Non-Extreme and Ultra-Extreme Domain Walls and Their Global Space-Times

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Non-extreme walls (bubbles with two insides) and ultra-extreme walls (bubbles of false vacuum decay) are discussed. Their respective energy densities are higher and lower than that of the corresponding extreme (supersymmetric), planar domain wall. These singularity free space-times exhibit non-trivial causal structure analogous to certain non-extreme black holes. We focus on anti-de Sitter–Minkowski walls and comment on Minkowski–Minkowski walls with trivial extreme limit, as well as walls adjacent to de Sitter space-times with no extreme limit.

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Domain walls can form as topological defects in the early Universe in theories with isolated minima of the matter potential [1] or as boundaries of true vacuum bubbles nucleating in a false vacuum [2]. The induced space-times of domain walls provide a fertile ground to study globally non-trivial space-times without singularities. Here we discuss certain domain walls which are natural generalizations of the planar, extreme domain walls with energy density \( \sigma_{\text{ext}} \equiv \sigma_{\text{susy}} \), that separate isolated supersymmetric vacua [3–8]. The non-extreme wall (\( \sigma_{\text{non}} > \sigma_{\text{ext}} \)) corresponds to a bubble with two insides; i.e., each side of the wall is inside a bubble. The ultra-extreme wall (\( \sigma_{\text{ultra}} < \sigma_{\text{ext}} \)) corresponds to the false vacuum decay [2,9] tunneling bubble [10]. We explain the relation between the non- and ultra-extreme domain wall bubbles [10] and the supersymmetric extreme walls [3–8]. The interesting global structure of the solutions with analogies to certain black holes is pointed out.

We choose to describe the gravitational field in the rest frame of the wall, i.e. we use comoving coordinates of observers sitting on the wall. Hence, the wall is placed at a fixed \( z \)-coordinate, and the metric is static in the \((t, z)\)-directions transverse to the wall. The metric is assumed to be homogeneous and isotropic in the \((\varrho, \phi)\) surfaces parallel to the wall [11]. Since the extrinsic curvature is independent of the wall’s proper time, one can show [12] that the metric is

\[
ds^2 = A(z) \left( dt^2 - dz^2 - \beta^{-2} \cosh^2 \beta t \ d\Omega^2 \right),
\]

with \( A(z) > 0 \) and \( d\Omega^2 \equiv [1-(\beta \varrho)^2]^{-1} d(\beta \varrho)^2 + (\beta \varrho)^2 d\phi^2 \).

In the extreme limit, \( \beta \to 0 \), the \((\varrho, \phi)\) surface becomes a plane with \( \varrho \) and \( \phi \) planar polar coordinates. When \( \beta \neq 0 \), the \((\varrho, \phi)\) hyperspace is the surface of a three-dimensional sphere, that is, its topology is \( S^2 \) [8]. In this case the coordinate \( \rho = \beta^{-1} \sin \theta \) is compact. The scalar curvature of the spatial \( S^2 \) is \( 2\beta^2 A(z)^{-1} \cosh^2 \beta t \). The constant \( z \) section with \( \beta \neq 0 \) is \( (2+1) \)-dimensional de Sitter space-time (\( dS_4 \)), which has the topology \( \mathbf{R}(\text{time}) \times S^2(\text{space}) \) [13]. \( dS_4 \) is completely covered by the coordinates \((t, \theta, \phi)\). Indeed, geodesic completeness for this \((2+1)\)-dimensional space-time requires the use of the compact spatial section in the \((t, z)\)-directions transverse to the wall [14]. The novel issues of geodesic completeness in the \((1+1)\)-dimensional space-time transverse to the wall, the \((t, z)\)-directions, will be addressed in this Letter.

The extreme walls (\( \beta = 0 \)) induce a static, conformally flat space-time \([4,5,15]\) classified in Ref. [5]: two types of \( \text{AdS}_4–\text{AdS}_4 \) walls (Type II and Type III) and an \( \text{AdS}_4–\text{M}_4 \) wall (Type I) where \( \text{AdS}_4 \) and \( \text{M}_4 \) denote the type of asymptotic geometry away from the wall, i.e. anti-de Sitter and Minkowski space-time, respectively. Here, we focus on space-times which are asymptotically \( \text{M}_4 \) in the direction transverse to the wall (Type I and its generalizations), and comment on the other possibilities at the end.

The extreme Type I wall energy density, \( \sigma_{\text{ext}} \), and the conformal factor, \( A(z) \), are

\[
\sigma_{\text{ext}} = 2\kappa^{-1} \alpha, \quad A(z) = \begin{cases} (\alpha z - 1)^{-2} & z < -w \\ 1 & z > +w \end{cases},
\]

where, without loss of generality, the wall is centered at \( z = 0 \). \( 2w > 0 \) is the width of the wall and \( \kappa \equiv 8\pi G \).

The \( \text{M}_4 \) side is chosen to be at \( z > w \), and the cosmological constant \( \Lambda \equiv -3\alpha^2 \) with \( \alpha \geq 0 \) on the \( z < w \) side. The \text{horo-spherical} coordinates used on the \( \text{AdS}_4 \) side are discussed in [5,8,16]. In the supersymmetric model the fields are governed by coupled first order rather than second order differential equations, thus allowing for a straightforward solution of the field equations for any thickness of the wall [4,5]. The coordinates of the metric (1) are not geodesically complete in the \((t, z)\) directions. Geodesic extensions have been provided with emphasis on the Type I walls in Ref. [7] and Type II walls in Ref. [8].

We now discuss walls with \( \beta > 0 \). Primarily we describe infinitely thin walls, \( w = 0 \), and thus employ Israel’s formalism of singular hypersurfaces [17]. Neverthef-
less, we motivate our analysis from an underlying scalar field theory and mention generic thick wall results where appropriate. Israel’s matching conditions at the wall are \( \kappa S^0_i = -[K^0_j] = \frac{\sigma}{n} + \delta^0_j [K^0_j] \). Here \( K^0_j \equiv -n^0_j \) is the wall’s extrinsic curvature, and \( n^\nu = \pm A^{-1/2} \delta^\nu \) is the space-like unit normal orthogonal to the wall’s four velocity \( u^\nu = A^{-1/2} \delta^\nu \). \( [K^0_j] \equiv -K_{(z=0^+)} = K_{(z=0^-)} \) signifies the discontinuity of the extrinsic curvature at the wall, and the Lanczos surface energy-momentum tensor \( \kappa S_i^0 = \sigma \delta^0_j \) is of the domain wall form [18]. The sign ambiguity of \( n^\nu \) is resolved by demanding a positive energy density for the wall and by using the underlying scalar field theory to identify the wall with a kink-like source. Then, Einstein’s field equations and Israel’s matching method yield two kinds of solutions with energy density and conformal factor

\[
\sigma_{\text{ultra}} = 2\epsilon^{-1}\left[\alpha^2 + \beta^2\right]^{1/2} \pm \beta \quad A(z) = \begin{cases} 
\beta^2 \alpha^{-2} \sinh(\beta z - \beta z')^{-2} & z < 0 \\
\frac{\epsilon^2 t^2 z^2}{\beta^2} & z > 0,
\end{cases}
\]

where \( e^{2\beta z'} = [\alpha^2 + 2\alpha^2 + 2\beta^2(\beta^2 + 2\alpha^2)^{1/2}] / \alpha^2 \equiv \eta \geq 1 \) is determined by \( A(0) \equiv 1 \).

The upper sign solution of Eq. (3) represents a non-extreme wall. In the non-extreme wall region the potential barrier associated with the scalar field is larger than in the corresponding extreme domain wall [19], which implies that \( \sigma_{\text{non}} > \sigma_{\text{ext}} \), and that \( A(z) \) falls off on the M4 side. Within \( N = 1 \) supergravity theory, it can be shown that such a wall can be realized as a wall interpolating between a supersymmetric M4 vacuum and an AdS4 vacuum with supersymmetry spontaneously broken.

At \( t = 0 \) the bubble has a radius \( \beta^{-1} \) which then increases as \( \cosh \beta t \). Additionally, since the radius of the bubble \( \beta^{-1} A(z)^{1/2} \cosh \beta t \) decreases as we move spatially away from the bubble in both \( z \) directions, observers on both sides are inside the bubble.

The non-extreme walls exhibit cosmological horizons on both the AdS4 and M4 sides. Namely, a particle with energy per unit mass \( E \geq 1 \), freely falling at constant velocity \( \theta = \phi = \infty \)-direction, has a finite proper time \( \tau = \alpha^{-1}(\arcsin\left[1 + (\epsilon/\beta^2)^{1/2} \eta + 1 - (\eta/\beta^2)\right]^{-1} - \arcsin\left[1 + (\epsilon/\beta^2)^{1/2} \right]^{-1}) \) and \( \tau = \beta^{-1} \alpha^{-1} \left( \cosh \beta t - \left( \cosh^2 \beta t - 1 \right)^{1/2} \right), \) respectively. As \( \beta \to 0 \), the cosmological horizon on the AdS4 side becomes a Cauchy horizon (as in the extreme wall space-time) with \( \tau = \alpha^{-1} \arcsin(1/\epsilon) \), while the M4 side becomes geodesically complete [7].

To investigate geodesically complete space-times for the non-extreme walls, we transform the metric (3) to the inertial spherical M4 and Einstein cylinder AdS4 coordinates on the respective sides. Introducing the radial Rindler coordinates \( \ell = \beta^{-1} e^{-\beta \ell} \sinh \beta t \) and \( z = \beta^{-1} e^{-\beta \ell} \cosh \beta t \) brings the line element on the M4 side to the spherically symmetric form \( ds^2 = dt^2 - dz^2 - \frac{r^2}{\ell^2} d\Omega_2^2 \). The \((\ell, \ell, \theta, \phi)\) coordinates define an inertial frame in which the bubble at \( z = 0^+ \) lives on the hyperbolic trajectory \( x^2 - y^2 = -\tan(u'/2) \tan(v'/2) = \beta^{-2} \) with constant acceleration \( \beta \), i.e. a Rindler trajectory [20]. Here \( u', v' = 2 \tan^{-1}[\beta(\ell \mp \ell')] \) are the usual compact null coordinates. On the AdS4 side, we map to the spherically symmetric Einstein cylinder coordinates [14]. This transformation is done in three steps: (i) \( \ln \Xi = \beta(z' - z) \). (ii) Radial Rindler transformation: \( T = \Xi \sinh \beta t \) and \( R = \Xi \cosh \beta t \). (iii) Compact time-like and radial coordinates: \( T \pm R = \tan((t \pm \psi)/2) \). The line element on the AdS4 side \((z < 0)\) becomes \( ds^2 = (a \cos \psi)^{-2}(dt^2 - dv^2 - \sin^2 \psi d\Omega_2^2) \), where \( -\pi \leq t \pm \psi \leq \pi \) and \( 0 \leq \psi \leq \pi/2 \). The center of symmetry is at \( \psi = R = 0 \). The bubble at \( z = 0^- \) again lives on a hyperbolic trajectory \( R^2 - T^2 = -\tan((t \pm \psi)/2) \tan((t + \psi)/2)/2 = \eta^{-1} \).

The (t, z)-chart is a conformal diagram which covers the space-time on both sides of the non-extreme wall region. To complete the space-time, we extend onto pure M4 and AdS4 on the respective sides, as shown in Fig. 1. On the AdS4 side, one may consider a symmetric, periodic extension yielding a lattice structure of walls. The Penrose diagram for this extension bears remarkable similarities to the one of a non-extreme \((m^2 G > c^2)\) Reissner-Nordström (RN) black hole; however, without singularities. The endpoints of the wall trajectories are on the affine boundary of AdS4 [7,8,21] and M4 and thus are not probed. The AdS4 (A) and M4 (M) diagrams are linked to each other at the wall regions.

The analogy between the space-time of the walls and that of black holes goes further. On the AdS4 side of the non-extreme walls, the metric in the \((t, z)\)-directions is identical to that of the \((t, r)\)-directions near the event horizon of non-extreme black holes. For the RN system in its \((t, r)\)-section, we have the line element \( ds^2 = a(r) d\theta^2 - a(r)^{-1} d\rho^2 \), where \( r \equiv r - r_+ \to 0^+ \), \( a(r) = \rho_0 \Delta r/r_+^2 \), \( r_\pm = G_0 \pm (m^2 - c^2 G^{-1})^{1/2} / \Delta r \equiv r_+ - r_-. \) Defining \( e^{2\beta z} \equiv \eta \epsilon / (\rho + \Delta r) \) along with \( 2\beta = \Delta r/r_+^2 \) and \( \alpha = 1/r_+ \), brings the above metric to the form \( ds^2 = A(z)(dt^2 - dz^2) \) where \( A(z) \) is the conformal factor on the AdS4 side of the wall (Eq. (3) for \( z < 0 \). As one approaches the extreme RN limit \((m^2 G \to c^2)\), \( \Delta r \to 0 \), and \( A(z) \) reduces to that of the extreme domain wall (Eq. (2) for \( z < 0 \). Furthermore, on the M4 side of the wall, the metric in the \((t, z)\)-directions (Eq. (3) for \( z > 0 \)) corresponds to the \((t, r)\)-directions of the Schwarzschild horizon [20]: \( ds^2 = a(r) d\theta^2 - a(r)^{-1} d\rho^2 \) with \( a(r) = \rho_0 (2m G)^{-1} \) and \( \rho_0 \equiv r - 2m G \to 0 \). Here we set \( \rho_0 \equiv (2m G)e^{-\beta \ell} \) and \( \beta^{-1} \equiv 4mG \).

The lower sign solution of Eq. (3) describes an ultra-extreme wall. For these walls the potential barrier associated with the scalar field is smaller than that of the extreme walls [19], which means \( \sigma_{\text{ultra}} < \sigma_{\text{ext}} \) and the metric blows up on the M4 side. Ultra-extreme walls exhibit the same causal structure on the AdS4 side as the non-extreme wall. However, the M4 side is geodesically incomplete in the \((t, z)\)-coordinates. The M4 side is
the complement of the $M_4$ side of the non-extreme wall (see Fig. 2). The two diagrams are linked at the wall region. The regions bounded by the curved trajectory of the wall and the null infinities are covered by the $(t, z)$-coordinates.

The Minkowski side is on the outside of the ultra-extreme bubble because the radius $\beta^{-1} A(z)^{1/2} \cosh \beta t$ increases with $z$ on the $z > 0$ side. On the $AdS_4$ side, however, the radius decreases away from the wall, and thus $AdS_4$ is on the inside just as for the non-extreme solution. Since $\sigma_{\text{ultra}} < 2 \kappa^{-1} \alpha$, i.e. below the Coleman-De Luccia bound [9,22], the ultra-extreme solution for $t \geq 0$ describes the classical evolution of a bubble [10] of true vacuum created by the quantum tunneling process of false vacuum decay [2,9]. At $t = 0$ the bubble is formed with radius $\beta^{-1}$, expands as $\cosh \beta t$, and inevitably hits all time-like observers on the $M_4$ side. If there were no Cauchy horizons, the $AdS_4$ side would collapse to a singularity [23]. However, as shown in Fig. 1 there are Cauchy horizons on the $AdS_4$ side. Thus, the conclusion of Ref. [23] that the $AdS_4$ space collapses does not apply.

For completeness, we also give results for the Israel matching of thin $AdS_4$–$AdS_4$ walls. Type II walls [5,15], with $\sigma_{\text{ext}} = 2 \kappa^{-1} (\alpha_1 + \alpha_2)$, have a unique non-extreme counterpart, with $\sigma_{\text{non}} = 2 \kappa^{-1} (\alpha_1^2 + \beta^2)^{1/2} / \alpha_2 + (\alpha_2^2 + \beta^2)^{1/2})$. Type III walls [5], with $\sigma_{\text{ext}} = 2 \kappa^{-1} (\alpha_1 - \alpha_2)$, have a unique ultra-extreme counterpart with $\sigma_{\text{ultra}} = 2 \kappa^{-1} (\alpha_1^2 + \beta^2)^{1/2} - (\alpha_2^2 + \beta^2)^{1/2})$. Both walls have analogous solutions for the metric coefficient $A(z)$ and the geodesic extensions [12].

The $M_4$–$M_4$ walls [18] and their geodesic extensions correspond to the $\alpha \to 0$ limit of the non-extreme $AdS_4$–$M_4$ walls with the metric (1). In this limit, Eq. (3) reduces to $\sigma_{\text{non}} = 4 \kappa^{-1} \beta$ and $A(z) = e^{2 \beta |z|}$ for $|z| > 0$. Such walls have a trivial extreme limit $\beta \to 0$ with $\sigma_{\text{non}} = 4 \kappa^{-1} \beta \to 0$. This is analogous to the case of the Schwarzschild space-time, which admits supersymmetry only in the trivial case of vanishing mass.

There are also walls separating de Sitter space ($dS_4$) from other vacua [24–26]. An $AdS_4$–$dS_4$ wall $A_2 = \pm 30_2$ has two wall solutions with $\sigma_\pm = 2 \kappa^{-1} (\alpha_1^2 + \beta^2)^{1/2} \pm (\alpha_2^2 + \beta^2)^{1/2}$, whereas a $dS_4$–$dS_4$ wall has the two solutions $\sigma_\pm = 2 \kappa^{-1} (\alpha_1^2 + \beta^2)^{1/2} \pm (\alpha_2^2 + \beta^2)^{1/2}$. $M_4$–$dS_4$ walls correspond to the special case $\alpha_1 = 0$. None of these walls have an extreme limit ($\beta \to 0$) since $\beta$ cannot be smaller than $\alpha$ of the $dS_4$ space(s) [24]. Additionally, they separate vacua which are unstable to quantum tunneling [9] except in the fine-tuned $dS_4$–$dS_4$ case $\alpha_1 = \alpha_2 = \alpha$ with $\sigma = 4 \kappa^{-1} (\alpha^2 + \beta^2)^{1/2}$.

Local and global properties of exact domain wall solutions have been analyzed. The non-extreme wall corresponds to a bubble with two insides. Its energy density is bounded from below by the one of the extreme wall, which is a planar supersymmetric configuration. Since the energy density of the extreme domain wall is equal to the Coleman-De Luccia bound [9], supersymmetry provides a lower bound [27] for a non-extreme domain wall. On the other hand, the ultra-extreme wall, which has energy density lower than the one of the extreme wall, corresponds to the classical evolution of a bubble [9,10,26] of true $AdS_4$ created by the decay of the false $M_4$ vacuum.

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[11] This is consistent with a scalar field source \( \Phi = \Phi(z) \), for which \( T^\mu_\nu = v \delta^\mu_\nu + k(\delta^\mu_\nu + 2n^\mu n_\nu) \), where \( n^\mu \equiv \pm A^{-1/2} \delta^\mu_z \), and \( v \) and \( k \) are the potential and kinetic energies, respectively.


[13] Since the topology of AdS4 is \( S^1/(\text{time}) \times \mathbb{R}^3/(\text{space}) \), the time-coordinate used in Eq. (1) is a de-compactification of the periodic time of AdS4 [5,8].


[19] Einstein’s equations, with a \( z \) dependent scalar field, are \( 3H^2 = 4[3\beta^2 + (k - v)A(z)] \), \( 3\partial_\phi H = -2(2k + v)A(z) \), where \( H \equiv \partial_t \ln A(z) \), and \( v \) and \( k \) are given in [11]. For \( z < -w, k = 0 \) and \( v = -3\kappa_2^2 \) and for \( z > w, k = 0 \) and \( v = 0 \). However, inside the wall (\( |z| < w \)), \( k > 0 \).


[22] In the case of non-positive cosmological constants \( \Lambda = -3\kappa_2^2 \), the Coleman-De Luccia bound [9] is \( \sigma < 2\kappa_2^{-1}[\alpha_1 - \alpha_2] \); cf. M. Cvetič, S. Griffies, and S.-J. Rey, Nucl. Phys. B389, 3 (1993).


