On the Clebsch - Gordan coefficients for the two - parameter quantum algebra $SU(2)_{p,q}$

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Abstract

We show that the Clebsch - Gordan coefficients for the $SU(2)_{p,q}$ - algebra depend on a single parameter $Q = \sqrt{pq}$, contrary to the explicit calculation of Smirnov and Wehrhahn [J.Phys.A 25 (1992),5563].

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Recently, the Clebsch-Gordan problem for the two-parameter quantum algebra $SU(2)_{p,q}$ was analyzed [Smirnov and Wehrhahn 1992]. It was claimed that the corresponding C.-G. coefficients do depend on the two deforming parameters $p$ and $q$.

In this Comment we show that the C.-G. coefficients depend effectively only on one parameter $Q = \sqrt{pq}$, and that $SU(2)_{p,q}$ is isomorphic to $SU(2)_Q$, both as algebras and Hopf co-algebras. Our results are in agreement with [Drinfeld 1989].

We recall the $SU(2)_{p,q}$ algebra defined in [Smirnov and Wehrhahn 1992] ($p$ and $q$ are real parameters):

\[
\begin{align*}
[J_0, J_\pm] &= \pm J_\pm \\
[J_+, J_-]_{p,q} &= J_+ J_- - pq^{-1}J_- J_+ = [2J_0]_{p,q} \\
[2J_0]_{p,q} &= \frac{q^{2J_0} - p^{-2J_0}}{q - p^{-1}} \\
(J_0)^\dagger &= J_0 \\
(J_\pm)^\dagger &= J_\mp
\end{align*}
\]

The coproduct $\Delta$ is:

\[
\begin{align*}
\Delta(J_\pm) &= J_\pm \otimes p^{-J_0} + q^{J_0} \otimes J_\pm \\
\Delta(J_0) &= J_0 \otimes 1 + 1 \otimes J_0
\end{align*}
\]

The finite dimensional unitary irreducible representation (IRREP) $D^j$ of spin $j$ contains the highest weight vector $|jj>$, satisfying

\[
\begin{align*}
J_0 |jj> &= j |jj> \\
J_+ |jj> &= 0 \\
<jj|jj> &= 1
\end{align*}
\]

The other orthonormalized states of IRREP $D^j$, $|jm>$, with $-j \leq m \leq j$, satisfy
\begin{align*}
J_+ \mid jm >_{p,q} &= (pq^{-\frac{1}{2}})^{(j-m-1)} \sqrt{[j-m]_{p,q}[j+m+1]_{p,q}} \mid jm + 1 >_{p,q} \\
J_- \mid jm >_{p,q} &= (pq^{-\frac{1}{2}})^{(j-m)} \sqrt{[j+m]_{p,q}[j-m+1]_{p,q}} \mid jm - 1 >_{p,q} \\
J_0 \mid jm >_{p,q} &= m \mid jm >_{p,q}
\end{align*}

Now, we define the $SU(2)_Q$ algebra with three generators $(J_\pm)_Q$ and $(J_0)_Q$:

\begin{align*}
[(J_0)_Q,(J_\pm)_Q] &= \pm (J_\pm)_Q \\
[(J_+)_Q,(J_-)_Q] &= [2(J_0)_Q]_Q \\
[n]_Q &= \frac{Q^n - Q^{-n}}{Q - Q^{-1}} = \left(\frac{p}{q}\right)^{\frac{1}{2}(n-1)}[n]_{p,q}
\end{align*}

with the coproduct

\begin{align*}
\Delta((J_\pm)_Q) &= (J_\pm)_Q \otimes Q^{-(J_0)_Q} + Q^{+(J_0)_Q} \otimes (J_\pm)_Q \\
\Delta((J_0)_Q) &= (J_0)_Q \otimes 1 + 1 \otimes (J_0)_Q
\end{align*}

The relations between the $SU(2)_{p,q}$ generators and the $SU(2)_Q$ generators are

\begin{align*}
J_+ &= \left(\frac{q}{p}\right)^{\frac{1}{2}(J_0 - \frac{1}{2})}(J_+)_Q \\
J_- &= \left(\frac{q}{p}\right)^{\frac{1}{2}(J_0 + \frac{1}{2})}(J_-)_Q \\
J_0 &= (J_0)_Q
\end{align*}

It is easy to show that relations (7) map equations (1) and equations (5) one into another. Moreover, the $SU(2)_{p,q}$ coproduct is identical to the $SU(2)_Q$ coproduct, $\Delta_{p,q} \equiv \Delta_Q$:

\begin{align*}
\Delta_{p,q}(J_0) &= \Delta_Q(J_0) = J_0 \otimes 1 + 1 \otimes J_0 \\
\Delta_{p,q}(J_\pm) &= \Delta_Q((J_\pm)_Q) = (\frac{q}{p})^{\frac{1}{2}(J_0 + \frac{1}{2})}(J_\pm)_Q \\
&= J_\pm \otimes p^{-J_0} + q^{J_0} \otimes J_\pm
\end{align*}

This is also true for the antipode $\gamma_{p,q} \equiv \gamma_Q$, the counit $\varepsilon_{p,q} \equiv \varepsilon_Q$, the states $\mid jm >_{p,q} \equiv \mid jm >_Q$ and the Casimir operator $(C_2)_{p,q} \equiv (C_2)_Q$. Thus we have
proved the Hopf-algebra isomorphism between $SU(2)_{p,q}$ and $SU(2)_Q$. As a consequence of this isomorphism, the $p,q$ C.-G. coefficients of $SU(2)_{p,q}$ should be identical to those of $SU(2)_Q$.

Returning to the Smirnov and Wehrhahn’s paper, one can immediately show, using our equation (5) and $[n]_{p,q} = (q/p)^{n-1}[n]_Q$, that all the equations in Section 2 of their paper can be reduced to the equations with single parameter $Q$. Particularly, the states in equation (2.7) can be written as $|jm>_{p,q} = |jm>_{Q}$. Therefore, the projection operator $P_{mm'}^j = |jm> <jm'|$ also has to be expressed in terms of the $Q$-parameter only. However, their projection operator, eq.(3.7), is wrong since it explicitly depends on the $(p/q)$ parameter and does not satisfy the projection property $(P_{mm'}^j)^2 = P_{mm}^j$. Hence, all the subsequent formulae in Section 5 (starting with the equation (5.3)) are incorrect. For example, the C.-G. coefficients given in Table 1 do not satisfy the orthonormality relations (4.10).

The correct formula for the projection operator should read

$$P_{mm'}^j = (\frac{p}{q})^{-\frac{1}{2}(j-m)(j-m-1)} \sqrt{\frac{(j+m)_{p,q}!}{[2j]_{p,q}!(j-m)_{p,q}!}} j^{j-m} P_{mm'}^j \times (\frac{p}{q})^{-\frac{1}{2}(j-m')(j-m'-1)} \sqrt{\frac{(j+m')_{p,q}!}{[2j]_{p,q}!(j-m')_{p,q}!}} j^{j'-m'}$$

$$\equiv (P_{mm'}^j)_Q$$ (9)

Using this projector we obtain the C.-G. coefficients which depend on the single parameter $Q$. In this way, these C.-G. coefficients are identical to those for $SU(2)_Q$:

$$<j_1 m_1 j_2 m_2 | JM >_{p,q} = <j_1 m_1 j_2 m_2 | JM >_{Q=\sqrt{pq}}$$ (10)

for all $j_1, m_1, j_2, m_2$, $J$ and $M$. 

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References

Drinfeld V G 1989 Algebra i analiz 1, 1 (in Russian)