Covariant - tensor method for quantum groups and applications I : $SU(2)_q$

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Short title: Covariant - tensor method for quantum groups  
Classification number: 02.20.+b
Abstract

A covariant - tensor method for $SU(2)_q$ is described. This tensor method is used to calculate q - deformed Clebsch - Gordan coefficients. The connection with covariant oscillators and irreducible tensor operators is established. This approach can be extended to other quantum groups.
1. Introduction

In recent years there has been considerable interest in q-deformations of Lie algebras (quantum groups) [1] and their applications in physics [2]. The main goal of these applications is a generalization of the concept of symmetry. The properties of quantum groups are similar to those of classical Lie groups with q not being a root of unity. However, it is still not clear to what extent the familiar tensor methods, used in the representation theory of Lie algebras, are applicable to the case of q - deformations.

Different types of the tensor calculus for $SU(2)_q$ were proposed and applied in references [3,4,6,9,10]. However, no simple covariant - tensor calculus for $SU(n)_q$ was presented. In this paper we propose a simple covariant - tensor method for $SU(2)_q$ which can be extended to the general $SU(n)_q$. Details for $SU(n)_q$ and especially for $SU(3)_q$ will be published separately.

The plan of the paper is the following. In Section 2 we recall the basics of the $SU(2)_q$ algebra, its fundamental representation and invariants. In Section 3 we construct the general $SU(2)_q$ - covariant tensors and invariants. In Section 4 we apply this tensor method to calculate q - deformed Clebsch-Gordan coefficients and in Section 5 we demonstrate their symmetries. We point out that this method is simpler than that used in previous calculations [5,6] and can be generalized to other quantum groups. Finally, in Section 6 we connect covariant tensors with covariant q - oscillators and construct unit irreducible tensor operators.

2. $SU(2)_q$ - algebra, its fundamental representation and invariants

Let us recall that three generators of $SU(2)_q$ obey the following commutation relations [1] (we take q real)

\[
[J^0, J^\pm] = \pm J^\pm
\]

\[
[J^+, J^-] = [2J^0]_q = \frac{q^{2J^0} - q^{-2J^0}}{q - q^{-1}}
\]

(2.1)

The coproduct $\Delta : SU(2)_q \rightarrow SU(2)_q \otimes SU(2)_q$ is defined as

\[
\Delta(J^\pm) = J^\pm \otimes q^{J^0} + q^{-J^0} \otimes J^\pm
\]

\[
\Delta(J^0) = J^0 \otimes 1 + 1 \otimes J^0
\]

(2.2)

Let $V_2$ be a two - dimensional space spanned by the basis $|e_a>$, $a = 1, 2$, and $|v> = \sum_a |e_a > v_a | \in V_2$. The $SU(2)_q$ generators $J^k (k = \pm, 0)$ act as
\[ J^k |e_a\rangle = \sum_b (J^k)_{ba} |e_b\rangle \]

\[ J^k |v\rangle = \sum_{a,b} (J^k)_{ba} |e_b\rangle v_a = \sum_b |e_b\rangle (J^k |v\rangle)_b = \sum_b |e_b\rangle v_b. \]  

(2.3)

In the fundamental representation of $SU(2)_q$ the generators $J^k$s are ordinary 2x2 Pauli matrices. Let $(V_2)^*$ be a dual space with the basis $<e_a| = (|e_a\rangle)^+$ and $<v| = (|v\rangle)^+ = \sum_a v_a^* < e_a|$. The dual basis is orthonormal, i.e. $<e_a|e_b> = \delta_{ab}$. We note that the components of the vector $|v\rangle$, $v_a$, (or $v_a^*$ of $<v|$) are not defined as real or complex numbers. Their algebraic properties will follow from $SU(2)_q$ -invariance requirements. Here we identify (for the spin $j = 1/2$)

\[ |e_a\rangle = |\frac{1}{2}, m_a\rangle \]

\[ \langle e_a| = \langle \frac{1}{2} | m_a | \]

\[ m_a = \pm \frac{1}{2} \]  

and the matrix elements of the generators $J_k$ are

\[ \langle e_a| J^0 |e_a\rangle = m_a \]

\[ \langle e_1| J^+ |e_2\rangle = \langle e_2| J^- |e_1\rangle = 1 \]

(2.4)

We define a scalar product as $< u| v > = \sum_a u_a^* v_a$ and the norm as $< v| v > = \sum_a v_a^* v_a$. This scalar product (and the norm) are not $SU(2)_q$ - invariant. Instead, the quantity

\[ \langle v| q^{-J^0} |v\rangle \]  

(2.5)

is invariant under the action of the coproduct (2.2) in the following sense:

\[ \Delta(J^\pm) \langle v| q^{-J^0} |v\rangle = (J^\pm \langle v|) |v\rangle + (q^{-J^0} \langle v|) J^\pm q^{-J^0} |v\rangle = -\langle v| J^\pm |v\rangle + \langle v| J^\mp |v\rangle = 0, \]

(2.6)

\[ \Delta(J^0) \langle v| q^{-J^0} |v\rangle = (J^0 \langle v|) q^{-J^0} |v\rangle + \langle v| J^0 q^{-J^0} |v\rangle = -\langle v| J^0 q^{-J^0} |v\rangle + \langle v| J^0 q^{-J^0} |v\rangle = 0. \]

(2.7)

The quadratic forms

\[ \sum_a u_a^* q^{-J^0} v_a = \sum_a u_a^* q^{-m_a} v_a \]

(2.8)

\[ \sum_a v_a q^{m_a} u_a^* \]

(2.9)
are $SU(2)_q$ -invariant. Note that the first quadratic form (2.8) can be written as

\[ <u|q^{-J_0}|v> \].

If we demand $\sum_v q^{m_\alpha} v_\alpha = \sum_v v_\alpha q^{m_\alpha} v_\alpha^*$, it follows that $v_1 v_1 = q v_1 v_1^*$

and $v_2^* v_2 = q^{-1} v_2 v_2^*$.

In addition to the $<u|q^{-J_0}|v>$ - invariant form we consider another form,

\[ \epsilon_{ab}|e_a\rangle|e_b\rangle \]

(2.11)

with

\[ \epsilon_{ab} = \begin{pmatrix} 0 & q^{1/2} \\ -q^{-1/2} & 0 \end{pmatrix} \]

(2.12)

where $\mathbf{T} = 2$ and $\overline{\mathbf{T}} = 1$. Note that the $q$ - antisymmetric combination $v_a v_b \epsilon_{ab}$ is

$SU(2)_q$ -invariant, showing that $v_a$ and $v_b$ do not commute. Instead, they $q$ - commute, i.e. $v_2 v_1 = q v_1 v_2$.

3. General $SU(2)_q$ - tensors and invariants

Let us consider the tensor - product space $(V_2)^{\otimes k} = V_2 \otimes ... \otimes V_2$ with the basis

$|e_{a_1} \rangle \otimes ... \otimes |e_{a_k} \rangle, a_1,...,a_k = 1,2$. Then we write an element of the tensor space $(V_2)^{\otimes k}$ as tensor $|T\rangle$ of the form

\[ |T\rangle = |e_{a_1}\rangle...|e_{a_k}\rangle T^{a_1}...T^{a_k} = |e_{a_1}...e_{a_k}\rangle T^{a_1...a_k} \]

(3.1)

We have the following proposition:

The tensor $|T\rangle$ transforms under the $SU(2)_q$ algebra as an irreducible representation of spin $j = k/2$ if and only if $T^2 T^1 = q T^1 T^2$.

Let us assume $T^2 T^1 = q T^1 T^2$. Then

\[ |T_{j=\frac{1}{2}}\rangle = |e_{a_1}...e_{a_k}\rangle T^{a_1...a_k} = \sum_{m=-j}^{+j} |j m\rangle T^{j m} \]

(3.2)

The vectors $|j m\rangle$ span the space $V_{2j+1}$ of the irreducible representation with spin $j$. From $T^2 T^1 = q T^1 T^2$ it follows that

\[ T^{a_1...a_k} = q^{c(a_1,...a_k)} : T^{a_1...a_k} : \]

(3.3)

where $: T :$ means the normal order of indices (1’s on the left of 2’s), i.e. $T^{11...22}$ and index 1 (2) appears $n_1(n_2)$ times, respectively. $c$ is the number of inversions with respect to the normal order. Hence,
\[ |jm\rangle = \{ e_{\{a_1...a_k\}} \} = \frac{1}{\sqrt{f}} q^{-\frac{M}{2}} \sum_{\text{perm}(a_1...a_k)} q^{\chi(a_1...a_k)} |e_{a_1...a_k}\rangle \]  

(3.4)

where the curly bracket \(\{a_1...a_k\}\) denotes the q-symmetrization. The summation runs over all the allowed permutations of the fixed set of indices \((n_1 1's\) and \(n_2 2's\) and

\[ M = n_1 n_2 = (j + m)(j - m) \]

\[ j = \frac{1}{2}(n_1 + n_2) \quad m = \frac{1}{2}(n_1 - n_2) \]  

(3.5)

\[ f = \left( \begin{array}{c} 2j \\ j + m \end{array} \right)_q = \frac{[2j]_q!}{[j + m]_q! [j - m]_q!} \]

The important relation is

\[ f = q^{-M} \sum_{\text{perm}(a_1...a_k)} q^{2\chi(a_1...a_k)} \]  

(3.6)

From equation (3.4) and the definition of the coproduct \(\Delta(J^\pm)\) (2.2) we can reproduce

\[ \Delta(J^\pm) |jm\rangle = \sqrt{[j + m]_q [j \pm m + 1]_q} |jm \pm 1\rangle \]

\[ \Delta(J^0) |jm\rangle = m |jm\rangle \]  

(3.7)

From (3.2) and (3.4) we immediately obtain the relation between \(T^{jm}\) and the components of \(T^{a_1...a_k}\):

\[ T^{jm} = q^{\frac{M}{2}} \sqrt{f} : T^{a_1...a_k} : \]

\[ T^{j-m} = q^{\frac{M}{2}} \sqrt{f} : T^{\bar{a}_1...\bar{a}_k} : \]  

(3.8)

where \(\bar{T} = 2, \bar{\Omega} = 1\) and \(T^{j-m} = (T^{jm})_{n_1 \leftrightarrow n_2}\). In the dual space \((V_2^{\otimes k})^*\) we define

\[ \langle e_{a_k...a_1} | = (|e_{a_1...a_k}\rangle)^+ \]

\[ \langle e_{a_k...a_1} | e_{b_1...b_k} \rangle = \delta_{a_1 b_1} \cdots \delta_{a_k b_k} \]

and in the dual space \((V_{2j+1})^*\) we define

\[ \langle jm| = (|jm\rangle)^+ = \langle e_{\{a_k...a_1\}}| = \]

\[ = \frac{1}{\sqrt{f}} q^{-\frac{M}{2}} \sum_{\text{perm}(a_1...a_k)} q^{\chi(a_1...a_k)} (|e_{a_1...a_k}\rangle)^+ = \]  

(3.10)

\[ = \frac{1}{\sqrt{f}} q^{-\frac{M}{2}} \sum_{\text{perm}(a_1...a_k)} q^{\chi(a_1...a_k)} \langle e_{a_k...a_1} | . \]
As a consequence of equations (3.4), (3.6) and (3.9) we obtain

\[ \langle jm_1 | jm_2 \rangle = \frac{1}{f} q^{-M} \sum_{\text{perm}(a_1...a_k)} q^{2\chi(a_1...a_k)} \delta_{m_1m_2} = \delta_{m_1m_2}. \] (3.11)

The \(SU(2)_q\) - invariant quantity built up of the tensors \( < T | \) and \( | U >\) of spin \( j = k/2\) is

\[ I = \langle T | q^{-J^0} | U \rangle = (T^{a_k...a_1})^* q^{-J^0} U^{a_1...a_k} = \sum_{m=-j}^{+j} (T^{jm})^* U^{jm} q^{-m}. \] (3.12)

The second type of the \(SU(2)_q\) - invariant quantity built up of the tensors \( | T >\) and \( | U >\) of spin \( j = k/2\) is

\[ I' = T^{a_k...a_1} U^{b_1...b_k} \epsilon_{a_1b_1} \epsilon_{a_2b_2} ... \epsilon_{a_kb_k} \] (3.13)

with \( \epsilon_{ab} \) given in (2.11). Of course, \( T^a T^b \epsilon_{ab} = 0 \) if \( T^a \) and \( T^b \) q-commute. Furthermore, using equation (3.3) we can also write

\[ I = q^{\chi(s)}(T^{a_k...a_1})^* q^{-J^0} U^{s(a_1...a_k)} \]
\[ I' = q^{\chi(s)}T^{a_k...a_1} U^{s(b_1...b_k)} \epsilon_{a_1b_1} ... \epsilon_{a_kb_k} \] (3.14)

where \( s \in S_k \) is a fixed permutation of the indices \( a_1...a_k \) and \( \chi(s) = \chi(a_1...a_k) - \chi(s(a_1...a_k)) \) is the number of inversions with respect to the \((a_1...a_k)\) order.

4. \( q \) - Clebsch-Gordan coefficients

Here we present a new simple method for calculating the q-deformed Clebsch-Gordan coefficients. It can be immediately extended and applied to \(SU(n)_q\) and other quantum groups. This method is a consequence of the previously described tensor method and construction of invariants. Our notation is

\[ | JM \rangle = \sum_{m_1, m_2} \langle j_1m_1 j_2m_2 | JM \rangle_q | j_1m_1 \rangle \langle j_2m_2 |. \] (4.1)

For \( q \in R \), \( C - G \) coefficients are real

\[ \langle j_1m_1 j_2m_2 | JM \rangle_q^* = \langle j_1m_1 j_2m_2 | JM \rangle_q \] (4.2)

and

\[ \langle j_1m_1 j_2m_2 | JM \rangle_q = \langle JM | j_1m_1 j_2m_2 \rangle_q \] (4.3)

Using the tensor notation \( | jm \rangle = | e_{(a_1...a_k)} \rangle \) ((3.4), (3.9) and (3.10)), we first calculate \( C - G \) coefficient for \( j_1 \otimes j_2 \rightarrow j_1 + j_2 \):
\[ \langle j_1 + j_2 \, m_1 + m_2 \mid j_1 m_1 \, j_2 m_2 \rangle_q = \]
\[ = \langle e_{\{b_1 \ldots b_k \ldots a_1 \}} \mid e_{\{a_1 \ldots a_k \}} e_{\{b_1 \ldots b_1 \}} \rangle = \]
\[ = \langle e_{\{b \}} \mid e_{\{a \}} e_{\{b \}} \rangle = \]
\[ = \frac{1}{\sqrt{f_1 f_2 f_3}} q^{-\frac{1}{2}(M_1 + M_2 + M_3)} \sum_{\text{perm}(a),(b)} q^{\chi(a) + \chi(b) + \chi(a,b)} = \] (4.4)
\[ = \sqrt{\frac{f_1 f_2}{f_3}} q^{\frac{1}{2}(M_1 + M_2 - M_3)} q^{(j_1 - m_1)(j_2 + m_2)} = \]
\[ = \sqrt{\frac{f_1 f_2}{f_3}} q^{j_1 m_2 - j_2 m_1} . \]

where we have used

\[ \chi(a,b) = \chi(a) + \chi(b) + (j_1 - m_1)(j_2 + m_2) \] (4.5)

and equation (3.6) together with the abbreviations

\[ k = 2j_1 \quad l = 2j_2 \quad j_3 = j_1 + j_2 \quad m_3 = m_1 + m_2 \]
\[ M_i = (j_i + m_i)(j_i - m_i) \]
\[ f_i = \left( \begin{array}{c} 2j_i \\ j_i + m_i \end{array} \right)_q \] (4.6)

\[ \frac{f_1 \cdot f_2}{f_3} = \frac{[2j_1]_q! [2j_2]_q! [j_3 + m_3]_q! [j_3 - m_3]_q!}{[j_1 + m_1]_q! [j_1 - m_1]_q! [j_2 + m_2]_q! [j_2 - m_2]_q! [2j_3]_q!} \]

The main observation is that any $C - G$ coefficient $\langle j_1 m_1 j_2 m_2 \mid JM \rangle$ can be written in the form (4.4). Namely, the $C - G$ coefficient $\langle j_1 m_1 j_2 m_2 \mid JM \rangle$ is projection of the state $\langle j_1 m_1 \otimes j_2 m_2 \rangle = \langle e_{\{a_1 \ldots a_2 j_1 \}} e_{\{b_1 \ldots b_2 j_2 \}} \rangle$ from the tensor product space $V^*_{2j_1 + 1} \otimes V^*_{2j_2 + 1}$ to the state $|JM \rangle = |e_{\{a_1 \ldots a_{2j_1 - n + 1} \ldots a_{2j_2} \ldots b_n \ldots b_{n + 1} \ldots b_{2j_2} \}} \rangle$ (with the appropriate symmetry of $2j_1 + 2j_2$ indices) in the space $V_{2J + 1} \subset V^*_{2j_1 + 1} \otimes V^*_{2j_2 + 1}$. Here, the square brackets $[\ldots]$ denote $q$-antisymmetrization and $n = 2j = j_1 + j_2 - J$. 

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Furthermore, the state $|e_{[a_1 [a_2...[a_n,b_n], b_2]b_1]} > \propto \epsilon_{a_1b_1}...\epsilon_{a_nb_n}$ transforms as a singlet, i.e. it is invariant under the coproduct action in the tensor product space $V_n \otimes V_n$. Hence, using the equation (3.4), we can write

$$\langle j_1 m_1 j_2 m_2 | JM \rangle_q = \mathcal{N} \sum_{\text{perm}(a,b),(c,d)} \langle e_{\{a,b\}} e_{\{c,d\}} | e_{\{a,d\}} \rangle \cdot (\epsilon_{b,c})_n$$

$$= \mathcal{N} q^{\frac{1}{2}(M_1 + M_2 + M_j)} \sqrt{f_1 \cdot f_2 \cdot f_j} \sum_{\text{perm}(a,b),(c,d)} q^{\chi(a,b) + \chi(c,d) + \chi(a,d)} (\epsilon_{b,c})_n$$

(4.7)

where the length of $b(c)$ is $n = j_1 + j_2 - J$, $(\epsilon_{b,c})_n = \epsilon_{b_1c_1}...\epsilon_{b_nc_n}$ and

$$\mathcal{N} = \left( \frac{[2 j_1]_q ![2 j_2]_q ![2 J + 1]_q}{[j_1 + j_2 - J]_q ![j_1 - j_2 + J]_q ![-j_1 + j_2 + J]_q ![j_1 + j_2 + J + 1]_q} \right)^{\frac{1}{2}}.$$  

Expression (4.7) is efficient for practical calculation of $C - G$ coefficients (see Appendix). We also present a simple derivation of the standard expression for $q - C - G$ coefficients [6]. Using the decomposition

$$< j_1 m_1 | = \sum_{m=-j}^{+j} < j_1 m_1 | j_1 - j m_1 - m, j m >_q < j_1 - j m_1 - m | < j m |$$

$$< j_2 m_2 | = \sum_{m=-j}^{+j} < j_2 m_2 | j - m, j_2 - j m_2 + m >_q < j - m | < j_2 - j m_2 + m |$$

(4.9)

$$| JM > = \sum_{m=-j}^{+j} < j_1 - j m_1 - m, j_2 - j m_2 + m | JM >_q | j_1 - j m_1 - m | < j_2 - j m_2 + m |$$

we immediately write

$$< j_1 m_1 j_2 m_2 | JM >_q = N \sum_{m=-j}^{+j} < j_1 m_1 | j_1 - j m_1 - m, j m >_q$$

$$\times < j_2 m_2 | j - m, j_2 - j m_2 + m >_q < j m, j - m | 00 >_q$$

(4.10)

$$\times < j_1 - j m_1 - m, j_2 - j m_2 + m | JM >_q$$

where $N$ is the norm depending on $j_1, j_2$ and $J$. Three of the four $C - G$ coefficients appearing on the right-hand side have the simple form (4.4). The fourth coefficient $< j m, j - m | 00 >$ also has a simple form. Namely, for $n = 2j$ we have

$$< j m, j - m | 00 >_q = \frac{1}{\sqrt{[n + 1]_q}} \epsilon_{a_1b_1}...\epsilon_{a_nb_n} =$$
The denominator $\sqrt{[2j+1]}$ comes from the orthonormality condition. Finally, inserting equations (4.4) and (4.11) into equation (4.10) we find

$$
\langle j_1 m_1 j_2 m_2 | J M \rangle_q = N \sum_{m=-j}^{+j} \left( \frac{(-)^{j-m} q^{j_1 m_2 - j_2 m_1}}{\sqrt{[2j+1]}_q} \right) 
\times q^{m(2J+2j+1)} \left( \begin{array}{c} 2j \\ j + m \end{array} \right)_q \left( \begin{array}{c} 2j_1 - 2j \\ j_1 - j + m_1 - m \end{array} \right)_q \left( \begin{array}{c} 2j_2 - 2j \\ j_2 - j + m_2 + m \end{array} \right)_q
\right)
$$

with $j_1 + j_2 - j = J + j$. This result agrees with the result found by Ruegg [6] if the normalization factor $N$ is taken as

$$
N = \left\{ \frac{[2j_1]_q! [2j_2]_q! [2J+1]_q! [j_1 + j_2 - J + 1]_q}{[j_1 + j_2 - J]_q! [j_1 - j_2 + J]_q! [-j_1 + j_2 + J]_q! [j_1 + j_2 + J + 1]_q!} \right\}^{\frac{1}{2}}
$$

We point out that our tensor method is simple and can be easily applied to $SU(n)_q$ for $n \geq 3$. We also mention that it can be applied to multiparameter quantum groups. For example, it can be shown [7] that $C - G$ coefficients for the two - parameter $SU(2)_{p,q}$ [8] depend effectively on one parameter only.

5. Symmetry relations

For completeness we rederive the known symmetry relations for $q - C - G$ coefficients and $q - 3 - j$ symbols. From equation (4.4) immediately follow symmetry relations

$$
< j_1 - m_1 j_2 - m_2 | j_1 + j_2 - m_1 - m_2 >_q = \\
= < j_2 m_2 j_1 m_1 | j_1 + j_2 m_1 + m_2 >_q = \\
= < j_1 m_1 j_2 m_2 | j_1 + j_2 m_1 + m_2 >_{q^{-1}}
$$

and

$$
< j_1 - m_1 j_1 + j_2 m_1 + m_2 | j_2 m_2 >_q = (-)^{j_1 + m_1} q^{-m_1} \\
\times \frac{[2j_2 + 1]_q}{[2j_1 + 2j_2 + 1]_q} < j_1 m_1 j_2 m_2 | j_1 + j_2 m_1 + m_2 >_q .
$$
Furthermore, from equation (4.11) we have
\[
< j - m \ j m \mid 00 \>_q = (-)^{2j} < jm \ j - m \mid 00 \>_q^{-1}
\]
\[
< jm \ 00 \mid jm \>_q = 1.
\]

(5.3)

The symmetry relations (5.1)-(5.3) are sufficient to derive the symmetries of the general \(C - G\) coefficients. From equation (4.10) we obtain
\[
< j_1 - m_1 \ j_2 - m_2 \mid J - M \>_q = < j_2 m_2 \ j_1 m_1 \mid JM \>_q = (-)^{j_1 + j_2 - J} < j_1 m_1 \ j_2 m_2 \mid JM \>_q^{-1}
\]

and
\[
< j_1 - m_1 \ JM \mid j_2 m_2 \>_q = (-)^{J - j_2 + m_1} q^{-m_1}
\]

\[
\times \sqrt{\frac{[2j_2 + 1]_q}{[2J + 1]_q}} < j_1 m_1 \ j_2 m_2 \mid JM \>_q.
\]

(One can deduce this directly from (4.7) )

We can define the \(q\)-deformed 3 \(- j\) symbol as
\[
\left( \begin{array}{ccc} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{array} \right)_q = q^{\frac{j(j_2 - m_1)}{2}} (-)^{j_1 - j_2 - m_3} \sqrt{\frac{[2j_2 + 1]_q}{[2J + 1]_q}} < j_1 m_1 \ j_2 m_2 \ j_3 - m_3 \>_q
\]

(5.6)

where the additional factor \(q^{1/3(m_2 - m_1)}\) comes from the requirement that symmetry relations for the \((3 - j)_q\) coefficients should not contain explicit \(q\)-factors:

\[
\left( \begin{array}{ccc} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3 \end{array} \right)_q = \left( \begin{array}{ccc} j_2 & j_1 & j_3 \\ m_2 & m_1 & m_3 \end{array} \right)_q = (-)^{j_1 + j_2 + j_3} \left( \begin{array}{ccc} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{array} \right)_q^{-1}
\]

(5.7)

and that the \((3 - j)_q\) coefficients are invariant under cyclic permutations. Note that the \(SU(2)_q\) invariant, built up of the three states \(|j_1 m_1 \rangle, |j_2 m_2 \rangle\) and \(|j_3 m_3 \rangle\), is

\[
\sum_{m_1, m_2, m_3} < j_3 - m_3 \ j_3 m_3 \mid 00 \>_q < j_1 m_1 \ j_2 m_2 \mid j_3 - m_3 \>_q |j_1 m_1 \rangle |j_2 m_2 \rangle |j_3 m_3 \>_q
\]

\[
= \sum_{m_1, m_2, m_3} q^{\frac{j(j_1 - m_1)}{2}} \left( \begin{array}{ccc} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{array} \right)_q |j_1 m_1 \rangle |j_2 m_2 \rangle |j_3 m_3 \>_q
\]

\[
= \sum_{m_1, m_2, m_3} N_{123}(\epsilon_{(b,c)})_{k_1}(\epsilon_{(d,e)})_{k_2}(\epsilon_{(a,f)})_{k_3}|e_{(a,b)} \rangle |e_{(c,d)} \rangle |e_{(e,f)} \>_q.
\]

(5.8)

Now we identify
\begin{align}
< j_1m_1 j_2m_2|j_3 - m_3 >_q < j_3 - m_3 j_3m_3|00 >_q &= \\
= q^2(1-m_3) \binom{j_1 \ j_2 \ j_3}{m_1 \ m_2 \ m_3}_q = \\
= N_{123}(\epsilon_{(b,c)})_{k_1}(\epsilon_{(d,e)})_{k_2}(\epsilon_{(a,f)})_{k_3}
\end{align}

where, for example, \((\epsilon_{(b,c)})_k = \epsilon_{b_1c_1}...\epsilon_{b_kc_k}\) with

\[ k_1 = j_1 + j_2 - j_3 \quad k_2 = -j_1 + j_2 + j_3 \quad k_3 = j_1 - j_2 + j_3 \]  

and \(N_{123}\) is the normalization factor fully symmetric in indices \((123)\). Equation (5.9) represents the connection with the tensor notation used.

6. Covariant \(q\) - oscillators and irreducible tensor operators

Let us define the \(q\)-bosonic operators \(a_i\) and \(a_i^+(i = 1, 2)\) such that \(|e_i> = a_i^+|0,0>_F\) and \(<e_i| =_F <0,0|a_i\), where \(|0,0>_F\) denotes the (Fock) vacuum state invariant under \(SU(2)_q\). Hence, \(a_1^+\) and \(a_2^+\) are covariant operators transforming as an \(SU(2)_q\) doublet. Therefore, analogously as in equation (3.2), they \(q\)-commute

\[ a_2^+ a_1^+ = q a_1^+ a_2^+. \]  

Furthermore, we define the projector \(P_{(j-k/2)}\) from the tensor space \((V_2)^\otimes_k\) to the totally \(q\)-symmetric space carrying an irreducible representation of spin \(j = k/2\)

\[ P_{(j-k/2)}|e_i,...,i_k> = \frac{1}{\sqrt{|k|_q!}} a_{i_1}^+...a_{i_k}^+|0,0>_F = \]

\[ = \frac{1}{\sqrt{|k|_q!}} q^{\chi(i_1...i_k)}(a_1^+)^{n_1}(a_2^+)^{n_2}|0,0>_F. \]  

We find from equation (3.4) that

\[ |jm> = q^M \frac{(a_1^+)^{n_1}(a_2^+)^{n_2}}{\sqrt{|n_1|_q!|n_2|_q!}}|0,0>_F \]

\[ j = \frac{1}{2}(n_1 + n_2) \quad m = \frac{1}{2}(n_1 - n_2) \]

We define the number operators \(N_i\) and \(N\) as

\[ N_i|jm\> = N_i|n_1, n_2\> = n_i|n_1, n_2\> \]

\[ N = N_1 + N_2 \quad [N, N_i] = 0 \quad [N_1, N_2] = 0 \]

\[ [N_i, a_j^+] = \delta_{ij} a_i^+ \quad [N_i, a_j] = -\delta_{ij} a_i \]

\[ [N, a_i^+] = a_i^+ \quad [N, a_i] = -a_i. \]  

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The action of $a_i^+$ and $a_i$ on the basis vectors $|jm>$ is

\[
\begin{align*}
    a_i^+ |jm> &= q^{-\frac{1}{2}n_2} \sqrt{[n_1+1]_q} |j+\frac{1}{2}, m+\frac{1}{2}> \\
    a_2^+ |jm> &= q^\frac{1}{2}n_1 \sqrt{[n_2+1]_q} |j+\frac{1}{2}, m-\frac{1}{2}> \\
    a_1 |jm> &= q^{-\frac{1}{2}n_2} \sqrt{[n_1]_q} |j-\frac{1}{2}, m-\frac{1}{2}> \\
    a_2 |jm> &= q^\frac{1}{2}n_1 \sqrt{[n_2]_q} |j-\frac{1}{2}, m+\frac{1}{2}>. 
\end{align*}
\]

(6.5)

The commutation relations between $a_i$ and $a_j^+$ follow immediately:

\[
\begin{align*}
    a_2^+ a_1^+ &= q a_1^+ a_2^+ \quad a_2 a_1 = q^{-1} a_1 a_2 \\
    a_2 a_1^+ &= a_1^+ a_2 \quad a_1 a_2^+ = a_2^+ a_1
\end{align*}
\]

(6.6)

and

\[
\begin{align*}
    a_1 a_1^+ &= q^{-N_2} [N_1+1]_q \quad a_1^+ a_1 = q^{-N_2} [N_1]_q \\
    a_2 a_2^+ &= q^{+N_1} [N_2+1]_q \quad a_2^+ a_2 = q^{+N_1} [N_2]_q \\
    H &= a_1^+ a_1 + a_2^+ a_2 = [N]_q
\end{align*}
\]

(6.7)

Then

\[
\begin{align*}
    a_1 a_1^+ - q a_1^+ a_1 &= q^{-N} \\
    a_2 a_2^+ - q^{-1} a_2^+ a_2 &= q^{+N}
\end{align*}
\]

(6.8)

and

\[
\begin{align*}
    a_1 a_1^+ - q^{-1} a_1^+ a_1 &= q^{2J^0} \\
    a_2 a_2^+ - q a_2^+ a_2 &= q^{2J^0}.
\end{align*}
\]

(6.9)

The generators $J^\pm$ and $J^0$ can be represented as

\[
\begin{align*}
    J^+ &= q^{-J^0+\frac{1}{2}} a_1^+ a_2 \\
    J^- &= q^{-J^0-\frac{1}{2}} a_2^+ a_1 \\
    2J^0 &= N_1 - N_2 \\
    [J^+, J^-] &= [2J^0]_q = [N_1 - N_2]_q \\
    [N, J^\pm] &= [N, J^0] = 0.
\end{align*}
\]

(6.10)

We point out that the oscillator operators $a_i$ and $a_i^+$ are covariant since the corresponding tensors $|e_{\{i_1..i_k\}}>$, equation (3.4), are covariant and irreducible by construction.

We note that the covariant q-Bose operators $a$, $a^+$ (6.1) are the same as in [9], where they were constructed using the Wigner $D^{(j)}$-functions. A different set of covariant operators was constructed in ref.[10]. Other constructions [11] are non-covariant in the sense that operators do not transform as $SU(2)_q$ doublet. In the non-covariant approach one has to solve an additional problem of constructing covariant, irreducible tensor operators [12].
The definition of the irreducible tensor operators of $SU(2)_q$ is

$$(J^\pm T_{km} - q^{-m}T_{km}J^\pm)q^{-J^0} =$$

$$= \sqrt{[k \mp m]_q[k \pm m + 1]_q} T_{m \pm 1}$$

$$[J^0, T_{km}] = m T_{km} \quad (6.11)$$

$$|jm > = T_{jm}|0,0 >_F$$

According to equations (6.1-6.3) we define a unit tensor operator as

$$T_{jm} = q^{j_1n_1n_2} \frac{(a_1^+)^{n_1}(a_2^+)^{n_2}}{\sqrt{[n_1]_q!![n_2]_q!!}} \quad (6.12)$$

which is covariant and irreducible by construction and satisfies the requirements (6.11) automatically. Note that $(T_{km})^+$ transforms as contravariant tensor. One can define the tensor

$$V_{k\mu} = (-)^{k-\mu} q^\mu T_{k-\mu}^+ \quad (6.13)$$

which transforms as covariant, irreducible tensor. In the tensor notation we have

$$V_{\{i_1...i_k\}}^{+} = \epsilon_{i_1j_1}...\epsilon_{i_kj_k} T_{\{j_1...j_k\}} = (-)^{n_2} q^{1/2(n_1-n_2)} T_{k-\mu} \quad (6.14)$$

For completeness, we present relations between the Biedenharn operators $b_i, b_i^+$ of ref.[11], $t_i, t_i^+$ of ref.[10] and $a_i, a_i^+$ of the present paper:

$$b_1 = q^{-N_2-1/2N_1} t_1 = q^{1/2} N_2 a_1$$
$$b_2 = q^{-1/2N_2} t_2 = q^{1/2} N_1 a_2$$
$$b_1^+ = t_1^+ q^{-N_2-1/2N_1} = a_1^+ q^{1/2} N_2$$
$$b_2^+ = t_2^+ q^{-1/2N_2} = a_2^+ q^{-1/2N_1} \quad (6.15)$$

We point out that the general covariant oscillators (e.g. $t_i, t_i^+$ and $a_i, a_i^+$) are characterized by the anyonic type $q$-commutation relation (6.1). Actually, equation (6.1) is a consequence of underlying braid group symmetry and can be also obtained from the $SU(2)_q$ R-matrix [10].

Finally, we give the Borel-Weil realization

$$a_i^+ \equiv X_i \quad a_1 \equiv D_i \quad i = 1, 2 \quad (6.16)$$

which is covariant automatically. The commutation relations are
\[ X_2 X_1 = q X_1 X_2 \quad D_2 D_1 = q^{-1} D_1 D_2 \]
\[ D_1 X_1 = q X_1 D_1 + q^{-N} \quad D_2 X_2 = q^{-1} X_2 D_2 + q^N \]  
\[ [D_i, X_j] = 0 \quad i \neq j \]  

or

\[ D_1 X_1 = q^{-1} X_1 D_1 + q^{2J_0} \]
\[ D_2 X_2 = q X_2 D_2 + q^{2J_0} \]  

where

\[ N_i = X_i \partial_i \]
\[ \partial_i = \frac{\partial}{\partial X_i} \]  

It follows that

\[ D_i X_i^n = [n]_q X_i^{n-1} \]
\[ D_1 = \frac{1}{X_1} [X_1 \partial_1]_q q^{-X_2 \partial_2} \]
\[ D_2 = \frac{1}{X_2} [X_2 \partial_2]_q q^{X_1 \partial_1} \]  

\[ 15 \]
Appendix

We demonstrate usefulness of the equation (4.7) for the practical calculations. Using equations (4.5) and (4.11) we write:

\[
\chi(a, b) = \chi(a) + \chi(b) + n_2(a)n_1(b)
\]
\[
\chi(c, d) = \chi(c) + \chi(d) + n_2(c)n_1(d)
\]
\[
\chi(a, d) = \chi(a) + \chi(d) + n_2(a)n_1(d)
\]
\[
\chi(b) = \chi(c)
\]
\[
(\epsilon_{b,c})_n = (-)^{n_2(b)}q^{\frac{1}{2}(n_1(b) - n_2(b))}
\]

where

\[
\begin{align*}
n &= 2j = j_1 + j_2 - J \\
n_1(b) &= n_2(c) = j + m \\
n_2(b) &= n_1(c) = j - m \\
n_1(a) &= j_1 - j + m_1 - m \\
n_2(a) &= j_1 - j - m_1 + m \\
n_1(d) &= j_2 - j + m_2 + m \\
n_2(d) &= j_2 - j - m_2 - m
\end{align*}
\]

After inserting equation (3.6) into equation (4.7), we immediately obtain the final result, equation (4.12):

\[
N q^{-\frac{1}{2}(M_1+M_2+M_J)} \sum_{n_1(b) = 0}^{2j} \sum_{\text{perm}(a)} \sum_{\text{perm}(b)} \sum_{\text{perm}(d)} q^{n_2(a)n_1(b)+n_1(b)n_1(d)+n_2(a)n_1(d)}
\]
\[
\times q^{2\chi(a)+2\chi(b)+2\chi(d)}(\epsilon_{b,c})_2j =
\]
\[
= N q^{-\frac{1}{2}(M_1+M_2+M_J)} \sum_{m=-j}^{+j} q^{n_2(a)n_1(b)+n_1(b)n_1(d)+n_2(a)n_1(d)}
\]
\[
\times f_a f_b f_d q^{n_1(a)n_2(a)+n_1(b)n_2(b)+n_1(d)n_2(d)}(\epsilon_{b,c})_2j
\]
\[
= N \sum_{m=-j}^{+j} (-)^{j-m} q^{j_1m_2-j_2j_1} q^{m(2J+2j+1)}
\]
\[
\times f_a f_b f_d \frac{1}{\sqrt{f_1 f_2 f_J}}.
\]

We extend this simple calculation of the \(SU(2)_q\) C.-G. coefficients to the \(SU(N)_q\) quantum groups in the forthcoming paper.
Acknowledgements

This work was supported by the joint Croatian-American contract NSF JF 999 and the Scientific Fund of Republic of Croatia.
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