Heavy-quark axial charges to non-leading order

I. INTRODUCTION

This paper announces results for the non-leading order (NLO) heavy-quark corrections to QCD axial charges for protons. We combine the renormalization group (RG) running of the axial charge with the Columbia-arguable heavy-quark corrections to the axial charge in the framework of the weak decay of the proton. The calculation is performed by allowing the heavy-quark corrections to the axial charge to be non-leading. We evaluate the axial charges in the framework of the NLO corrections to the weak decay of the proton.

Many techniques for calculating the NLO corrections to the axial charge have been developed. We follow the approach of Ref. [1], where the axial charge is expressed in terms of the anomalous dimensions. The calculation is performed by allowing the heavy-quark corrections to the axial charge to be non-leading.

We find that, when first and finally corrected, the results are non-leading at the one-loop level. We assume that the remnant of the axial charge is neglected. The results are non-leading at the one-loop level.

We follow the approach of Ref. [2], where the axial charge is expressed in terms of the anomalous dimensions. The calculation is performed by allowing the heavy-quark corrections to the axial charge to be non-leading. We find that, when first and finally corrected, the results are non-leading at the one-loop level.

We conclude with the renormalization group (RG) running of the axial charge in the framework of the weak decay of the proton. We assume that the remnant of the axial charge is neglected. The results are non-leading at the one-loop level.

We conclude with the renormalization group (RG) running of the axial charge in the framework of the weak decay of the proton. We assume that the remnant of the axial charge is neglected. The results are non-leading at the one-loop level.
A scale-invariant current $(S_{\mu5})_f$ is obtained when \((6)\) is multiplied by
\[
E_f(\alpha_f) = \exp \int_0^\alpha dx \frac{\gamma_f(x)}{\beta_f(x)}
\]

Up to \(O(m_b^{-1})\) corrections, the invariant singlet charge \((3)\) is given by
\[
g_{A}^{(3)} \Big|_{\text{inv}} = E_3(\alpha_3)(\Delta u + \Delta d + \Delta s)_{\text{inv}}
\]
Flavour-dependent, scale-invariant axial charges \(\Delta s_{\text{inv}}\) such as
\[
\Delta s_{\text{inv}} = \frac{1}{3} \left( g_{A}^{(3)} - g_{A}^{(8)} \right)
\]
can then be obtained from linear combinations of \((8)\) and
\[
g_{A}^{(3)} = \Delta u - \Delta d = (\Delta u - \Delta d)_{\text{inv}}
\]
\[
g_{A}^{(8)} = \Delta u + \Delta d - 2\Delta s = (\Delta u + \Delta d - 2\Delta s)_{\text{inv}}
\]
Here \(g_{A}^{(3)} = 1.267 \pm 0.004\) is the isotriplet axial charge measured in neutron beta-decay, and \(g_{A}^{(k)} = 0.58 \pm 0.03\) is the octet charge measured independently in hyperon beta decay. Taking \(\alpha_u = 0.1, \alpha_d = 0.2\) and \(\Delta_c = 0.35\) in \((5)\), we find a small heavy-quark correction factor \(\rho = -0.02\), with LO terms.

Our results extend and make more precise the well-known work of Collins, Wilczek and Zee [8] and Kaplan and Manohar [2], where heavy-quark effective theory was used to estimate \(g_{A}^{(3)}\) in leading order (LO) for sequential decoupling of \(t, b\) and \(t, b, c\) respectively. Our analysis is also influenced by a discussion of [6] by Chetyrkin and Kühn [16], who considered some aspects of NLO decoupling of the quark from the neutral current and in particular, the requirement that the result be scale invariant.

The plan of this paper is as follows. Section 2 is a brief review of Witten’s application of improved Callan-Symanzik equations \([13]\) to the decoupling of a heavy quark in mass-independent renormalization schemes. In Section 3, we combine it with matching conditions \([15]\) to deal with next-to-leading-order (NLO) calculations involving axial-vector currents. Section 4 is then a direct derivation of \((5)\) from the formula \((1)\) for the neutral current. Our concluding remarks in section 5 indicate the result of extending \((5)\) to simultaneous decoupling of \(t, b, c\) — done not only for numerical reasons, but also to check that the \(t, b\) contributions cancel for \(m_t = m_b\).

II. WITTEN’S METHOD

In mass-independent schemes such as \(\overline{\text{MS}}\), renormalized masses behave like coupling constants. This key property is exploited in Witten’s method.

Let \(\mu\) be the scale used to define dimensional regularization and renormalization. Then the \(\overline{\text{MS}}\) scale is
\[
\mu = \sqrt{4\pi \alpha} e^{-\gamma/2}, \quad \gamma = 0.5772\ldots
\]

We choose the same scale \(\mu\) irrespective of the number of flavours \(f\) being considered, and so hold \(\mu\) fixed as the heavy quarks (masses \(m_b\)) decouple:
\[
F \rightarrow f \text{ flavours, } m_b \rightarrow \infty
\]

Also held fixed in this limit are the coupling \(\alpha_f\) and light-quark masses \(m_{\ell_f}\) of the residual \(f\)-flavour theory, and all momenta \(p\). Feynman diagrams for amplitudes
\[
\mathcal{A}_F = \mathcal{A}_F(p, \bar{p}, \alpha_f, m_{\ell_f}, m_b)
\]
give rise to power series in \(m_b^{1}\) modified by polynomials in \(\ln(m_b/\mu)\). We consider just the leading power \(\mathcal{A}_F\):
\[
\mathcal{A}_F = \mathcal{A}_F \{1 + O(1/m_b)\}
\]

As \(m_b\) tends to infinity, logarithms in \(\mathcal{A}_F\) can be produced by any \(1\)PI (one-particle irreducible) subgraph which contains at least one heavy-quark propagator and whose divergence by power counting is at least logarithmic. The effect is equivalent to shrinking all contributing \(1\)PI parts of each diagram to a point. This means \([14]\) that the \(f\)-flavour amplitudes \(\mathcal{A}_F\) are the same as amplitudes \(\mathcal{A}_f\) in the residual \(f\)-flavour theory, apart from \(m_{\ell_f}\)-dependent renormalizations of the coupling constant, light masses, and amplitudes:
\[
\mathcal{A}_F(p, \bar{p}, \alpha_f, m_{\ell_f}, m_b) = \sum_{\mathcal{A}} Z_{\mathcal{A}}(\alpha_f, m_{\ell_f} / \mu) \mathcal{A}_f(p, \bar{p}, \alpha_f, m_{\ell_f})
\]

Eventually, we will have to invert \((15)\), i.e. use \(\alpha_f\) and \(m_{\ell_f}\) as dependent variables instead of \(\alpha_f\) and \(m_{\ell_f}\), because we hold \(\alpha_f\) and \(m_{\ell_f}\) fixed as \(m_b \rightarrow \infty\).

For any number of flavours \(f\) (including \(F\)), let
\[
D_f = \frac{\partial}{\partial \alpha_f} + \beta_f(\alpha_f) \frac{\partial}{\partial \alpha_f} + \delta_f(\alpha_f) \sum_{k=1}^{f} \frac{m_{\ell_k}}{m_{\ell_k}} \frac{\partial}{\partial m_{\ell_k}}
\]
be the corresponding Callan-Symanzik operator. Then the amplitude \(\mathcal{A}_F\) and hence its leading power \(\mathcal{A}_F\) both satisfy an \(F\)-flavour improved Callan-Symanzik equation:
\[
\left\{ D_F + \gamma_f(\alpha_f) \right\} \mathcal{A}_F = 0
\]
In general, both \(\gamma_f\) and \(Z = (Z_{\mathcal{A}})\) are matrices.

If we substitute \((14)\) in \((17)\) and change variables,
\[
D_F = \frac{\partial}{\partial \alpha_f} + (D_F \alpha_f) \frac{\partial}{\partial \alpha_f} + \sum_{k=1}^{f} \left( D_F m_{\ell_k} \right) \frac{\partial}{\partial m_{\ell_k}}
\]
the result is an improved Callan-Symanzik equation for each residual amplitude,
\[ \{ D_{\gamma} \beta_{\gamma}(a_{\gamma}) \} A_{\gamma} = 0 \]  
(19)
where the functions \[ \beta_{\gamma}(a_{\gamma}) = D_{\gamma} a_{\gamma} \]  
(20)
\[ \delta_{\gamma}(a_{\gamma}) = D_{\gamma} \ln m_{\ell} \]  
(21)
\[ \gamma_{\gamma}(a_{\gamma}) = \mathcal{Z}^{-1} \left( D_{\gamma}(a_{\gamma} P) + D_{\gamma} \mathcal{Z} \right) \]  
(22)
depend on \( a_{\gamma} \) alone. The lack of \( m_{\ell} \) dependence of the renormalization factors in (14) and (15) ensures mass-independent renormalization for the residual theory.

Although these equations hold for any \( f < F \), their practical application is straightforward only when heavy quarks are decoupled at a time. So we set \( F = f + 1 \), where just one quark \( h \) is heavy. Then it is convenient to introduce a running coupling [12]
\[ \tilde{a}_{h} = \tilde{a}_{h}(a_{F}, \ln(m_{h}/\bar{\mu})) \]  
(23)
associated with the \( \mathcal{Z}_{F} \) renormalized mass \( m_{h} \):
\[ \ln(m_{h}/\bar{\mu}) = \int_{a_{F}}^{\tilde{a}_{h}} dx \left( 1 - \delta_{F}(x) \right) / \beta_{F}(x) \]  
(24)
It satisfies the constraints
\[ \tilde{a}_{h}(a_{F}, 0) = a_{F} \quad \tilde{a}_{h}(a_{F}, \infty) = 0 \]  
(25)
the latter being a consequence of the asymptotic freedom of the \( F \) flavour theory \( F \leq 16 \). Also, eqs. (16), (20) and (24) imply that \( \tilde{a}_{h} \) is renormalization group (RG) invariant:
\[ \int D_{F} \tilde{a}_{h} = 0 \]  
(26)
Witten’s solution of (22) for the matrix \( \mathcal{Z} \) is
\[ \mathcal{Z}(a_{F}, m_{h}/\bar{\mu}) = \exp \left\{ \int_{a_{F}}^{\tilde{a}_{h}} dx \left[ \frac{\gamma_{F}(x)}{\beta_{F}(x)} \right] \right\} \mathcal{Z}(\tilde{a}_{h}, 1) \]
\[ \times \exp \left\{ \int_{a_{F}}^{\tilde{a}_{h}} dx \left[ \frac{\gamma_{F}(x)}{\beta_{F}(x)} \right] \right\end{array} \]  
(27)
where “ord” indicates ordering of matrix integrands in the exponentials. Note that it is the \textit{relative} scaling between the initial and residual theories which matters.

For our NLO calculation, we need the formulas
\[ \beta_{F}(x) = \frac{x^{2}}{3\pi} \left( \frac{33}{2} - f \right) - \frac{x^{3}}{12\pi^{2}}(153 - 19f) + O(x^{4}) \]
\[ \gamma_{F}(x) = \frac{x^{2}}{3\pi} f + \frac{x^{3}}{9\pi^{2}}(177 - 2f) + O(x^{4}) \]
\[ \delta_{F}(x) = -\frac{2x}{\pi} + O(x^{2}) \]  
(28)
where \( \gamma_{F} \) refers to the \( F \)-flavour singlet current (6) and includes the three-loop term found by Larin [19] and Chetyrkin and Kühn [16].

### III. MATCHING PROCEDURE

Our task is to evaluate to NLO accuracy the quantities \( \tilde{a}_{h}, a_{F}(\tilde{a}_{h}, 1) \) and \( \mathcal{Z}(\tilde{a}_{h}, 1) \) in (27), such that the answers depend on \( a_{F} \) and not \( a_{F} \).

Bernreuther and Wetzel [16] applied the Appelquist-Carrazone decoupling theorem [14] to the gluon coupling constant \( a_{Q}^{\text{MS}} \) renormalized at space-like momentum \( Q \).
\[ a_{Q}^{\text{MS}}(b) = a_{Q}^{\text{MS}}(m_{h} + O(m_{h}^{-1})) \]  
(29)
and compared calculations of \( a_{Q}^{\text{MS}} \) in the \( F = f + 1 \) and \( f \) flavour MS theories. This reduces to a determination of the leading power of the one-\( h \)-loop MS\(_{F} \) gluon self-energy. The result is a matching condition
\[ a_{F}^{-1} - a_{F}^{-1} = C_{L0} \ln \frac{m_{h}}{\bar{\mu}} + C_{NLO} + O(a_{F}, m_{h}^{-1}) \]  
(30)
with \( a_{F} \)-independent LO and NLO coefficients given by
\[ C_{L0} = \frac{1}{3\pi} \quad C_{NLO} = 0 \]  
(31)
As a result, we find:
\[ a_{F}(\tilde{a}_{h}, 0) = \tilde{a}_{h} + O(\tilde{a}_{h}^{-1}) = \tilde{a}_{h} \]  
(32)
Bernreuther and Wetzel showed that it is possible to deduce all LO and NLO terms in (30) from (31) and \( \beta_{F} \) and \( \delta_{F} \) in (28). We have done the calculation explicitly:
\[ a_{F}^{-1} = \frac{1}{3\pi} \ln \frac{m_{h}}{\bar{\mu}} + C_{F,\text{LO}} + \left[ 1 + \frac{\alpha_{F}}{3\pi} \ln \frac{m_{h}}{\bar{\mu}} \right] \frac{1}{2\pi^{2}} \left[ 153 - 19f \right] + \left[ 1 + \frac{a_{F}}{3\pi} \left( \frac{33}{2} - f \right) \right] \ln \frac{m_{h}}{\bar{\mu}} \]  
(33)
From (24), we have also found \( \tilde{a}_{h} \) in NLO,
\[ \tilde{a}_{h}^{-1} = \frac{1}{3\pi} \ln \frac{m_{h}}{\bar{\mu}} + \frac{1}{2\pi^{2}} \left[ 153 - 19f \right] \ln \frac{m_{h}}{\bar{\mu}} + \left[ 1 + \frac{a_{F}}{3\pi} \left( \frac{33}{2} - f \right) \right] \ln \frac{m_{h}}{\bar{\mu}} \]  
(34)
where \( \tilde{m}_{h} \) is Witten’s RG invariant mass:
\[ \tilde{m}_{h} = m_{h} \exp \int_{a_{F}}^{\tilde{a}_{h}} dx \delta_{F}(x)/\beta_{F}(x) \]  
(35)
If desired, \( \ln(\tilde{m}_{h}/\bar{\mu}) \) can be eliminated by substituting
\[ \ln \frac{m_{h}}{\bar{\mu}} = \frac{1}{2\pi^{2}} \left[ 153 - 19f \right] \ln \frac{m_{h}}{\bar{\mu}} + \frac{1}{2\pi^{2}} \left[ 1 + \frac{a_{F}}{3\pi} \left( \frac{33}{2} - f \right) \right] \ln \frac{m_{h}}{\bar{\mu}} \]  
(36)
Therefore the asymptotic formula for \( \tilde{a}_{h} \) as \( m_{h} \to \infty \) is
\[ \tilde{a}_{h} \sim 3\pi \left\{ \left( \frac{33}{2} - f \right) \ln \frac{m_{h}}{\bar{\mu}} + k_{f} \ln \frac{m_{h}}{\bar{\mu}} + O(1) \right\} \]
\[ k_{f} = \frac{3(153 - 19f)}{2(33 - 2f)} - \frac{6(33 - 2f)}{31 - 2f} \]  
(37)
To find the matrix $\mathcal{Z}(\tilde{\alpha}_b, 1)$ in NLO, we need a matching condition for the MS amplitude $\Gamma_{\mu \bar{\nu}}$ for $\tilde{\ell} \gamma_\mu \gamma_5 \tilde{h}$ to couple to a light quark $\ell$. We have calculated the leading power due to the two-loop diagram \(\Box^2\):

$$\Gamma_{\mu \bar{\nu}} = \left( \frac{\alpha_s}{\pi} \right)^2 \gamma_\mu \gamma_5 \left( \ln \frac{m_b}{\mu} + \frac{1}{8} \right) + O(\alpha_s^3, m_b^{-1}) \quad (38)$$

Consequently, there is a NLO term $\tilde{\alpha}_b^2 / 8\pi^2$ in $\mathcal{Z}(\tilde{\alpha}_b, 1)$ for $\tilde{h} \gamma_\mu \gamma_5 \tilde{h}$ to produce $\tilde{\ell} \gamma_\mu \gamma_5 \ell$ as $m_b \to \infty$.

### IV. Heavy Quarks Decoupled from $J^Z_{\mu \bar{\nu}}$

Let us adopt the shorthand notation $q_f$ for MS currents $(\bar{q}_f \gamma_\mu \gamma_5 q_f)_f$ in the $f$-flavour theory, e.g. the neutral current $J^Z_{\mu \bar{\nu}}$ and the scale-invariant singlet current $(\bar{S}_{\mu \bar{\nu}})_f$:

$$J^Z_{\mu \bar{\nu}} = \frac{1}{2} (t - b + c + s + u - d)_{\mu \bar{\nu}} \quad (39)$$

$$S_f = E_f(a_f (u + d + s + \ldots))_{\mu \bar{\nu}} \quad (40)$$

We begin by decoupling the $t$ quark. Because of

$$(c - s + u - d)_{\mu \bar{\nu}} = (c - s + u - d)_{\mu \bar{\nu}} + O(1/m_b) \quad (41)$$

we see that (27) is non-trivial only for

$$(t - b)_{\mu \bar{\nu}} = \mathcal{Z}_{6 \to 5}(u + d + s + c + b)_{\mu \bar{\nu}} + \frac{1}{8} (u + d + s + c - 4b)_{\mu \bar{\nu}} + O(1/m_b)_{\mu \bar{\nu}} \quad (42)$$

Since $(t - b)_{\mu \bar{\nu}}$ is scale invariant, we have $\gamma_F = 0$ in (27):

$$\mathcal{Z}_{6 \to 5}(\alpha_F, m_b / \mu)_{\mu \bar{\nu}} \overset{\text{NLO}}{=} \mathcal{Z}_{6 \to 5}(\tilde{\alpha}_b, 1)_{\mu \bar{\nu}} \exp \left\{ \int_{\alpha_F}^{\tilde{\alpha}_b} \frac{\gamma_5(x)}{\beta_5(x)} \right\} \quad (43)$$

The operator matching condition (38) corresponds to

$$t_{\mu \bar{\nu}} = \frac{\alpha_s^2}{\pi} \left( \ln \frac{m_b}{\mu} + \frac{1}{8} \right) (u + d + s + c + b)_{\mu \bar{\nu}} + O(\alpha_s^3, m_b^{-1}) \quad (44)$$

and so we conclude

$$\mathcal{Z}_{6 \to 5}(\tilde{\alpha}_b, 1)_{\mu \bar{\nu}} = - \frac{1}{5} + \frac{1}{8\pi^2} \alpha_s^2 + O(\alpha_s^3) \quad (45)$$

Eq. (43) is to be expanded about $\tilde{\alpha}_b = 0$ with $\alpha_5$ held fixed. In that limit, the exponent tends to the constant factor $E_5(\alpha_5)$ of (7). This factor combines with the singlet current in (42) to form the scale-invariant operator $S_5\bar{\mu}$, as required by RG$_{f = 5}$ invariance. The full NLO result is then obtained by writing

$$(t - b)_{\mu \bar{\nu}} = \mathcal{Z}_{6 \to 5}(\tilde{\alpha}_b, 1)_{\mu \bar{\nu}} \exp \left\{ - \int_{\alpha_F}^{\tilde{\alpha}_b} \frac{\gamma_5(x)}{\beta_5(x)} \right\} S_5 + \frac{1}{5} (u + d + s + c - 4b)_{\mu \bar{\nu}} \quad (46)$$

and expanding in $\tilde{\alpha}_b$, keeping all quadratic terms:

$$(t - b)_{\mu \bar{\nu}} = \left\{ - \frac{1}{5} + \frac{6}{23\pi} \left( \frac{1}{36} - \frac{612}{3312\pi} \right) \right\} S_5 + \frac{1}{5} (u + d + s + c - 4b)_{\mu \bar{\nu}} + O(1/m_b) \quad (47)$$

Next we decouple the $b$ quark. Here, it is natural to define five-flavour quantities $\tilde{\alpha}_b$ and $\tilde{m}_b$, analogous to the six-flavour running coupling $\tilde{\alpha}_b$ and mass $\tilde{m}_b$ for the top quark:

$$\ln \frac{\tilde{m}_b}{\mu} = \int \frac{\tilde{\alpha}_5}{\beta_5(x)} dx - \frac{1}{2} \frac{\delta_5(x)}{\beta_5(x)} \quad (48)$$

Eqs. (20) and (21) imply that $\tilde{\alpha}_b$ and $\tilde{m}_b$ are both RG$_{f = 5}$ and RG$_{f = 6}$ invariant.

$$\mathcal{D}_5 \tilde{\alpha}_b = 0 = \mathcal{D}_6 \tilde{\alpha}_b, \quad \mathcal{D}_5 \tilde{m}_b = 0 \mathcal{D}_6 \tilde{m}_b \quad (49)$$

and hence physically significant in the original six-flavour theory. So we write $\tilde{\alpha}_b$ and $\tilde{m}_b$ for $\tilde{\alpha}_b$ and $\tilde{m}_b$.

Consider decoupling the $b$ quark from (47). The NLO matching condition (38) becomes

$$b_5 = \frac{\alpha_s^2}{\pi} \left( \ln \frac{\tilde{m}_b}{\mu} + \frac{1}{8} \right) (u + d + s + c)_{\mu \bar{\nu}} + O(\alpha_s^3, m_b^{-1}) \quad (50)$$

so the non-singlet current in (47) can be written

$$(u + d + s + c - 4b)_{\mu \bar{\nu}} = \left\{ - \frac{\alpha_s^2}{\pi} \left( \frac{1}{8} - \frac{612}{3312\pi} \right) \right\} S_5 + O(\tilde{\alpha}_b^3, \tilde{m}_b^{-1}) \quad (51)$$

for the singlet current $S_5$ in (47). We find

$$S_5 = E_5(\tilde{\alpha}_b) \left\{ 1 + \frac{\alpha_s^2}{\pi} \right\} \frac{E_5^{-1}(\tilde{\alpha}_b) S_5}{E_5^{-1}(\tilde{\alpha}_b) S_5} + O(\tilde{\alpha}_b^3, \tilde{m}_b^{-1}) \quad (52)$$

taking into account the definitions (7) and (40). Then we expand (51) and (52) in $\tilde{\alpha}_b$, keeping quadratic terms:

$$(t - b)_{\mu \bar{\nu}} = \left\{ - \frac{1}{5} + \frac{6}{23\pi} \left( \frac{1}{36} - \frac{612}{3312\pi} \right) \right\} S_5 + \frac{1}{5} (u + d + s + c - 4b)_{\mu \bar{\nu}} \quad (53)$$

The same technique can be applied to decouple the $c$ quark from $S_5$ in (53) and $(c - s + u - d)_{\mu \bar{\nu}}$ (the result of decoupling $b$ from (41)). That yields the final results (4) and (5) given in the Introduction.

### V. Remarks

Our results depend on two key features:

1. Like previous workers in this area, we decouple heavy quarks sequentially, i.e. one at a time.
2. Our running couplings $\tilde{\alpha}_t$, $\tilde{\alpha}_b$, and $\tilde{\alpha}_c$, which correspond to Witten’s prescription [12], are all renormalization group invariant.

The restriction to sequential decoupling is numerically reasonable for the $t$ quark, but dubious for the $b$ and $c$ quarks, because it amounts to an assumption that \( \ln(m_c/\bar{p}) \) is negligible compared with \( \ln(m_b/\bar{p}) \). This inhibits detailed comparison of NLO results with data, which ought to be carried out with NLO accuracy [20].

There is also a theoretical issue here: one would like to check that, in the limit \( m_t = m_b \), the $t$ and $b$ contributions cancel. However, that is outside the region of validity \( \ln(m_t/\bar{p}) \gg \ln(m_b/\bar{p}) \) for sequential decoupling.

For these reasons, we have extended our analysis to the case of simultaneous decoupling, where the mass logarithms are allowed to grow large together:

\[
\ln(m_c/\bar{p}) \sim \ln(m_b/\bar{p}) \sim \ln(m_t/\bar{p}) \to \text{large}
\]

This requires a considerable theoretical development of matching conditions and the renormalization group, which we will present separately. It involves the construction of running couplings \( \alpha_t, \alpha_b, \alpha_c \) with the following properties:

1. They are renormalization group invariant.

2. They are defined for \( m_t \geq m_b \geq m_c \), and can have a non-trivial dependence on more than one heavy-quark mass.

3. In the special case of sequential decoupling, they agree with \( \tilde{\alpha}_t, \tilde{\alpha}_b \) and \( \tilde{\alpha}_c \) to NLO.

4. For the case of equal masses, they coincide, e.g.

\[
\alpha_t = \alpha_b \text{ for } m_t = m_b
\]

Then we find that the result for the simultaneous decoupling of the $t$, $b$, $c$ quarks from the neutral current is of the same form (4) as the sequential answer, but with the sequential running couplings in (5) replaced by our simultaneous couplings $\alpha_t, \alpha_b, \alpha_c$:

\[
\mathcal{P} = \frac{6}{23\pi} \left( \frac{m_b}{m_t} \right) \left( 1 + \frac{125663}{82900\pi \alpha_b} + \frac{6167}{3312\pi \alpha_t} - \frac{22}{75\pi \alpha_c} \right)
\]

\[
- \frac{6}{23\pi \alpha_c} - \frac{181}{48 \pi^2} \alpha_c^2 + O(\alpha_c^3)
\]

Notice the factorization of the terms depending on $\alpha_t$ and $\alpha_b$. Given (54), the factor $\alpha_b - \alpha_c$ ensures that all contributions from $b$ and $t$ quarks cancel (as they should) for $m_t = m_b$.

Acknowledgments

This work was supported by the Australian Research Council and the Austrian FWF. FMS is supported by contract number PV-IFT/005. RJC thanks Professor Wojtek Zakrzewski for his hospitality at Durham. SDB thanks Professor Dietmar Kuhl and the HEP group for their hospitality at Innsbruck.

---

[20] This includes matching conditions for the $b$ and $c$ masses, to be discussed elsewhere.