $c_n$ at level $n$ is given by

$$\sum_{n=0}^{\infty} c_n x^n = \prod_{n=1}^{\infty} \frac{1}{(1 - x^n)^{D-2}} \quad (2.8)$$
A similar analysis holds for the closed string.

Given the number of states we can calculate the cosmological constant of the closed string and test its modular invariance. This quantity is proportional to

$$\Delta = \int \frac{d^2 \tau}{Im \tau} \int d^D p Tr [z^{L_0 - 1} \bar{z}\bar{L}_0 - 1]$$

(2.9)

where $z = e^{2\pi i \tau}$. Assuming the value of $\alpha^2$ required to give $c = 26$ and equation (2.8), we find that

$$\Delta \propto \int \frac{d^2 \tau}{(Im \tau)^2} F(\tau)$$

(2.10)

where

$$F(\tau) = \frac{1}{(Im \tau)^{D-2}} |\eta(\tau)|^{-2(D-2)}$$

(2.11)

and

$$\eta(\tau) = z^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - z^n)$$

(2.12)

Using the standard modular transformations of $\eta$, namely

$$\eta(\tau + 1) = e^{\frac{i\pi}{12}} \eta(\tau)$$

$$\eta(-\frac{1}{\tau}) = \sqrt{-i\tau} \eta(\tau)$$

(2.13)

we find that the cosmological constant is modular invariant for any $D$. In the $W_3$ string we encounter bosonic-like sectors, which do not have intercept 1 or $c = 26$. In fact, they have intercept 1 or $\frac{15}{16}$ and $c = \frac{51}{2}$. It follows from the above remarks that a straightforward light-cone quantization will not apply in these sectors as the Lorentz algebra will not close. As a consequence, the count of states will not be given by equation (2.8). The purpose of the next section is to find the count of states in these sectors.

3. The States

The simplest method of finding the number states at a given level for the usual bosonic string is to use the light-cone formalism; however, in our case this formalism does not work in a straightforward way as was pointed out in the previous section. Our method of finding the spectrum of states in this section will be to
construct the spectrum generating algebra. Let us first explain this technique\cite{11} for the open bosonic string in a way which will be useful for the W-string. We introduce the operators.

$$A^\mu_n = A^\mu_n - \frac{n}{2} k^\mu F_n$$

(3.1)

where

$$A^\mu_n = \oint dz : i \partial x^\mu e^{ink \cdot x} :$$
$$F_n = \oint dz : \frac{k \cdot \partial x}{k \cdot x} e^{ink \cdot x} :$$

(3.2)

and $k^\mu$ is a momentum which satisfies $k^2 = 0$. The operators $A^\mu_n$ commute with $\tilde{L}_n$ and obey the algebra

$$[A^\mu_n, A^\nu_m] = (n \eta^\mu_\nu + 2n^3 k^\mu k^\nu) p.k \delta_{n+m,0} + mk^\mu A^\nu_{n+m} - nk^\nu A^\mu_{n+m}.$$ (3.3)

We next introduce the light-cone coordinates, given by

$$V^\pm = \frac{1}{\sqrt{2}} (V^{D-1} \pm V^0).$$ (3.4)

for any vector $V^\mu \; \mu = 0, 1, \ldots, D - 1$. We choose the momentum $k^\mu$ to have the components $k^- = 1, \; k^+ = k^i = 0$.

The operator $A^+_n = p^+ \delta_{n,0}$ and plays no further role. It is appropriate to make one further redefinition; let

$$B_n^i = A_n^i, \; i = 1, \ldots, D - 2$$

(3.5)

$$B_n = -A^-_n - l_n + 1$$

(3.6)

where $l_n = \frac{1}{2} : \sum_{i=1}^{D-2} \sum_p B^i_{-p} B^i_{n+p} :$. In this expression the operators are normal ordered with respect to the $B_n^i$ and not with respect to the underlying oscillators. The algebra of these operators, which, of course, also commute with $\tilde{L}_n$, is

$$[B^i_n, B^j_m] = n \delta_{n+m,0} \delta^{ij}$$

$$[B_n, B_m] = (n - m) B_{n+m} + \left(2 - \frac{D-2}{12}\right) n(n^2 - 1) \delta_{n+m,0}$$

$$[B_n, B^i_m] = 0.$$ (3.7)
We can consider these operators to act on the tachyon state \( |p, 0\rangle \) which we choose to have momentum \( p^+ = 1 = p^-; p^i = 0 \). They have a well defined action since \( k.p = 1 \) and so the general such state is of the form

\[
\prod_{j} B_{-n_j} \prod_{p} B_{-n_p}^{k_p} |p, 0\rangle.
\] (3.8)

Clearly, these states are all physical states in the sense that they satisfy the physical state conditions \( L_n |\psi\rangle = 0, \ n \geq 1 \ (L_0 - 1)|\psi\rangle = 0 \). It is also obvious that states with no \( B_n \)'s are of positive norm. For \( D = 26 \), the \( B_n \) algebra has no central term (i.e. \( c = 0 \)) and the \( B_n \)'s act on states which are seen by them as being of highest weight, \( h = 0 \), since \( B_0 = -p^- + 1 \). Consequently, the norm of any such states with \( B_n \) oscillators is zero.

In fact, the above states are all the physical states. To see this, we add to our collection of oscillators

\[
\phi_n = \oint d z z^{-1} e^{i n k . x}.
\] (3.9)

These have the relations

\[
[L_p, \phi_m] = -p \phi_m \ , \ [B_p, \phi_m] = m \phi_{m + p} \ ,
\]

\[
[\phi_n, \phi_m] = 0 \ , \ [B^i_m, \phi_p] = 0 .
\] (3.10)

One can show that the oscillators \( B_n^i, B_n \) and \( \phi_n \) do span the same Hilbert space as the original oscillators \( \alpha_\mu^i, \mu = 0, 1, \ldots, 25 \). The proof is similar to the proof of the no ghost theorem of reference \[12 \]. It is also straightforward to show that any state which contains a \( \phi_{-n} \) factor is not a physical state.

We now give the analogous construction for the \( W_3 \) string which consists of the bosonic fields \( \varphi, x^\mu \; \mu = 0, 1, \ldots, 24 \) and the fermion field \( \psi \). As we explained earlier, we are considering only the states of standard ghost type which are in \( \tilde{H} \). We construct the operators

\[
C^i_n = A^i_n, \ i = 1, \ldots, 23
\]

\[
C_n = -A^-_n + r_n
\] (3.11)
and
\[ G_s = \oint dz :\left(ik\partial_x \right)^{\frac{1}{2}} e^{iskx} \psi(z) : \]  
(3.12)

where
\[ r_n = -\left\{ \frac{1}{2} \sum_{i=1}^{23} \sum_p C^i_{-p} C^i_{n+p} + \frac{1}{2} \sum_r G_{-r} G_{n+r} \left( r + \frac{n}{2} \right) + \delta \right\} + 1 . \]  
(3.13)

They obey the algebra
\[ [C^i_n, C^j_m] = \delta_{n+m,0} \delta_{i,j}, \quad [C^i_n, C^m_n] = 0, \quad [C^i_n, G^r_r] = 0, \]
\[ \{G^r_r, G^s_s\} = \delta_{r+s,0}, \quad [C^i_n, C^m_n] = (n-m)C^i_{n+m} + \frac{n(n^2-1)}{24} \delta_{n+m,0} \]  
(3.14)

The index on the \( G^r_r \) can take either half integer or integer values corresponding to the Neveu-Schwarz and Ramond sectors, while the parameter \( \delta \) takes different values depending on the sector. It is zero in the Neveu-Schwarz sector, but \( \frac{1}{16} \) in the Ramond sector. The operators \( C^i_n, C^i_n \) and \( G^s_s \) all commute with the Virasoro operators \( \tilde{L}_n \) and hence we can use them to create physical states by acting on the tachyon state. The intercepts in the two sectors are \( a^i = (1, \frac{15}{16}) \) and as such we choose for our tachyonic momentum to be \( p^+ = 1, p^- = a^i - \delta, p^i = 0 \).

We note that \( p.k = 1 \). The states
\[ \prod_{n_k} C^i_{-n_k} \prod_{r_j} G^j_{-r_j} \prod_{n_i} C^i_{-n_i} \langle 0, p \rangle \]  
(3.15)

are physical states. Those states with no \( C^i_{-n}'s \) are clearly of positive definite norm. The operators \( C^i_{-n} \) obey a Virasoro algebra which has a central charge \( \tilde{c} = \frac{1}{2} \) and they act on highest weight states with weight \( h_i = -p^- + 1 - \delta = 1 - a^i \); that is 0 and \( \frac{1}{16} \). These states correspond to certain of the Ising model states which, being a model in the unitary minimal series [13], has states with only positive norm. Consequently the theory is unitary in these two sectors in the sense that it satisfies a no ghost theorem. The number of these states which have positive definite norm is encoded through the Kac determinant in the Ising Model characters[14], which are defined by
\[ \chi_h(z) = \sum_{n=0}^{\infty} z^{h - \frac{1}{16}} \dim V_{n+h} = z^{h - \frac{1}{16}} \hat{\chi}_h(z) \]  
(3.16)
where \( h \) is weight of the highest weight state and \( V_q \) is the dimension of the space with weight \( q \). The results are[14]

\[
\hat{\chi}_0(z) = \prod_{n=1}^{\infty} \frac{1}{(1-x^n)}
\]

\[
\left\{ 1 + \sum_{\substack{m = 0, 3. \mod 4}} (-1)^m x^{\frac{m(3m-1)}{4}} + \sum_{\substack{m = 0, 1. \mod 4}} (-1)^m x^{\frac{m(3m+1)}{4}} \right\}
\]

\[
\hat{\chi}_1(z) = \prod_{n=1}^{\infty} \frac{1}{(1-x^n)}
\]

\[
\left\{ 1 + \sum_{\substack{m = 1, 2. \mod 4}} (-1)^{m+1} x^{\frac{m(3m-1)}{4}} + \sum_{\substack{m = 2, 3. \mod 4}} (-1)^{m+1} x^{\frac{m(3m+1)}{4}} \right\}
\]

\[
\hat{\chi}_{16}(z) = \prod_{n=1}^{\infty} (1 + x^n) = \prod_{n=1}^{\infty} \frac{1}{1-x^n} \sum_{p \in \mathbb{Z}} (-1)^p x^{p(\frac{3p+1}{2})}
\]  

(3.17)

where \( m = 1, 2, 3, \ldots \).

In fact, the states of equation (3.15) are all the physical states. One can introduce the operator \( \phi_n \) of equation (3.19) and by a similar argument convince oneself that it completes the Fock space to that generated by \( \alpha_n^\mu \) and \( \phi_n \) and that any state with a \( \phi_n \) oscillator is not a physical state. The states arising from \( G_r \) oscillators can belong to the Neveu-Schwarz or Ramond sector. In the former case, they will contribute a factor

\[
\prod_{r=\frac{1}{2}}^{\infty} (1 + x^r)
\]  

(3.18)

and in the latter case a factor

\[
\prod_{m=0}^{\infty} (1 + x^m)
\]  

(3.19)
One could GSO project [15] in either of these sectors in the usual way. Let us refer to these factors generically as $\hat{N}(x)$ which should be taken to represent either sector, project or not.

We can now write down the number of states for the open $W_3$ string in the two sectors. The number of states $c_n$ at level $n$ is given for intercept $a^1 = 1$ by

$$\sum c_n x^n = \prod_{n=1}^{\infty} \frac{1}{(1-x^n)^{2\theta}} \hat{\chi}_0(x) \hat{N}(x)$$

(3.20)

and for intercept $a^2 = \frac{15}{16}$

$$\sum c_n x^n = \prod_{n=1}^{\infty} \frac{1}{(1-x^n)^{2\theta}} \hat{\chi}_\frac{15}{16}(x) \hat{N}(x)$$

(3.21)

The closed $W_3$ string has its number of states at level $c_n$ given by the above expressions times a similar factor with $x$ replaced by $\bar{x}$.

We now repeat the calculation of the spectrum, but for the $W_3$ string constructed from $D + 1$ scalars string described in the introduction. The first step is to take account of the background charge $\alpha^\mu$ when constructing the spectrum generating algebra. In doing this we will often use the same symbol as before, but the reader will understand that its definition has been modified. One finds that if the operator $A^\mu_n$ of equation (3.2) are modified to become.

$$A^\mu_n = A^\mu_n - \frac{1}{2}(nk^\mu - 2i\alpha^\mu)F_n$$

(3.22)

they will commute with $\tilde{L}_n$ provided $k \cdot (k - 2i\alpha) = 0$. If we further take $k^2 = 0$ then they obey the algebra.

$$[A^\mu_n, A^\nu_m] = (n!^\mu^\nu + 2n^3 k^\mu k^\nu) p.k\delta_{n+m,0} + mk^\mu A^\nu_{n+m} - nk^\nu A^\mu_{n+m}$$

$$-i\alpha^\mu k^\nu m^2 \delta_{n+m,0} p \cdot k + i\alpha^\nu k^\mu m^2 \delta_{n+m,0} p \cdot k.$$  \hspace{1cm} (3.22)

We next define the operators

$$C_n = -A^\mu_n - \left\{ \sum_p \sum_i : C^i_{n-p} C^i_p : -i(n+1)\alpha \cdot C_n \right\} + 1$$
\[ C_n^i = A_n^i + i \alpha^i \delta_{n,0} \]  

(3.24)

We have taken the background charge \( \alpha^\mu \) to be transverse ie. \( \alpha^+ = 0 \) and \( \alpha^- = 0 \) and have chosen \( k^- = 1 \), \( k^+ = k^i = 0 \). These new operators have the advantage that they have particularly simple commutation relations, namely the same as those of equation (3.14) without the \( G_{-r} \). Thus the \( C_n \) obey a Virasoro algebra with central charge \( \tilde{c} = \frac{1}{2} \).

The tachyon is chosen to have its momentum \( p^\mu \) in the direction \( p^+ = 1 \), \( p^- = a^i \), \( p^i = 0 \), \( i = 1, \ldots, D-2 \), where \( a^i \) is the intercept corresponding to the sector we are considering. We note that \( \alpha \cdot p = 0 \), \( p \cdot k = 1 \) and \( p^2 = 2a^i \) and consequently \( \frac{1}{2} p \cdot (p - 2i\alpha) = a^i \) as it should. Since \( C_n \) and \( C_n^i \) commute with \( \tilde{L}_n \) we conclude that they generate physical states which are of the form

\[ C_{-m_1} \ldots C_{-m_q} C_{-n_1} \ldots C_{-n_p} |0, p) \]  

(3.25)

where \( |0, p) \) is the tachyon state and is annihilated by \( \alpha_n^\mu \), \( n \geq 1 \). The states are similar to those that occurred for the \( W_3 \) string considered above, except that there is no \( G_r \) and we may conclude that they are the only physical states in the Fock space generated by \( \alpha_n^\mu \). These states have positive norm. The operators \( C_n \) act on a highest weight state of weight \( h_i = 1 - a_i \) and consequently the number of states \( c_n \) at level \( n \) in the sector with intercept \( a^i \) is given by

\[ \prod_{n=1}^{\infty} \frac{1}{(1 - x^n)^{D-2}} \tilde{\chi}_{h_i}(x) \]  

(3.26)

It is straight-forward to verify that the count of states given by equation (3.26) for the first two levels is the same as that found in reference [9] by explicitly solving the physical states conditions.

The background charge \( \alpha^\mu \) was chosen to lie in the transverse direction. In fact any choice which satisfies \( k^2 = 0 = k \cdot \alpha \), \( p \cdot k = 1 \), \( \frac{1}{2} p \cdot (p - 2i\alpha) = a^i \) can be used and leads to the same results for the spectrum. When showing that the \( C_n \) had a central charge of \( 1/2 \) we used the value of \( \alpha^\mu \) of equation (1.3), however the result would in general be that the \( C_n \) obeyed a Virasoro algebra with central charge \( \tilde{c} = 26 - D - 12a^2 \). We note that for a critical bosonic string \( \tilde{c} = 0 \) and as \( C_0 \) acts on states of highest weight 0 we find that \( C_n \) creates only null states. Consequently, we recover for the critical bosonic string the light-cone count of
states of section 2. In general, however, one can imagine such states arising in other theories, for example in the higher $W_N$ string theories. Demanding that there be no negative norm states implies that $\tilde{c}$ must be one of the central charges of the minimal unitary series if $\tilde{c} \leq 1$ and that the intercepts are related to the weights $h_{r,s}$ of the corresponding minimal unitary series by $a_{r,s} = 1 - h_{r,s}$. This observation explains, at least from the viewpoint of unitarity, why the $W_N$ strings involve the minimal models.

For the $W_3$ string the above means that the only other intercept, apart from 1 and $15/16$, is $1/2$. In this latter case, the previous analysis applies, but the $C_n$ acts on states generated by $C^{i_n}$ which have a highest weight of $1-1/2 = 1/2$. As such, the count of states is given by equation (3.26) with the Ising character for weight 1/2.

4. Discussion of Modular Invariance

Having calculated the spectrum of states of the $W_3$ strings in the two sectors with intercepts $a^i = (1, \frac{15}{16})$ we can examine whether or not they lead to a cosmological constant for the closed string that is modular invariant. The cosmological constant is of the form

$$\int \frac{d^2\tau}{Im\tau} \int d^D p Tr[z^{L_0-a^i} \bar{z}^{\bar{L}_0-a^j}]$$

$$= \int \frac{d^2\tau}{(Im\tau)^2} F_{ij}$$

where

$$F_{ij}(\tau) = \frac{1}{(Im\tau)^{D-2}} Tr[z^{N-a^i+\frac{a^2}{2}} \bar{z}^{\bar{N}-a^j+\frac{a^2}{2}}]$$

and

$$N = L_0 - \frac{p \cdot (p - 2i\alpha)}{2}$$

and similarly for $\bar{N}$.

The modular invariance of the cosmological constant must be examined separately for the two $W_3$ strings whose spectrum of states we found in section 3. We begin with the $D+1$ scalar $W_3$ string since in this case the result is particularly easy to find. For this string, we have a background charge, but on the other hand
we only have to worry about a sum over sectors with different intercepts $a^i$. Using equation (3.26) we find that

$$F_{ij}(\tau) = \frac{1}{(Im\tau)^{\frac{D-2}{2}}} \frac{1}{|\eta(\tau)|^{2(D-2)}} \chi_{h_i}(z)\chi_{h_j}(\bar{z})$$

$$= F^B F^I_{ij}$$ (4.4)

where $F^I_{ij} = \chi_{h_i}(z)\chi_{h_j}(\bar{z})$.

It is instructive to verify that the powers of $z$ and $\bar{z}$ are contained in $\eta$ and $\chi_{h_i}$ in the correct way. We find in $F_{ij}$ a $z$ prefactor to the power $\left(-\frac{D-2}{24}\right) + \left(-\frac{1}{48} + h_i\right)$ which one verifies is equal to $\frac{z^2}{2} - a^i$. In fact, $F^B$ is by itself a modular invariant factor and consequently if the cosmological constant is to be a modular invariant then we must sum over the sectors in such a way that for constants $e_{ij}$, $\sum_{ij} F^I_{ij} e_{ij}$ is modular invariant. This is equivalent to requiring that we build a modular invariant Ising model. However, it is well known that there is only one modular invariant combination of the Ising model characters namely:

$$|\chi_0|^2 + |\chi_{\frac{1}{16}}|^2 + |\chi_{\frac{1}{2}}|^2$$ (4.5)

The cosmological constant above includes a sum over only the weight 0 and 1/16 Ising characters and consequently modular invariance demands the presence in the spectrum of additional states whose spectrum form the character $\chi_{\frac{1}{2}}$ of the Ising model.

We now examine the modular invariance of the $W_3$ string constructed from $\varphi$, 25 scalars and one fermion. For this string we have no background charge, apart from in the $\varphi$ direction, but now the above trace may be decomposed into a product of traces over the bosonic and fermionic oscillators. From the previous section on the spectrum we find the cosmological constant to be given by a sum over terms of the form

$$F_{ij}(\tau) = \frac{1}{(Im\tau)^{\frac{D}{2}}} \frac{1}{|\eta(\tau)|^{46}} \chi_{h_i}(z)\chi_{h_j}(\bar{z})N(z)\bar{N}(\bar{z}) = F^B F^K_{ij}$$ (4.6)

where $F^B = \frac{1}{(Im\tau)^{\frac{D}{2}}} \frac{1}{|\eta(\tau)|^{46}}$.

The symbol $N(z)$ denotes the contribution to the trace from the left-handed fermionic sector. In this sector we must take account of whether the particles going
around the loop are of Neveu Schwarz or Ramond type and are projected or not. This corresponds to the sum over spin structures [16]; in the usual notations $(-, \pm)$ corresponds to Neveu-Scharwz particles in the loop and the $+$ and $-$ as to whether we insert a projector or not respectively; similarly $(+, \pm)$ correspond to Ramond particles in the loop and $+$ and $-$ as to whether they are projected or not.

We have, in obvious notation, the correspondence

$$N_{(-,-)} = \left( \frac{\theta_3(0|\tau)}{\eta(\tau)} \right)^{\frac{1}{2}}, \quad N_{(-,+)} = \left( \frac{\theta_4(0|\tau)}{\eta(\tau)} \right)^{\frac{1}{2}}$$
$$N_{(+,-)} = \left( \frac{\theta_2(0|\tau)}{\eta(\tau)} \right)^{\frac{1}{2}}, \quad N_{(+,+)} = 0 \quad (4.7)$$

The intercept has been divided in the above process between the traces in the different sectors, in other words into the $\eta$, $\chi$ and $N$ factors. We took $\frac{23}{24}$ to be associated with $\eta$, that is with the bosonic $A^i_n$ oscillator sector, $\frac{1}{48}$ in $\chi_0$, $-\frac{1}{24}$ in $\chi_{1/16}$ and $-\frac{1}{24}$ in the fermionic sector if it is Ramond, but $\frac{1}{48}$ if it is Neveu-Schwarz. One can readily verify that these terms do indeed give the actual intercepts $a^i = (1, \frac{15}{16})$.

Before ascertaining whether the cosmological constant is modular invariant, we must list the modular transformations of its building blocks; for the $\theta$ functions these are the following:

$$\theta_2(0|\tau + 1) = e^{\frac{4\pi i}{\tau}} \theta_2(0|\tau), \quad \theta_3(0|\tau + 1) = \theta_4(0|\tau), \quad \theta_4(0|\tau + 1) = \theta_3(0|\tau) \quad (4.8)$$

and

$$\theta_2(0|\frac{-1}{\tau}) = (-i\tau)^{\frac{1}{2}} \theta_4(0|\tau), \quad \theta_3(0|\frac{-1}{\tau}) = (-i\tau)^{\frac{1}{2}} \theta_3(0|\tau),$$
$$\theta_4(0|\frac{-1}{\tau}) = (-i\tau)^{\frac{1}{2}} \theta_2(0|\tau) \quad (4.9)$$

While the Ising characters $\chi_{h_i}$ transform as [17]

$$\chi_0(\tau + 1) = e^{-\frac{24\pi i}{48}} \chi_0(\tau), \quad \chi_{\frac{1}{16}}(\tau + 1) = e^{\frac{24\pi i}{48}} \chi_{\frac{1}{16}}(\tau),$$
$$\chi_{\frac{1}{2}}(\tau + 1) = e^{\frac{46\pi i}{48}} \chi_{\frac{1}{2}}(\tau), \quad (4.10)$$

and

$$\chi_0\left(\frac{-1}{\tau}\right) = \frac{1}{2}(\chi_0(\tau) + \chi_{\frac{1}{2}}(\tau)) + \frac{1}{\sqrt{2}} \chi_{\frac{1}{16}}(\tau)$$
\[ \chi_{\frac{1}{16}}(-\frac{1}{\tau}) = \frac{1}{\sqrt{2}}(\chi_0(\tau) - \chi_{\frac{1}{2}}(\tau)) \]
\[ \chi_{\frac{1}{2}}(-\frac{1}{\tau}) = \frac{1}{2}(\chi_0(\tau) + \chi_{\frac{1}{2}}(\tau)) - \frac{1}{\sqrt{2}}\chi_{\frac{1}{16}}(\tau) \] (4.11)

These transformations of the characters could be deduced from those of the \( \theta \) functions when we recognise that
\[ \chi_0(\tau) \pm \chi_{\frac{1}{2}}(\tau) = N_{(-,\pm)}(\tau) \]
and
\[ \chi_{\frac{1}{16}}(\tau) = \frac{1}{\sqrt{2}}N_{(+,-)}(\tau) \] (4.12)

When constructing a modular invariant partition function, we should allow not only for all possible sums over spin structures in the fermionic sector, but also sum over the different sectors corresponding to the different intercepts. However, since we only have two different intercepts, we can only use the corresponding characters \( \chi_{h_i} \) where \( h_i = 1 - a^i \). In fact, \( P^B \) is modular invariant by itself and so demanding that the cosmological constant to be modular invariant is equivalent to requiring that there exist a modular invariant system which is the tensor product of two Ising models with the condition that in one of the Ising models the states associated with the character with weight 1/2 is missing. Thus we now investigate all possible modular invariant two Ising models and examine whether any of them satisfy this condition.

The two Ising model partition function is of the form of a sum of terms of the form
\[ \chi_{h_i}(z)\chi_{h_j}(z)\chi_{h_k}(\bar{z})\chi_{h_l}(z) \] (4.13)

Using equation (4.12) we may rewrite this in the form
\[ \frac{1}{|\eta(\tau)|^2} \sum_{i,j,k,l} y^*_{ij} y^*_{kl} y^i y^j = \frac{1}{|\eta(\tau)|^2} P \] (4.14)

where \( y^i = \theta_{i+1}^{\frac{1}{2}}, i = 1, 2, 3 \). Invariance under \( \tau \rightarrow -\frac{1}{\tau} \) demands that \( P \) should be invariant under
\[ y_2 \rightarrow y_2, \ y_1 \leftrightarrow y_3 \] (4.15)
while $\tau \leftarrow \tau + 1$ implies that $P$ should be invariant under

$$y_2 \leftrightarrow y_3, \ y_1 \leftarrow e^{\frac{i\pi}{8}} y_1$$

(4.16)

A lengthy calculation shows that there are only two distinct matrices $b_{ij}^{kl}$ that are preserved by the transformations induced by equations (4.15) and (4.16). As such the only invariants are

$$\frac{1}{|\eta(\tau)|^2} \sum_i |y_i|^4$$

(4.17)

and

$$\frac{1}{|\eta(\tau)|^2} (|y_1|^2|y_2|^2 + |y_2|^2|y_3|^2 + |y_3|^2|y_1|^2).$$

(4.18)

It is straight-forward to re-express these two invariants in terms of the Ising characters and one finds that they do involve the character for weight 1/2 in both copies of the Ising model. Thus as for the other $W_3$ string we require the additional states associated with this character to gain modular invariance.

5 Conclusion and discussion

In this paper we have found all physical states of $W_3$ strings of standard ghost type. This was achieved by constructing the corresponding spectrum generating algebra which for the multi-scalar $W_3$ string consisted of the operators $A_n^\mu$ and $C_n$. These operators and one other $\phi_n$ spanned the same Hilbert space as that generated by the $\alpha_n^\mu$. Those involving the $\phi_n$ were not physical. The remaining states, generated by $A_n^\mu$ and $C_n$ are physical, but only some of those involving $C_n$ are null. The $C_n$ obeyed a Virasoro algebra with central charge 1/2 and one finds that the partition function for these states in the sectors with intercepts 1 and 15/16 involve Ising model characters corresponding to the weights 0 and 1/16 respectively.

We then examined whether the cosmological constant was modular invariant. It emerged that the states found above were not sufficient and one required, in addition, states corresponding to the Ising character 1/2. Following through the derivation of the count of states it was clear that precisely these states would arise from a sector whose physical states obeyed the conditions of equation (1.5), but with intercept 1/2.

In a previous paper[18] on $W_3$ string scattering, it also emerged that the consistency of the theory demanded the existence of additional states beyond those
of standard ghost type. In fact, there do exist additional states, in the cohomology
of $Q$, some physical states of non-standard ghost number were first found in
reference [19], in the context of the two scalar $W_3$. These states had ghost
number 2, in the convention where the states of standard ghost number type have
ghost number 0. These authors also realised that their effective intercept was 1/2
and that at the level of phenomenological number matching, discovered previously
[7],[8], these states should be associated with the weight 1/2 of the Ising model,
so completing the set of relevant Ising operators appearing in the $W_3$ string. The
generalisation of discrete states of two dimensional string theory to the two scalar
$W_3$ string were discussed in reference [20], these discrete states occur for states with
a number of different ghost numbers, however, also in this reference, examples of
ghost number 1, level one physical states in the 3 scalar $W_3$ string with continuous
momentum were given.

As we now explain, the cohomology of $Q$ necessarily involves such states
of non-standard ghost type. This follows from the observation that certain null
states vanish automatically in the free field representations used to construct string
theories. Such a state, being null, can also be written as $Q|\psi\rangle$ for some state $|\psi\rangle$. It
follows that $|\psi\rangle$ is a state which is annihilated by $Q$, has a non-standard ghost
numbers, but is not in general BRST exact. Clearly, for every vanishing null state
one can construct such a BRST exact state. An example of this phenomenon, in
the bosonic string, is the state $L_{-1}|0,p\rangle$ which vanishes for the special momentum
$p^\mu = 0$. It can, however, be written as $b_{-1}|0,0\rangle$ which is a states well known to
belong to the cohomology of $Q$. One can also use the vanishing null states to find
the discrete states in two dimensional string theory [21].

We now apply the above argument to the multi-scalar $W_3$ string to find
states of non-standard ghost type which are annihilated by $Q$. At level one the
null states are of the form [9]

$$(W_{-1} \pm \frac{i}{\sqrt{522}}L_{-1})|\beta,p\rangle$$

provided $L_n|\beta,p\rangle = 0, n \geq 1, W_n|\beta,p\rangle = 0, n \geq 1$ and with $L_0$ and $W_0$
eigenvalues of 3 and $\pm(-i)\sqrt{\frac{2}{201}}$ respectively. If we now consider those states
$|\beta,p\rangle$ which contain no $\varphi$ oscillators, then these physical state conditions become

$$\tilde{L}_n|\beta,p\rangle = 0, n \geq 1, (\tilde{L}_0 - (3 - 1/2\beta(\beta - 2i\alpha))|\beta,p\rangle = 0$$

(5.2)
and the $W_0$ condition implies only that $\beta = \frac{11iQ}{7}, \frac{6iQ}{7}$ or $\frac{4iQ}{7}$ and $\beta = \frac{10iQ}{7}, \frac{8iQ}{7}$ or $\frac{3iQ}{7}$ for the upper and lower signs respectively. For the choices $\beta = \frac{4iQ}{7}$ and $\beta = \frac{3iQ}{7}$ it is straight-forward to verify that the corresponding null states vanish for all states that satisfy the conditions of equation (5.2). It then follows that the two states
\[
(d_{-1} \pm \frac{i}{\sqrt{522}}b_{-1})|\beta, p\rangle
\]
where $b_n, c_n; d_n, e_n$ are the ghost fields, are annihilated by $Q$ for the above corresponding two values of $\beta$.

The effective intercepts for these non-standard number states can be read off from equation (5.2) to be $1/2$ for $\beta = \frac{4iQ}{7}$ and $15/16$ for $\beta = \frac{3iQ}{7}$. Consequently, it follows that the cohomology of $Q$ does indeed contain all the additional states associated with the Ising model character $1/2$ and so the $W_3$ string is indeed modular invariant. We note that we also find another set of the $15/16$ sector states.

States of the above type also exist at level 2 where the null state is of the form [9]
\[
\left(\frac{2}{261}L_{-2} + \frac{9}{522}L_{-1}^2 + W_{-1}^2\right)|\beta, p\rangle
\]
provided the state $|\beta, p\rangle$ is annihilated by $L_n$, $n \geq 1$ and $W_n$, $n \geq 0$ and has a $L_0$ eigenvalue of 2. Taking the states $|\beta, p\rangle$ to contain no $\varphi$ oscillators, then these physical state conditions become
\[
\tilde{L}_n|\beta, p\rangle = 0 \quad n \geq 1 \quad (\tilde{L}_0 - (2 - 1/2\beta(\beta - 2iQ))|\beta, p\rangle = 0
\]
and the $W_0$ condition implies only that $\beta = \frac{12iQ}{7}, iQ, \frac{2iQ}{7}$

In fact for $\beta = \frac{2iQ}{7}$, the above null states vanish identically for any state satisfying equation (5.5) and consequently the states
\[
\left(\frac{2}{261}b_{-2} + \frac{9}{522}L_{-1}b_{-1} + W_{-1}d_{-1}\right)|\beta = \frac{2iQ}{7}, p\rangle
\]
are annihilated by $Q$. The effective intercept for these states is again $1/2$ and so we find in the cohomology of $Q$ another set of the required additional states.

Given some states in the cohomology of $Q$ we can often find new states in the cohomology of $Q$ by applying the commutator of $Q$ with the free fields of the
theory [22]. Also the vanishing of null states is a general phenomenon and the above states are just two examples of an infinite number. These mechanisms are likely to lead to even more copies of the above states. In fact, even for the 26 dimensional bosonic string one finds two copies of the physical states, but in that case we apply a ghost condition to eliminate the additional states. As we have seen modular invariance for the general $W_3$ string only requires one copy of each sector with intercepts 1, 1/2 and 15/16. One might speculate that the cohomology of $Q$ consists of only these three sectors, possible discrete states, and copies of them and that there is an appropriate ghost constraint which removes the copies. It is also possible that there are conditions arising from global $W_3$ transformations and these may place further conditions on the spectrum. Whether this is the case or not requires a much better understanding of $W_3$ moduli than we have at present. On going work on the cohomology of $Q$ which includes further details of the above non-standard number states will be presented elsewhere[23].

It is interesting to note that the states with intercept 1/2 in the 25 scalar and one fermion $W_3$ string would imply the possibility of an additional massless state namely a scalar. Whether this occurs or not depends on the projections used to gain modular invariance.

It is clear from the arguments advanced in this paper, that the pattern found for $W_3$ will generalise to the $W_N$ strings; namely the count of states will involve characters from the corresponding minimal model and that modular invariance will require non-standard ghost number states which will be present in the cohomology of $Q$. Indeed the $W_N$ string can be constructed from $D+N-2$ scalars using an extended Miura construction. It is expected that the oscillators of $N-2$ of these scalars will be absent from positive definite physical states. This will leave an effective Hilbert space $\tilde{H}$ generated by the action of $D$ oscillators with a background charge $\alpha^\mu$ such that $12\alpha \cdot \alpha = 26 - (1 - \frac{6}{N(N+1)}) - D$. It is straight-forward to verify that the physical spectrum is generated by $C^i_n, i = 1, 2, ..., D - 2$ and by operators $C_n$, which obey a Virasoro algebra with central charge of $1 - \frac{6}{N(N+1)}$, and act on highest weight states with weights that are those of the primary field of the diagonal entries for the corresponding minimal model.

A final point concerns the well known A,$D$,$E$ classification of the modular invariants for minimal models [17]. For any Lie algebra we may construct a $W$ al-
gebra and then a $W$ string theory. As we have learnt from this paper the spectrum of states will involve a corresponding minimal model characters for which we must construct a modular invariant combination in order that $W$ string be modular invariant. Thus $W$ strings could provide a path from Lie groups to modular invariants that could explain the A,D,E correspondence. We note that the $W_3$ string studied in this paper originated from the group $SU(3)$ and it did indeed have the corresponding minimal model modular invariant $(A_2, A_3)$ in its cosmological constant. We also note that the $SU(N)$ $W_N$ string is associated with the unitary minimal model with central charge $c = 1 - \frac{6}{N(N+1)}$ and should have the modular invariant $(A_{N-1}, A_N)$.

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